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Pitfalls of downside performance measures with arbitrary targets

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Abstract

The Sharpe ratio has been criticized with regard to the assumptions of mean-volatility portfolio selection. Downside performance measures were developed to resolve this critique; they are consistent with expected utility under less restrictive assumptions. The most prominent family of downside performance measures is known as Kappa ratios and puts above target returns into relation to lower partial moments. While the Sharpe ratio of a mutual fund examines whether portfolios of mutual fund and risk-free asset dominate risk-adjusted passive portfolios of benchmark and risk-free asset, this characteristic cannot be transferred to downside performance measures with arbitrary targets. We show that Kappa ratios assign different values to passive strategies with varying fractions of benchmark and risk-free asset if the target differs from the risk-free rate. This effect can lead to reverse rankings of inferior and superior performing mutual funds. In addition, even the ratio of excess return and excess downside risk of passive portfolios is not constant in general. Therefore, downside performance measures turn out to be only applicable in asset management if the target is set equal to the risk-free rate.

Keywords

Downside risk, Kappa ratios, lower partial moment, performance measurement, Sharpe ratio

JEL classification

D81, G11

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1. Introduction

The classical Sharpe (1966) ratio, i.e. expected rate of return of a financial asset above the risk-free rate divided by its volatility, has been criticized with regard to the assumptions of mean-volatility based investment decisions. Some authors link the Sharpe ratio with quadratic utility, that exhibits increasing risk aversion. Actually, quadratic utility is only required for consistency of mean-variance decisions with expected utility theory if arbitrary distribution functions are allowed, especially if continuously as well as discretely distributed returns are evaluated by the investor (Johnstone and Lindley, 2011). However, in many cases of performance measurement and other applications of the mean-variance criterion on the capital market, we can assume that assets' returns belong to the same class of distribution functions. No assumption about the properties of the utility function is needed if a normal distribution can be assumed because a normal distribution is completely characterized by mean and variance. Of course, the class of distribution functions that can be described by mean and variance is not limited to the normal distribution.

Other authors criticize the Sharpe ratio because it measures risk in a symmetric way. They argue that a symmetric risk measure does not properly reflect the risk attitude of investors. This ignores that symmetric measurement of risk does not imply symmetric evaluation of profits and losses. Expected utility combines "beliefs" about the distribution of the return on financial assets and "tastes" in terms of risk preferences. Therefore, a symmetric risk measure can be combined with a utility function with decreasing marginal utility. There are multiple applications of expected utility in finance where, for example, the assumptions of normally distributed rates of return and exponential utility of the investor are combined. Other applications link log-normally distributed returns with a power utility function. A mean-variance criterion results in both cases and, of course, exponential and power utility functions exhibit positive, decreasing marginal utility.

Based on this criticism but also – and more important – based on weaker assumptions on "tastes" and "beliefs", downside-oriented performance measures were developed (Sortino and Price, 1994; Sortino et al., 1999; Dowd, 2000; Shadwick and Keating, 2002; Argawal and Naik, 2004; Darsino and Satchell, 2004; Kaplan and Knowles, 2004; Kazemi et al., 2004; Farinelli and Tibiletti, 2008). These ratios evaluate the return-risk trade-off of a financial asset, for example a mutual fund, in a downside-oriented framework. Here, return corresponds to expected rate of return above a predetermined target, also referred to as threshold rate of return, and risk is measured based on lower partial moments. Lower partial moments (LPM) measure the risk of realizing a rate of return R below target τ and are defined for continuously distributed returns as follows:

$$(1) \qquad \mathrm{LPM}_{n,\tau}(R) \equiv \int_{-\infty}^{\tau} (\tau - R)^n \, dF(R) = \mathrm{E}\left(\max\{\tau - R; 0\}\right)^n$$

where E(.): expected value

$F(R)$: cumulative distribution function

n: order of lower partial moment

Although not necessary from a theoretical perspective, n frequently is a natural number in applications. The focus on natural numbers allows a close relationship between lower partial moments and stochastic dominance criteria. For example, an investment decision based on lower partial moment of order one corresponds to second order stochastic dominance. A decision based on second order stochastic dominance is, in turn, consistent with expected utility maximization if the investor is risk averse in terms of positive and decreasing marginal utility. Moreover, a decision based on lower partial moment of order two, analogously, corresponds to third order stochastic dominance and is consistent with expected utility maximization if the investor exhibits decreasing risk aversion (Bawa, 1978).

Recent research proves that downside performance measures and the Sharpe ratio can be monotonically transformed into each other if portfolio rates of return belong to the same location-scale family of distributions (Chen et al., 2011; Schuhmacher and Eling, 2012; Schuhmacher and Breuer, 2014). This result is based on the work of Chamberlain (1983), Owan and Rabinovitch (1983), and Meyer (1987), who show that so-called multivariate q-radial (elliptical) distributions of the return of single assets correspond to location-scale distributions of the return of portfolios and location and scale parameters can be expressed, if existing, by mean and variance of the underlying distribution function. Empirical studies support this finding by observing similar rankings of investment funds (Eling and Schuhmacher, 2007; Eling, 2008) when different performance measures are used. Both theoretical and empirical papers in this field employ a target at the level of the risk-free rate. Ornelas et al. (2012) observe a decreasing rank correlation of performance measures when a different target is used.

However, an identical meaning of different performance measures does not imply that a mutual fund with higher downside performance ratio and Sharpe ratio, respectively, provides a higher level of expected utility. Instead, “the best portfolio will be the one giving the best boundary; clearly this is the one for which [the Sharpe ratio] is the greatest” (Sharpe, 1966, p. 122). This means that not in any case the mutual fund itself provides a higher expected utility, but an expected utility maximizing combination of the particular mutual fund and the risk-free asset exists. With downside performance measures with arbitrary target, portfolios of mutual fund and target do not exist if the target differs from the risk-free rate because the target represents a required minimum return and not a tradable security. This causes distortions in downside-oriented performance measurement.

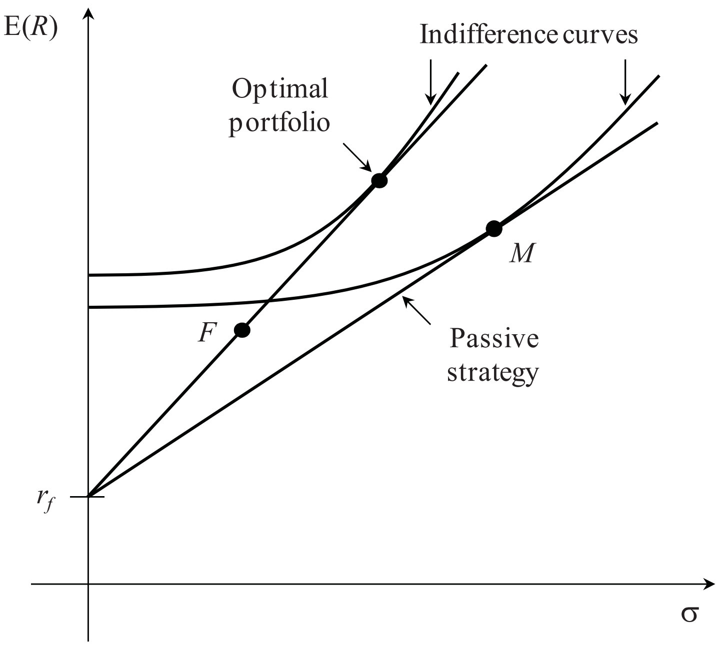
The remainder of this paper is organized as follows. Section 2 illustrates the functioning of the Sharpe ratio and describes well-known downside performance measures. In Section 3, we discuss possible wrong investment decisions if downside risk is measured by lower partial moment of order one with arbitrary target. Section 4 analyzes the general case of downside performance measurement based on lower partial moments of higher order. Section 5 briefly concludes.

2. Sharpe ratio and downside performance measures

The Sharpe ratio measures the return-risk trade-off of a financial investment in mean-volatility space and compares its ratio of expected excess return (above the risk-free rate r_f) and volatility σ to the corresponding ratio of the benchmark. Benchmark strategies are represented by a passive buy-and-hold portfolio of (preferably efficient) market index M and risk-free asset. Clearly, if the Sharpe ratio of a mutual fund F is higher than the Sharpe ratio of the benchmark, mutual fund F shows superior performance.

As indicated above, this does not imply that an investment solely in mutual fund F generates a higher expected utility than investing in benchmark M in any case. But with the help of mutual fund F the investor can reach a higher expected utility by combining F with the risk-free asset. In other words, if a mutual fund exhibits a higher Sharpe ratio than the benchmark, portfolios of mutual fund and risk-free asset dominate the corresponding risk-adjusted portfolios of benchmark and risk-free asset. Of course, this property can be transferred to performance rankings of mutual funds. Figure 1 illustrates this characteristic, where the mutual fund's Sharpe ratio is represented by the slope of the line through r_f and F. Sharpe (1966) applies volatility – instead of variance – as risk measure to obtain this identity of performance ratio and slope of the corresponding portfolio line.

Figure 1: *Functioning of the Sharpe ratio*



Downside performance measures also compute the ratio of expected excess return (here, above target τ) and downside risk where downside risk is determined on the basis of lower partial moments. Kaplan and Knowles (2004) introduce a general class of downside performance measures named Kappa ratios κ, also referred to as Sortino and Satchell (2001) ratios:

$$\text{(2)} \qquad \kappa_{n,\tau}(R) \equiv \frac{\mathrm{E}(R) - \tau}{\sqrt[n]{\mathrm{LPM}_{n,\tau}(R)}}$$

For n = 1, the Omega ratio of Shadwick and Keating (2002) can be transformed as follows (Kaplan and Knowles, 2004):

$$\text{(3)} \qquad \Omega_\tau(R) \equiv \frac{\int_\tau^\infty 1 - F(R)\,dR}{\int_{-\infty}^\tau F(R)\,dR} = \kappa_{1,\tau}(R) + 1$$

For n = 2, $\kappa_{2,\tau}(R)$ equals the Sortino ratio of Sortino and Price (1994). Besides, the upside-potential ratio (U-P) of Sortino et al. (1999) can also be expressed in terms of Kappa ratios:

$$\text{(4)}\qquad \begin{aligned} \text{U-P}_\tau(R) &\equiv \frac{\int_\tau^\infty R-\tau\, dR}{\sqrt{\mathrm{LPM}_{2,\tau}(R)}} = \frac{\mathrm{E}(R)-\tau+\mathrm{LPM}_{1,\tau}(R)}{\sqrt{\mathrm{LPM}_{2,\tau}(R)}} \\ &= \kappa_{2,\tau}(R)+\frac{\mathrm{LPM}_{1,\tau}(R)}{\sqrt{\mathrm{LPM}_{2,\tau}(R)}} = \kappa_{2,\tau}(R)\cdot\left(1+\frac{1}{\kappa_{1,\tau}(R)}\right) \end{aligned}$$

Hence, Omega ratio, Sortino ratio, and upside-potential ratio inherit the informative properties of Kappa ratios. For this reason, we concentrate on Kappa ratios in the following. Analogously to the Sharpe ratio, Kappa ratios can be geometrically illustrated by slopes of straight lines in $(\mathrm{E}(R), \sqrt[n]{\mathrm{LPM}_{n,\tau}(R)})$-space with intercept τ. But portfolio lines of risk-free asset and benchmark (or any other risky asset) appear in different shapes if lower partial moments of order one (Section 3) or higher order (Section 4) are used.

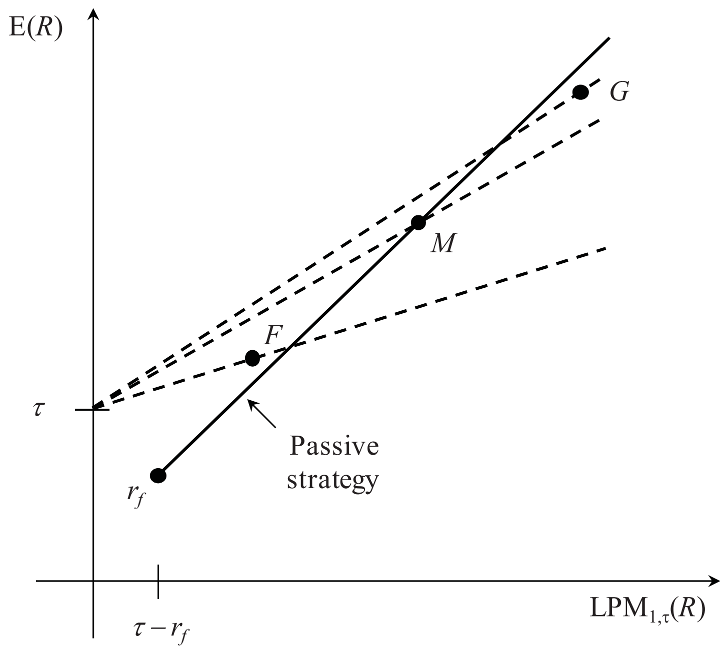
3. Kappa ratios of order one

Before we analyze the case of arbitrary targets, we refer to Bawa and Lindenberg (1977) who show that portfolios consisting of risk-free asset and benchmark can be positioned on a straight line in $(\mathrm{E}(R), \sqrt[n]{\mathrm{LPM}_{n,r_f}(R)})$-space if the target equals the risk-free rate (see also our appendix on the shape of portfolio lines in mean-lower partial moment space that covers this result as a special case). The intercept of this line corresponds to both target and risk-free rate. Therefore, the characteristics of the Sharpe ratio can be transferred to Kappa ratios $\kappa_{n,r_f}(R)$.

In addition to the Bawa and Lindenberg (1977) situation, we have to distinguish two more cases if the target differs from the risk-free rate. If the target exceeds the risk-free rate, the latter shows a positive lower partial moment; it amounts to $\mathrm{LPM}_{1,\tau}(r_f) = \tau - r_f$ according to formula (1). Therefore, the intercept of the particular portfolio line is neither the risk-free rate nor the target in this case. If the target is set below the risk-free rate, the intercept of the portfolio line remains the risk-free rate but does not equal the target. Both cases lead to distortions in downside performance measurement and corresponding ranking of investment alternatives.

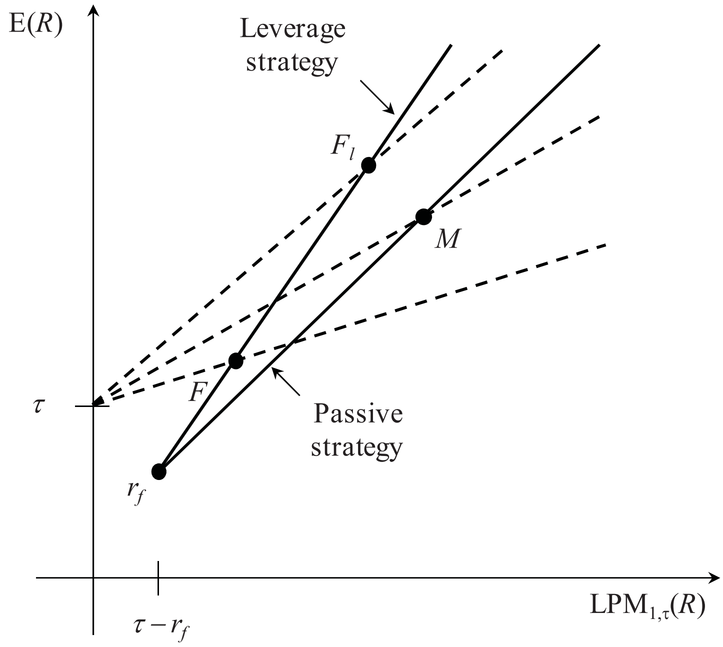
Figure 2 illustrates ranking reversals in $(\mathrm{E}(R), \mathrm{LPM}_{1,\tau}(R))$-space if the target exceeds the risk-free rate. In this figure, mutual fund *F* shows superior performance compared to the passive strategy because it is positioned above the line connecting benchmark *M* and risk-free asset. In contrast, mutual fund *G* exhibits inferior performance. At the same time, the Kappa ratio of order one, i.e. slope of the dashed lines in Figure 2, wrongly ranks mutual fund *G* superior to benchmark *M*, $\kappa_{1,\tau}(R_G) > \kappa_{1,\tau}(R_M)$, and *F* inferior to *M*, $\kappa_{1,\tau}(R_M) > \kappa_{1,\tau}(R_F)$. The case $\tau > r_f$ can be of practical relevance, e.g., for pension funds and life insurance companies in periods of low interest rates. The seemingly beneficial strategy of selecting investments according to the Kappa ratio of order one obviously leads to wrong decisions in this situation because investments would be selected that are dominated by the passive strategy (fund *G* in Figure 2).

Figure 2: *Ranking reversal by Kappa ratio of order one ($\tau > r_f$)*



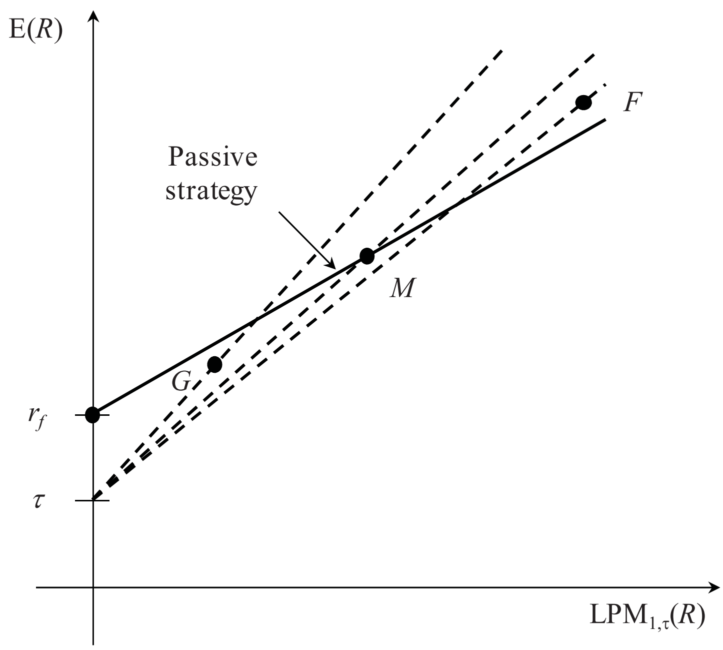
Moreover, the Kappa ratio of order one does not allow a ranking of mutual funds because it can be increased simply via levering. In Figure 3, portfolio F_l is realized by borrowing at the risk-free rate and investing in mutual fund *F*. In contrast, the Kappa ratio undesirably ranks F_l superior to *F*, $\kappa_{1,\tau}(R_{F_l}) > \kappa_{1,\tau}(R_F)$. Therefore, Kappa ratios with arbitrary targets in general fail to meet the requirement of being invariant to a change in portfolio leverage.

Figure 3: *Ranking reversal of Kappa ratio of order one by levering*



A similar picture occurs if the target is set below the risk-free rate. Figure 4 visualizes that the target is not a traded asset. Instead, the risk-free rate represents the intercept of the portfolio line in this situation. Compared to the previous figures, we changed the positions of investments *F* and *G*. Still, *G* exhibits inferior performance and *F* shows superior performance. The Kappa ratio of order one misleadingly attributes a reverse ranking to these funds.

Figure 4: *Ranking reversal by Kappa ratio of order one* ($\tau < r_f$)



As Figures 2 to 4 illustrate, Kappa ratios of order one only comply with the rationale of the Sharpe ratio if the target is set equal to the risk free rate. If the investor wishes to base her decision on the $(\mathrm{E}(R), \mathrm{LPM}_{1,\tau}(R))$-criterion for a different target, she should still select superior performing investments – dominating the passive strategy – and, subsequently, maximize expected utility by choosing her optimal mix with the risk-free asset. But maximization of a Kappa ratio with arbitrary target does not represent a suitable tool for asset management.

4. Kappa ratios of higher order

In $(\mathrm{E}(R), \sqrt[n]{\mathrm{LPM}_{n,\tau}(R)})$-space with $n > 1$, portfolio lines combining risk-free asset and benchmark (or any other risky asset) show a nonlinear shape (for a mathematical proof see the appendix). For $\tau > r_f$, once more, the lower partial moment of the risk-free asset is positive, $\sqrt[n]{\mathrm{LPM}_{n,\tau}(r_f)} = \tau - r_f$ (see, again, formula (1)). Therefore, curved lines similar to portfolios of imperfectly correlated risky assets in standard mean-volatility portfolio selection appear. The property of the Sharpe ratio, i.e. being invariant to portfolio leverage, cannot be transferred to downside performance measures in this situation. Kappa ratios again assign different values to passive strategies with different fractions of benchmark and risk-free asset.

Figure 5 illustrates the case $\tau > r_f$. As above, F dominates the passive strategy while G is dominated by the passive strategy. Again, Kappa ratios wrongly assign a reverse ranking, $\kappa_{n,\tau}(R_G) > \kappa_{n,\tau}(R_M) > \kappa_{n,\tau}(R_F)$. In contrast to Section 3, the risk-free asset does not represent the minimum lower partial moment portfolio here. Starting with the risk-free asset, downside risk can be initially reduced by including the benchmark into the passive portfolio. Thus, Kappa ratios vary with different weights of benchmark and risk-free asset in the passive portfolio. Transferred to evaluation of mutual funds, reverse rankings of superior and inferior performing funds may occur.

Figure 5: *Ranking reversal by Kappa ratios of higher order (τ > r_f)*

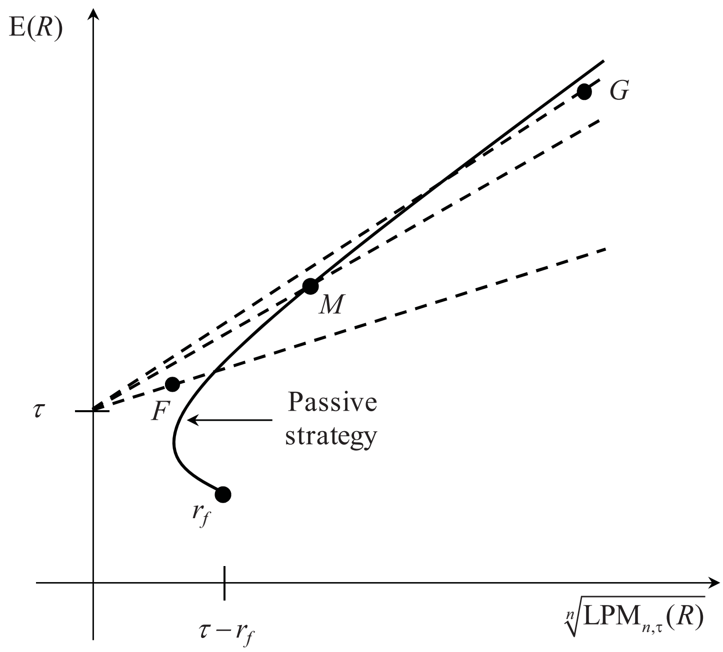
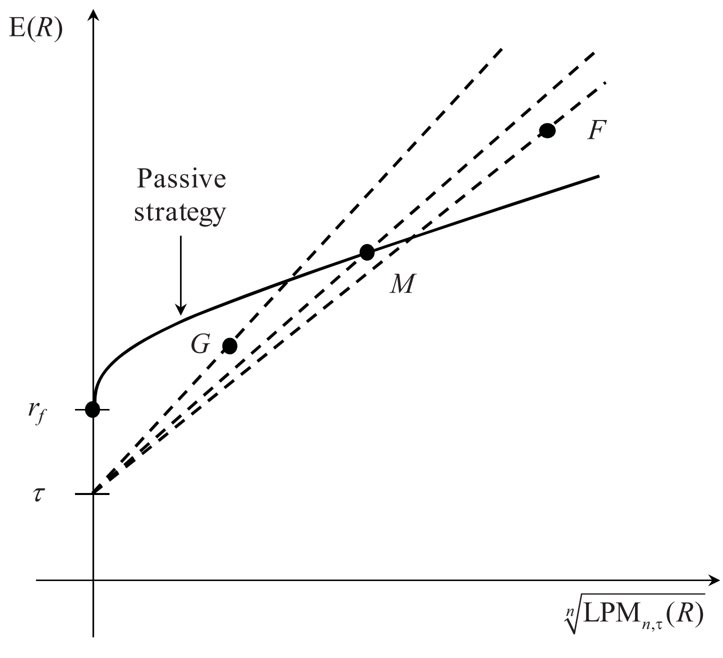


Figure 6 completes our visualizations and illustrates the situation when the target falls below the risk-free rate. The risk-free asset represents the minimum lower partial moment portfolio in this case. Nevertheless, passive portfolios are still positioned on a curved line. Therefore, the effects described above also emerge here.

Figure 6: *Ranking reversal by Kappa ratios of higher order (τ < r_f)*



Kappa ratios of higher order behave similar to Kappa ratios of order one regarding ranking reversals. Therefore, maximizing Kappa ratios of higher order may not lead to an improved portfolio position compared to the passive strategy. As the portfolio line of passive strategies is a curved line, the same holds for portfolios of mutual fund F and risk-free asset. In contrast to the $\text{LPM}_{1,\tau}$-framework, even the ratio of expected excess return, $\text{E}(R) - r_f$, and excess downside risk of higher order, $\sqrt[n]{\text{LPM}_{n,\tau}(R)} - \tau + r_f$, depends on portfolio composition.

Expected excess return over excess downside risk can be graphically illustrated by the slope of a line connecting mutual fund F and the risk-free asset. This ratio is not suitable as a performance measure in $(\text{E}(R), \sqrt[n]{\text{LPM}_{n,\tau}(R)})$-space for arbitrary targets because portfolios of F and the risk-free asset dominate positions on that line. Also this pitfall can be avoided by fixing the target at the level of the risk-free rate.

5. Conclusion

Reward-to-risk downside performance measures relate expected above target returns to below target risk measured by lower partial moments. Downside performance measures possess a broader decision-theoretic foundation, i.e. less restrictive assumptions, than the classical Sharpe ratio. On the other hand, a higher Sharpe ratio of, for example, a mutual fund compared to the benchmark implies that portfolios of mutual fund and risk-free asset dominate corresponding risk-adjusted portfolios of benchmark and risk-free asset in mean-volatility space. A main finding of our paper is that this characteristic of the Sharpe ratio can only be transferred to downside performance measurement if the target equals the risk-free rate.

Kappa ratios represent a well-known family of reward-to-risk downside performance measures where the order of Kappa ratios is determined by the order of the applied lower partial moment. Graphically illustrated, the Kappa ratio of a mutual fund corresponds to the slope of a straight line that connects the positions of mutual fund and target in mean-downside risk space. Our results show that this line does not depict a portfolio line for arbitrary targets as the target represents a required minimum return and not a tradable asset. This property in general leads to different Kappa ratios of passive strategies with varying portfolio fractions of benchmark and risk-free asset. In addition, the Kappa ratio of portfolios of mutual fund and risk-free asset depends on portfolio leverage. This effect hampers rankings of mutual funds.

Furthermore, ranking reversals of superior and inferior performing mutual funds may occur if performance is measured by Kappa ratios with arbitrary target. Finally, the shape of portfolio lines of a risky and the risk-free asset in mean-lower partial moment space is distinct from the familiar linear shape of this portfolio line in mean-volatility space. For downside risk measures of higher order curved portfolio lines result if the target differs from the risk-free rate. We conclude as a result of these effects that downside performance measures should only be applied in asset management if the target is set equal to the risk-free rate.

Appendix

Portfolios *P* consisting of benchmark *M* (or any other risky asset) and risk-free asset show the following rate of return:

(A.1) $R_P = x \cdot R_M + (1-x) \cdot r_f$

where x: portfolio fraction of *M*

We show the following properties for such portfolios in $(\mathrm{E}(R), \sqrt[n]{\mathrm{LPM}_{n,\tau}(R)})$-space within one proof:

(1) If the target equals the risk-free rate, portfolios of *M* and r_f can be positioned on a straight line connecting *M* and τ (Bawa and Lindenberg, 1977). The intercept of this line is $\tau = r_f$.

(2) If the order of the lower partial moment equals one, portfolios of *M* and r_f can be positioned on a straight line connecting *M* and r_f. This is a result of Harlow and Rao (1989). If $\tau > r_f$, the risk-free asset's lower partial moment amounts to $\mathrm{LPM}_{1,\tau}(r_f) = \tau - r_f$. The intercept of the portfolio line is neither r_f nor τ in this case.

(3) If $n > 1$, $\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)}$ is convex in x. Therefore, portfolios of *M* and r_f in $(\mathrm{E}(R), \sqrt[n]{\mathrm{LPM}_{n,\tau}(R)})$-space are positioned on a curved line similar to portfolios of two (imperfectly correlated) risky assets in standard mean-volatility portfolio selection. This extends the result of Harlow and Rao (1989) – who show the convexity of $\mathrm{LPM}_{n,\tau}(R_P)$ in x – because a concave transformation of a convex function can be convex, concave or linear in general.

We follow the idea of the proof of Harlow and Rao (1989) but analyze the n-th root of $\mathrm{LPM}_{n,\tau}(R_P)$:

$$\text{(A.2)} \quad \sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)} = \left(\int_{-\infty}^{\tau} (\tau - R_P)^n \; dF(R_P) \right)^{\frac{1}{n}}$$

Solving R_P in formula (A.1) for $R_M = \dfrac{R_P - (1-x)\cdot r_f}{x}$ and substituting, then the upper limit of the integral in formula (A.2) becomes $\varphi(x) = \dfrac{\tau - (1-x)\cdot r_f}{x}$. Subsequently, taking the first derivate of (A.2) with respect to x yields:

$$\text{(A.3)} \quad \frac{d}{dx}\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)} = \frac{1}{n}\cdot \mathrm{LPM}_{n,\tau}(R_P)^{\frac{1}{n}-1} \cdot \frac{d}{dx} \underbrace{\int_{-\infty}^{\varphi(x)} (\tau - x\cdot R_M - (1-x)\cdot r_f)^n \; dF(R_M)}_{\equiv I(x)}$$

We used here that for a positive linear transformation of the return ($x > 0$), the corresponding density functions are related as follows:

$$f(R_M) = x\cdot f(R_P) \Rightarrow \frac{dF(R_M)}{dF(R_P)} = \frac{f(R_M)}{f(R_P)}\cdot\frac{dR_M}{dR_P} = 1$$

Applying Leibniz' rule for parameter integral $I(x)$ in formula (A.3) yields:

$$\text{(A.4)} \quad \begin{aligned} \frac{dI(x)}{dx} &= \int_{-\infty}^{\varphi(x)} n\cdot(\tau - x\cdot(R_M - r_f) - r_f)^{n-1}\cdot(r_f - R_M)\; dF(R_M) \\ &\quad + \underbrace{(\tau - x\cdot\varphi(x) - (1-x)\cdot r_f)}_{=0}{}^n \cdot \frac{d}{dx}\varphi(x) \end{aligned}$$

$$\Rightarrow \frac{d}{dx}\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)} = \mathrm{LPM}_{n,\tau}(R_P)^{\frac{1}{n}-1} \cdot \int_{-\infty}^{\varphi(x)} (\tau - x\cdot(R_M - r_f) - r_f)^{n-1}\cdot(r_f - R_M)\; dF(R_M)$$

Utilizing Leibniz' integral rule again, the second derivate of (A.2) reads as follows:

$$\frac{d^2}{dx^2}\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)}$$

$$=(1-n)\cdot \mathrm{LPM}_{n,\tau}(R_P)^{\frac{1}{n}-2}\cdot\left(\int_{-\infty}^{\varphi(x)}(\tau-x\cdot(R_M-r_f)-r_f)^{n-1}\cdot(r_f-R_M)\;dF(R_M)\right)^2$$

$$+\mathrm{LPM}_{n,\tau}(R_P)^{\frac{1}{n}-1}\cdot\int_{-\infty}^{\varphi(x)}(n-1)\cdot(\tau-x\cdot(R_M-r_f)-r_f)^{n-2}\cdot(r_f-R_M)^2\;dF(R_M)$$

(A.5)

$$=(1-n)\cdot\underbrace{\mathrm{LPM}_{n,\tau}(R_P)^{\frac{1}{n}-2}}_{>0}\cdot\left\{\left(\int_{-\infty}^{\varphi(x)}(\tau-x\cdot(R_M-r_f)-r_f)^{n-1}\cdot(r_f-R_M)\;dF(R_M)\right)^2\right.$$

$$-\left(\int_{-\infty}^{\varphi(x)}(\tau-x\cdot(R_M-r_f)-r_f)^{n}\;dF(R_M)\right)$$

$$\left.\cdot\left(\int_{-\infty}^{\varphi(x)}(\tau-x\cdot(R_M-r_f)-r_f)^{n-2}\cdot(r_f-R_M)^2\;dF(R_M)\right)\right\}$$

Clearly, for n = 1, $\frac{d^2}{dx^2}\mathrm{LPM}_{1,\tau}(R_P)$ becomes zero. Therefore and because the expected value is a linear operator, portfolios of M and r_f can be positioned on a straight line in (E(R), $\mathrm{LPM}_{1,\tau}(R)$)-space. This is property (2).

For $\tau = r_f$, the term in curly brackets in formula (A.5) simplifies to:

$$\left(\int_{-\infty}^{r_f}x^{n-1}\cdot(r_f-R_M)^n\;dF(R_M)\right)^2-$$

$$\left(\int_{-\infty}^{r_f}x^{n}\cdot(r_f-R_M)^n\;dF(R_M)\right)\cdot\left(\int_{-\infty}^{r_f}x^{n-2}\cdot(r_f-R_M)^n\;dF(R_M)\right)=0$$

Hence, if the target is set equal to the risk-free rate, portfolios of M and r_f can be positioned on a straight line in (E(R), $\sqrt[n]{\mathrm{LPM}_{n,r_f}(R)}$)-space (property (1)).

Finally, the term in curly brackets in formula (A.5) is negative in general because rewriting and applying the Cauchy-Schwarz inequality yields:

$$\int_{-\infty}^{\varphi(x)} \left((\tau - x\cdot(R_M - r_f) - r_f)^{\frac{n}{2}} \right)^2 dF(R_M)$$
$$\cdot \int_{-\infty}^{\varphi(x)} \left((\tau - x\cdot(R_M - r_f) - r_f)^{\frac{n-2}{2}} \cdot (r_f - R_M) \right)^2 dF(R_M)$$
$$\geq \left(\int_{-\infty}^{\varphi(x)} (\tau - x\cdot(R_M - r_f) - r_f)^{n-1} \cdot (r_f - R_M)\, dF(R_M) \right)^2$$

Therefore, $\frac{d^2}{dx^2}\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)} \geq 0$ for $n > 1$ (property (3)), which completes our proof.

References

Agarwal, V., and Naik, N.Y. (2004). Risk and portfolio decisions involving hedge funds. *Review of Financial Studies* 17, 63–98.

Bawa, V.S. (1978). Safety-first, stochastic dominance, and optimal portfolio choice. *Journal of Financial and Quantitative Analysis* 13, 255–271.

Bawa, L., and Lindenberg, B. (1977). Capital market equilibrium in a mean-lower partial moment framework. *Journal of Financial Economics* 5, 189–200.

Chamberlain, G. (1983). A characterization of the distributions that imply mean-variance utility functions. *Journal of Economic Theory* 29, 185–201.

Chen, L., He, S., and Zhang, S. (2011). When all risk-adjusted performance measures are the same: In praise of the Sharpe ratio. *Quantitative Finance* 11, 1439–1447.

Darsino,T., and Satchell, S. (2004). Generalizing universal performance measures. *Risk* 17(6), 80–84.

Dowd, T.K. (2000). Adjusting for risk: An improved Sharpe ratio. *International Review of Economics and Finance* 9, 209–222.

Eling, M. (2008). Does the measure matter in the mutual fund industry? *Financial Analysts Journal* 64(3), 54–66.

Eling, M., and Schuhmacher, F. (2007). Does the choice of performance measure influence the evaluation of hedge funds? *Journal of Banking and Finance* 31, 2632–2647.

Farinelli, S., and Tibiletti, L. (2008). Sharpe thinking in asset ranking with one-sided measures. *European Journal of Operational Research* 185, 1542–1547

Harlow, W.V., and Rao, R.K.S. (1989). Asset pricing in a general mean-lower partial moment framework: Theory and evidence. *Journal of Financial and Quantitative Analysis* 24, 285–311.

Johnstone, D.J., and Lindley, D.V. (2011). Elementary proof that mean-variance implies quadratic utility. *Theory and Decision* 70, 149–155.

Kaplan, P.D., and Knowles, J.A. (2004). Kappa: A generalized downside risk-adjusted performance measure. *Journal of Performance Measurement* 8(1), 42–54.

Kazemi, H., Schneeweis, T., and Gupta, R. (2004). Omega as a performance measure. *Journal of Performance Measurement* 8(1), 16–25.

Meyer, J. (1987). Two-moment decision models and expected utility maximization. *American Economic Review* 77, 421–430.

Ornelas, J.R., Silva, A.F., and Fernandes, J.L. (2012). Yes, the choice of performance measure does matter for ranking of US mutual funds. *International Journal of Finance and Economics* 17, 61–72.

Owen, J., and Rabinovitch, R. (1983). On the class of elliptical distributions and their applications to the theory of portfolio choice. *Journal of Finance* 38, 745–752.

Schuhmacher, F., and Breuer, W. (2014). When all risk-adjusted performance measures are the same: In praise of the Sharpe ratio – A comment. *Quantitative Finance* 14, 775–776.

Schuhmacher, F., and Eling, M. (2012). A decision-theoretic foundation for the reward-to-risk performance measures. *Journal of Banking and Finance* 36, 2077–2082.

Shadwick,W.F., and Keating, C. (2002). A universal performance measure. *Journal of Performance Measurement* 6(1), 59–84.

Sharpe, W.F. (1966). Mutual fund performance. *Journal of Business* 39, 118–138.

Sortino, F., and Price, L. (1994). Performance measurement in a downside risk framework. *Journal of Investing* 3(3), 1519–1533.

Sortino, F., and Satchell, S. (ed.) (2001). *Managing downside risk in financial markets*, Oxford.

Sortino, F., van der Meer, R., and Plantinga, A. (1999). The Dutch triangle: A framework to measure upside potential relative to downside risk. *Journal of Portfolio Management* 26(1), 50–58.

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