Costs of capital under credit risk

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Abstract

Credit risk analysis represents a growing field in financial research since decades. However, in company valuation – to be more precise, in cost of capital computations – credit risk is merely taken into consideration at the level of the debt beta approach. Our paper proves that applications of the debt beta approach suffer from unrealistic assumptions. As an advantageous approach, we develop an alternative framework to determine costs of capital based on Merton's model. We present (quasi-) analytic formulas for costs of equity and debt which are consistent with Modigliani-Miller theory in continuous-time and discrete-time settings without taxes. Our framework is superior to the debt beta approach regarding the quantity and quality of required data in peer group analysis. Since equity and debt are represented by options in Merton's model, we compute expected option rates of return without resorting to betas. Thereby, our paper also contributes to the option pricing literature.

Keywords

Company valuation, debt beta, expected option return, Merton's model, WACC

JEL classification

G13, G32, G33

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1.   Introduction

The discounted cash flow method of company valuation applies the present value method under risk where expected future cash flows are discounted by risk-adjusted rates of return. Following the opportunity cost principle, this rate of return represents cost of capital from the company's side and required expected rate of return from the investors' side. In the classical approach based on the capital asset pricing model (CAPM), the required rate of return on equity is set equal to the beta-adjusted expected rate of return of a portfolio which consists of the stock market index and the risk-free asset.

Peer group analysis – based on Hamada (1972) – is applied in practice because the beta of a non-publicly traded company is unknown. Since the classical CAPM-based approach does not consider credit risk, the required rate of return on debt is set equal to the risk-free rate. The sum of required rates of return on equity and debt, weighted by the corresponding equity and debt ratios, results in the weighted average cost of capital (WACC) which is used in entity approaches of company valuation to discount expected future cash flows. Our paper analyzes single costs of capital and WACC in different frameworks under credit risk.

Under credit risk, i.e., limited liability of the equity holders, part of the risk is shifted from the equity holders to the debt holders. Therefore, the cost of debt is higher and the cost of equity is lower than in the classical framework. In the CAPM framework under credit risk, the equity beta is reduced and the debt beta is introduced. The debt beta approach has been developed since the 1970s (Haugen and Pappas, 1971; Conine, 1980; Harris and Pringle, 1985; Kaplan and Stein, 1990). While Haugen and Pappas (1971) and Harris and Pringle (1985) at least mention that a combined stock-corporate bond index is needed to compute the debt beta, more recent literature omits this fact and mainly focuses on tax shield analyses under credit risk (Ruback, 2002; Fernandez, 2004; Arzac and Glosten, 2005). Other authors suggest to employ the implied debt beta (Cohen, 2008) consistent with the company's debt interest rate.

This paper clarifies that the debt beta approach is not feasible in company valuation unless a (preferably return-risk efficient) index of stocks and corporate bonds is used to compute the betas. In the absence of historical data for a combined stock-corporate bond index, the debt beta approach appears to be practically useless. In addition, we show that the implied debt beta approach based on the company's interest rate is tautological and, therefore, leads to improper cost of capital computations in applications.

Besides the debt beta approach, Merton's (1974) model is widely used in credit risk evaluation. Merton (1974) represents the origin of structural credit risk models with an enormous amount of research on this topic. However, we were not able to find a contribution that applies the pure option-based approach to calculate costs of capital in terms of expected rates of return. To our knowledge, only Galai and Masulis (1976) deal with the expected rate of return on equity in a combined option theory-CAPM approach, and Cooper and Davydenko (2007) address the expected return on risky debt in Merton's framework.

We close this gap by deriving (quasi-) analytic formulas for instantaneous, continuously compounded, and simple per-period costs of equity and debt. In contrast to the literature on expected option returns (e.g., Cox and Rubinstein, 1985, 189–190 and 210; Coval and Shumway, 2001; Jones, 2006) which at least partially use CAPM betas, we stay in the Black-Scholes-Merton framework and, thus, beta is absent in our option-based formulas. In this regard, we also contribute to the body of option pricing literature. Moreover, we show that without capital structure effects – in the absence of discriminatory taxation in particular – Modigliani-Miller (1958) proposition I holds according to which the WACC in the levered company equals the expected equity return in the, apart from that, identical unlevered company. As our paper contributes to the starting point of the company valuation procedure, we refrain from analyzing a tax shield.

Our paper is organized as follows. Section 2 repeats the classical Modigliani-Miller CAPM approach to determine WACC without credit risk. It serves to introduce the notation and illustrate the single steps of computing costs of capital. Our statements in this section are deliberately detailed to prepare our argumentation in the subsequent sections. Section 3 describes the debt beta approach where credit risk is taken into consideration within the CAPM framework. We lay emphasis on the assumptions of the debt beta approach with respect to the market portfolio and show that serious distortions occur if betas are computed with respect to a stock index. In addition, we argue that the implied debt beta approach based on the company's debt interest rate does not fully reflect credit risk since the debt interest rate, precisely because of credit risk, is higher than the expected return on debt. We consider the debt beta approach as a reference model to evaluate whether an option-based cost of capital approach performs better.

Sections 4 and 5 contain our main findings. Section 4 starts with the classical Black-Scholes-Merton stochastic calculus to price derivative contracts. Since equity mirrors a call option and debt parallels a risk-free bond minus a put option in Merton's model, we insert

partial derivations ("Greeks") of option pricing formulas into the arising stochastic processes for equity and debt. This enables us to compute single costs of capital and WACC in terms of instantaneous expected rates of return. Subsequently, we contrast the data requirements in peer group analysis of our option-based approach with those of CAPM-based frameworks.

Instantaneous rates of return are relevant for continuous-time portfolio decisions. However, a discrete-time framework is predominantly employed in company valuation. Therefore, we determine per-period costs of capital in terms of continuously and discretely compounded expected rates of return in section 5. Although we consider the option-based approach for cost of capital computations to be superior to CAPM-based approaches, we merge the option-based and the debt beta approach – based on Galai and Masulis (1976) – in section 6 to analyze whether a combination of both frameworks is able to overcome short-comings of the debt beta approach. Section 7 concludes our paper.

2. The classical Modigliani-Miller CAPM approach

The entity approach of company valuation, in particular for non-publicly traded companies, starts by valuing the corresponding as-if pure equity (unlevered) company to eliminate capital structure effects. These effects result from, among other things, different taxation of cost of equity and debt. Tax effects are taken into consideration in a later step of the company valuation procedure. Our paper contributes to the first steps of this process. Therefore, we ignore taxes. If there are no capital structure effects, the cost of equity in the as-if unlevered company – which is unknown – is equal to the WACC as a result of the famous Modigliani-Miller theorem (Modigliani and Miller, 1958, proposition I):

$$\text{WACC} = \text{E}(R_E^u) = \text{E}(R_E^l) \cdot \frac{E}{E+D} + \underbrace{\text{E}(R_D)}_{=r} \cdot \frac{D}{E+D} \quad \text{where}$$

(1)

$\text{E}(R_E^u) = $ required rate of return (of equity holders) in the as-if unlevered company,

$\text{E}(R_E^l) = $ required rate of return of equity holders in the levered company,

$\text{E}(R_D) = $ required rate of return of debt holders,

$r = $ risk-free rate,

$E = $ market value of equity,

$D = $ market value of debt.

Using the risk-free rate as the required rate of return of debt holders implies that there is no credit risk, i.e., liability of equity holders is not limited to the amount invested in the

company. Disregarding term structure effects, the market value of debt amounts to its nominal value if the debt interest rate equals the risk-free rate. Hence, the market value of debt is equated with its book value in this situation. If there are no capital structure effects, i.e., the WACC is independent of the debt-equity ratio, and if there is no credit risk, we refer to this situation as the classical Modigliani-Miller framework.

For non-publicly traded companies, the required equity return $E(R_E^l)$ in formula (1) is unknown. To estimate $E(R_E^l)$ based on market data (fair value accounting), the well-known CAPM is used in the basic company valuation approach:

$$
E(R_E^l) = r + \beta_E \cdot \left(E(R_M) - r \right) \quad \text{where}
$$

(2)
$$E(R_M) = \text{ expected rate of return of stock market index } M,$$
$$\beta_E = \text{ beta of equity in the levered company.}$$

The idea behind this is that the required equity return equals the expected return of a comparable alternative investment on the capital market, i.e., a risk-adjusted portfolio of the stock market index and the risk-free asset. The fraction of the stock market index in this portfolio is set equal to the beta of the company to be valued to achieve the same level of systematic risk.[1]

If the particular company is not publicly traded, a time series of historical data to estimate its beta is not available. Therefore, an unlevering-relevering routine via betas of publicly traded peer group companies is applied in practice. Starting from the basic leverage formula

$$
R_E^l = R_A + \left(R_A - r \right) \cdot \frac{D}{E} \quad \text{where}
$$

(3)
$$A = E + D = \text{ market value of assets,}$$
$$R_A = \text{ rate of return on assets,}$$

the equity beta reads

(4)
$$
\beta_E = \frac{\text{Cov}\left(R_E^l, R_M \right)}{\text{Var}(R_M)} = \underbrace{\frac{\text{Cov}\left(R_A, R_M \right)}{\text{Var}(R_M)}}_{=\beta_A} \cdot \left(1 + \frac{D}{E} \right) \Leftrightarrow \beta_A = \beta_E \cdot \frac{E}{E+D}.
$$

---

[1] This only holds if the market index used to compute the beta is return-risk efficient (Roll's critique). We neither advocate applying the CAPM in company valuation nor ignore the criticism brought forward by, i.a., Fernandez (2015); we use the CAPM-based approach as a benchmark for the option-based approach that we carry out in sections 4 and 5.

where the assets beta of the particular company is set equal to the assets beta of the peer group: $\beta_A = \beta_A^{\text{peer}} = \beta_E^{\text{peer}} \cdot \dfrac{E^{\text{peer}}}{E^{\text{peer}} + D^{\text{peer}}}$. Formula (4) represents Hamada's (1972) equation without taxes.

Relevering is not necessary in the absence of capital structure effects because the WACC equals $E\left(R_E^u\right) = r + \beta_A^{\text{peer}} \cdot \left(E(R_M) - r\right)$. However, if a tax shield is subsequently taken into consideration in WACC computation, single costs of capital have to be determined in the first step. Therefore, as preparation for further steps in the company valuation procedure, we compute single costs of capital throughout this paper.

We note that the classical Modigliani-Miller framework to determine the WACC does not necessarily rely on the CAPM. Rearranging leverage formula (3) in terms of expected values yields WACC formula (1):

(5) $\quad E\left(R_E^l\right) = E\left(R_A\right) + \left(E\left(R_A\right) - r\right) \cdot \dfrac{D}{E} \Leftrightarrow E\left(R_A\right) = E(R_E^l) \cdot \dfrac{E}{E + D} + r \cdot \dfrac{D}{E + D}$.

Formula (5) can be used for unlevering-relevering if the expected equity return of peer group companies is directly estimated:

$$E\left(R_A\right) = E\left(R_A^{\text{peer}}\right) = E\left(R_E^{l,\text{peer}}\right) \cdot \frac{E^{\text{peer}}}{E^{\text{peer}} + D^{\text{peer}}} + r \cdot \frac{D^{\text{peer}}}{E^{\text{peer}} + D^{\text{peer}}}.$$

The CAPM approach is frequently used in company valuation and preferred to direct estimation of the expected equity return because estimation errors of average returns are high in general. Nevertheless, differences in direct and beta-based estimations of the expected equity return basically result from different time series. The market risk premium is usually estimated from a long time series, whereas betas are frequently estimated from shorter time series. However, the CAPM-based approach ignores that the regression to estimate the beta may exhibit an alpha. Instead, the intercept is fixed at the level of the (current) risk-free rate in applications.[2]

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[2] To be more precise, betas are usually estimated via regression based on excess rates of return over risk-free rates. According to the CAPM, the intercept of the regression line (alpha) should be zero. For this reason, moving to expected rates of return, the intercept at the level of the risk-free rate is not adjusted by alpha.

By construction, the WACC equals the required return in the as-if unlevered company within the classical Modigliani-Miller CAPM framework only if the debt interest rate equals the risk-free rate:

(6)
$$\text{WACC}^{\text{CAPM}} = \underbrace{\left(r + \beta_A^{\text{peer}} \cdot \left(1 + \frac{D}{E} \right) \cdot \left(\text{E}(R_M) - r \right) \right)}_{= \text{E}\left(R_E^l \right)} \cdot \frac{E}{E+D} + \text{E}(R_D) \cdot \frac{D}{E+D}$$

$$\underset{\text{iff } \text{E}(R_D)=r}{=} \; r + \beta_A^{\text{peer}} \cdot \left(\text{E}(R_M) - r \right) = \text{E}\left(R_E^u \right).$$

In case of credit risk, the debt interest rate i_D of the company to be valued includes a credit spread and, therefore, is higher than the risk-free rate. However, inserting i_D into the CAPM-based WACC formula (6) is inconsistent with the classical Modigliani-Miller theory. The debt beta approach discussed in the following section was developed to overcome this difficulty.

3. The debt beta approach

In case of credit risk, the return to debt holders is risky. Hence, leverage formula (3) changes to

(7)
$$R_E^l = R_A + (R_A - R_D) \cdot \frac{D}{E} \quad \text{where}$$
$$R_D = \text{ rate of return on debt.}$$

Thus, the equity beta differs from formula (4) and reads under credit risk

(8)
$$\beta_E = \beta_A \cdot \left(1 + \frac{D}{E} \right) - \beta_D \cdot \frac{D}{E} \quad \text{where}$$
$$\beta_D = \frac{\text{Cov}(R_D, R_M)}{\text{Var}(R_M)} = \text{ debt beta.}$$

Formula (8) represents Conine's (1980) equation, also referred to as extended Hamada (1972) equation and Fernandez' (2004) equation, respectively, without taxes. This formula was earlier developed by Haugen and Pappas (1971). Compared with the classical Modigliani-Miller CAPM framework, the equity beta is adjusted because under credit risk part of the risk, i.e., risk of additional cover by the equity holders in case of overindebtedness, is transferred to the debt holders.

Formulas (7) and (8) are valid by definition. However, expected returns on assets and debt depending on the respective beta are required to determine costs of capital. Therefore, theoretical support of the following formula is needed:

(9) $\quad E(R_D) = r + \beta_D \cdot \left(E(R_M) - r\right)$ and $E(R_A) = r + \beta_A \cdot \left(E(R_M) - r\right)$.

If formula (9) holds, the WACC also in this situation fulfills the Modigliani-Miller property that it equals the required equity return in the as-if unlevered company. As we assume that neither equity nor debt of the company to be valued is publicly traded, formula (9) is used for the unlevering-relevering peer group procedure. To estimate the peer group debt beta, Haugen and Pappas (1971), Bierman and Oldfield (1979), and Harris and Pringle (1985) suggest to measure the sensitivity of debt returns to a generalized market portfolio that consists of both stocks and corporate bonds. This approach requires that the equity beta also has to be calculated with respect to this stock-bond market portfolio to obtain the assets beta. In addition – assuming that the peer group market value of debt can be deduced from corporate bond prices – the market value of debt of the company to be valued is needed in the relevering step of peer group analysis.

Figure 1 relates to the peer group of publicly traded companies. It illustrates that even under restrictive assumptions, like return-risk efficiency of the stock index in the stock universe, and if mean-volatility (E(R)-σ) analysis is appropriate and applicable to evaluate corporate bonds, the debt beta cannot be calculated with respect to the stock index. In figure 1, the market portfolio M consists of stocks and corporate bonds. The WACC can be determined based on the betas of equity and debt with respect to the stock-corporate bond market portfolio and the assets beta equals the weighted average of equity and debt betas because the CAPM exhibits the property of value additivity. However, exchanges do not provide price data for a (preferably return-risk efficient) stock-corporate bond market portfolio. A time series of historical price data does not exist.

If stock index I is return-risk efficient in the stock universe, the equity beta with respect to I can be found at the intersection of the horizontal line through equity position E and the tangent to stock index I. However, depending on the degree of inefficiency of the stock index with respect to the stock-corporate bond market portfolio, the systematic risk of debt with respect to stock index I can be almost everywhere on a horizontal line through debt position D as a result of Roll and Ross (1994) – therefore, we did not indicate it in the figure. The approach of Cornell and Green (1991) which is often cited in the debt beta literature does not

remedy this problem because the authors run regressions of corporate bond returns on stock index and treasury bill returns. However, treasury bill returns do not reflect (systematic) corporate credit risk.

Figure 1: Stock index versus stock-corporate bond market portfolio
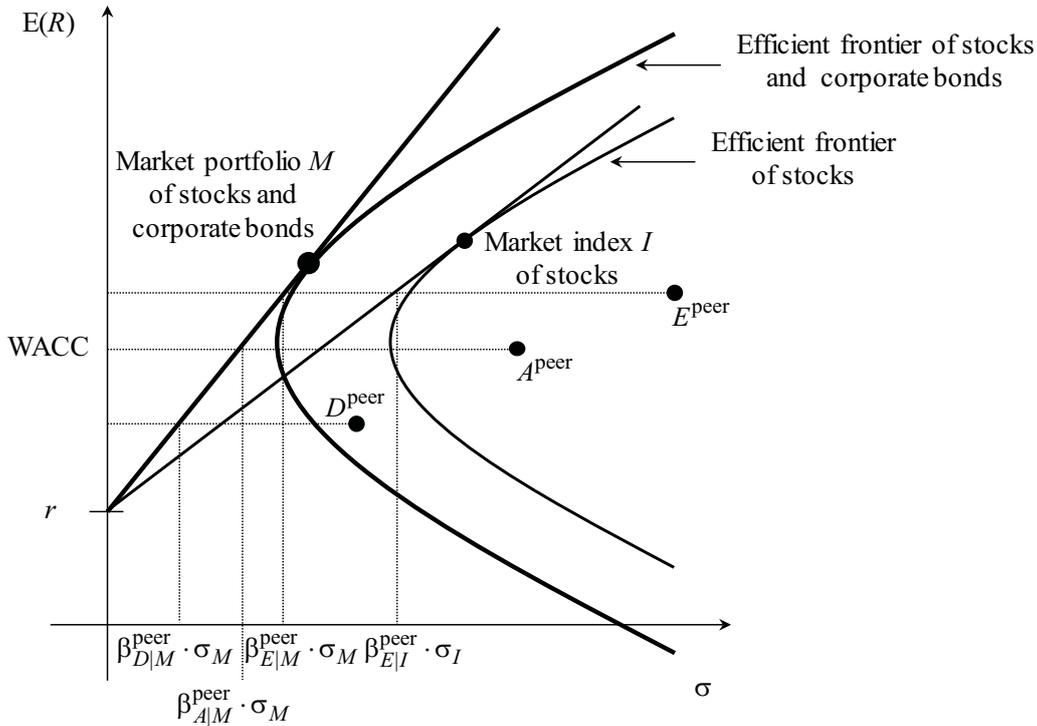
As a result, the WACC cannot be calculated via stock index betas. This is the reason why, i.a., Kaplan and Stein (1990) and Damodaran (2012, 411) have to make an assumption about the systematic risk of corporate bonds in terms of stock market risk. In either case, this method implies an inconsistent debt-equity ratio. To overcome these difficulties, Cohen (2008) and Benninga (2014, 599), for example, suggest to apply the implied debt beta

(10) $\quad \beta_D^{\text{implied}} = \dfrac{E(R_D) - r}{E(R_M) - r}.$

If definition (10) is used, the costs of equity and debt are consistent with Modigliani-Miller proposition I. However, definition (10) requires knowing the cost of debt which we are looking for. Therefore, it is suggested to use the debt interest rate i_D of the company to be valued as a proxy for the expected return to debt holders in applications. Inserting definition

(10) into formula (8) where $E(R_D)$ is substituted by i_D will lead to WACC at the level of the required equity return in the as-if unlevered company:

$$\text{WACC}^{\text{debt beta}} = \left(r + \underbrace{\left[\underbrace{\beta_A \cdot \left(1 + \frac{D}{E}\right) - \underbrace{\frac{i_D - r}{E(R_M) - r}}_{\beta_D} \cdot \frac{D}{E}}_{\beta_E} \right] \cdot \left(E(R_M) - r\right)}_{} \right) \cdot \frac{E}{E + D}$$

(11)
$$+ \left(r + \underbrace{\frac{i_D - r}{E(R_M) - r}}_{\beta_D} \cdot \left(E(R_M) - r\right) \right) \cdot \frac{D}{E + D}$$

$$= r + \beta_A \cdot \left(E(R_M) - r\right) = E\left(R_E^u\right).$$

This is the reason why the debt beta approach is popular in the industry. Formula (11) holds for arbitrary i_D because formula (10) represents a definition rather than a result from asset pricing theory. Therefore, formula (11) is tautological. However, formula (11) implies a certain debt-equity ratio for a given debt interest rate. However, this ratio does not reflect the true capital structure either for peer group companies or the company to be valued since the debt interest rate differs from the expected return of debt holders. This is why some arbitrary adjustments occur in practice.

The debt interest rate has to cover credit risk. Therefore, the debt holders do not expect to receive interest payments at the level of i_D in case of credit risk. At least, they have to take the probability of default into consideration. Simply adjusting i_D for the probability of default would lead to a lower bound of the expected debt holders' return because in case of default, there might be a positive recovery rate. Loss given default would not be 100 percent in this case. Therefore, the expected loss that combines default probability and loss given default should be taken into consideration.[3]

Even if we would be able to determine the expected return of debt, the assets beta still has to be determined with respect to the combined market portfolio of stocks and corporate bonds. In this light, the debt beta approach suffers from a circularity problem. These considerations motivate applying an option-based approach to determine costs of capital.

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[3] The expected loss can be computed within the option-based approach presented in sections 4 and 5.

4. Instantaneous costs of capital in the option-based approach

Option pricing theory was originally developed to value corporate liabilities. In the basic option-based approach of company valuation (Merton's model), debt is simply represented by a single zero-coupon bond with face value K and time to maturity T. If the value of assets A_T at maturity exceeds the face value of the bond, the debt holders receive the redemption amount K. If the value of assets at maturity falls below K, the debt holders get the remaining assets. Therefore, the pay-off to debt holders D_T shows option-like characteristics.

The equity pay-off E_T at maturity of the bond is the residual between assets and debt. Thus, equity also possesses an option-style pay-off. Figure 2 illustrates (on the right-hand side) the well-known property that equity in Merton's model represents a call option on the company's assets, whereas debt under credit risk equals a risk-free zero-coupon bond minus a put option on the assets. This put option values the expected loss of the debt holders and reflects credit risk, which can be seen by comparing both parts of the figure.

Figure 2: Equity and debt pay-offs in the option-based approach of company valuation
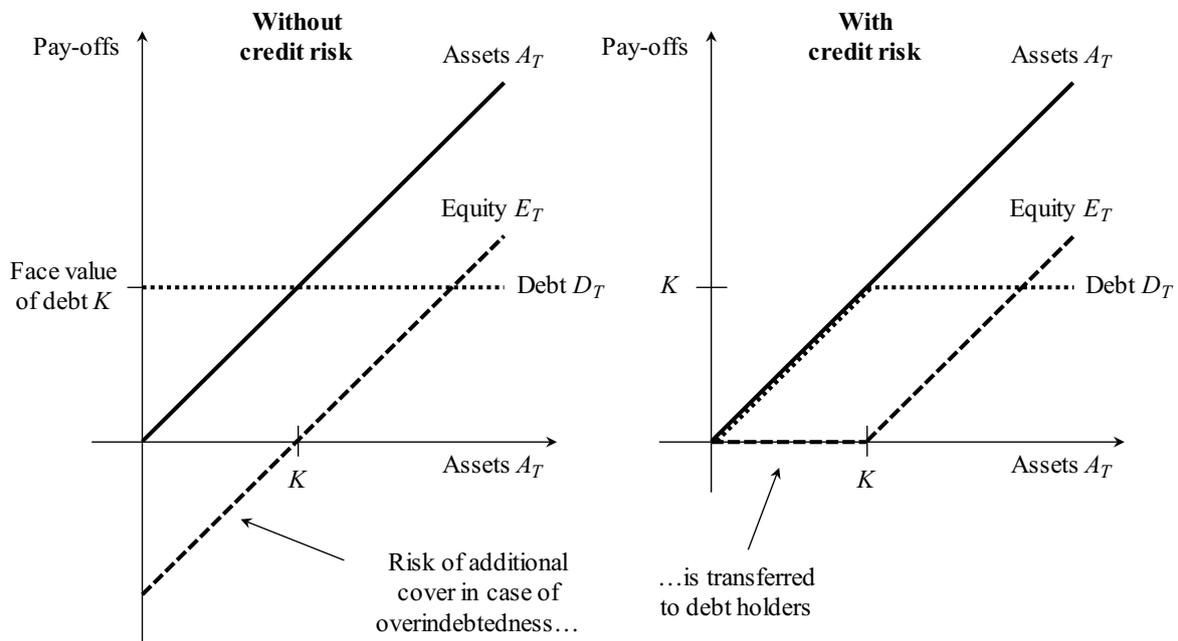
In Merton's model, the values of equity and debt are calculated by using the famous Black-Scholes formula for a vanilla call option:

$$E = A \cdot \mathrm{N}(d_1) - K \cdot e^{-r \cdot T} \cdot \mathrm{N}(d_2)$$

$$D = A - E = A \cdot \mathrm{N}(-d_1) + K \cdot e^{-r \cdot T} \cdot \mathrm{N}(d_2) \quad \text{where}$$

(12)
$$d_1 = \frac{\ln \dfrac{A}{K} + \left(r + \dfrac{\sigma^2}{2} \right) \cdot T}{\sigma \cdot \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma \cdot \sqrt{T},$$

$\mathrm{N}(\cdot) = $ cumulative standard normal distribution function.

As the pay-off of assets equals the sum of the pay-offs of equity and debt, according to put-call parity, this relation also holds in terms of current values. Therefore, the value of assets is independent of K, i.e., capital structure. This implies that the Modigliani-Miller property is also valid under credit risk (if there are no additional costs in case of default).[4]

We extend the Black-Scholes-Merton approach by computing expected rates of return on equity and debt in terms of options to compute costs of capital. We stay in the Black-Scholes-Merton framework and assume a geometric Brownian motion for the value of the company's assets

(13)
$$dA = \mu \cdot A \, dt + \sigma \cdot A \, dW \quad \text{where}$$
$\mu = $ expected continuously compounded rate (drift rate) of return on assets,
$dW = $ standard Wiener process.

Applying Itô's lemma for any European-style derivate contract $f(A,t)$ yields

(14)
$$df = \underbrace{\frac{\partial f}{\partial A}}_{=\Delta_f \text{ (delta)}} dA + \underbrace{\frac{\partial f}{\partial t}}_{=\Theta_f \text{ (theta)}} dt + \frac{\sigma^2}{2} \cdot A^2 \cdot \underbrace{\frac{\partial^2 f}{\partial A^2}}_{=\Gamma_f \text{ (gamma)}} dt \,.$$

Inserting formula (13) into formula (14) results in

(15)
$$df = \left(\Delta_f \cdot \mu \cdot A + \Theta_f + \frac{\sigma^2}{2} \cdot A^2 \cdot \Gamma_f \right) dt + \Delta_f \cdot \sigma \cdot A \, dW \,.$$

Hence, f follows an Itô process $\dfrac{df}{f} = \mu_f(f,t) \, dt + \sigma_f(f,t) \, dW$ with drift rate (instantaneous expected rate of return)

(16)
$$\mu_f(f,t) = \frac{\Delta_f \cdot \mu \cdot A + \Theta_f + \dfrac{\sigma^2}{2} \cdot A^2 \cdot \Gamma_f}{f}$$

---

[4] Stiglitz (1969) was the first to prove this characteristic without using put-call parity, which was published by Stoll (1969) in the same year.

and local volatility

$$(17) \quad \sigma_f(f,t) = \frac{\Delta_f \cdot \sigma \cdot A}{f}.$$

Formulas (13) to (17) represent the standard Black-Scholes-Merton framework for pricing derivative contracts. We insert the required partial derivations (Greeks) for equity and debt into formula (16) to determine costs of capital. As equity represents a call option on the company's assets, the corresponding Greeks are (e.g., Haug, 1997, 11–15)

$$\Delta_E = \Delta_{\text{call}} = N(d_1), \quad \Theta_E = \Theta_{\text{call}} = -\frac{A \cdot n(d_1) \cdot \sigma}{2 \cdot \sqrt{T}} - r \cdot K \cdot e^{-r \cdot T} \cdot N(d_2), \text{ and}$$

$$(18) \quad \Gamma_E = \Gamma_{\text{call}} = \frac{n(d_1)}{A \cdot \sigma \cdot \sqrt{T}} \quad \text{where}$$

$n(\cdot) = $ standard normal density.

As debt parallels a risk-free zero-coupon bond minus a put option, the Greeks of a put option determine the Greeks of debt. Additionally, we have to take into consideration that the risk-free bond with face value K and present value $K \cdot e^{-r \cdot T}$ exhibits a theta:

$$(19) \quad \Delta_{\text{put}} = N(d_1) - 1, \quad \Theta_{\text{put}} = r \cdot K \cdot e^{-r \cdot T} \cdot N(-d_2) - \frac{A \cdot n(d_1) \cdot \sigma}{2 \cdot \sqrt{T}}, \text{ and } \Gamma_{\text{put}} = \frac{n(d_1)}{A \cdot \sigma \cdot \sqrt{T}}$$

$$\Rightarrow \Delta_D = -\Delta_{\text{put}}, \quad \Theta_D = r \cdot K \cdot e^{-r \cdot T} - \Theta_{\text{put}}, \text{ and } \Gamma_D = -\Gamma_{\text{put}}.$$

Inserting formulas (18) and (19) into formula (16), the instantaneous expected rates of return on equity and debt read

$$(20) \quad \mu_E = \mu \cdot \frac{A}{E} \cdot N(d_1) - r \cdot \frac{K \cdot e^{-r \cdot T}}{E} \cdot N(d_2) = \frac{\mu \cdot A \cdot N(d_1) - r \cdot K \cdot e^{-r \cdot T} \cdot N(d_2)}{A \cdot N(d_1) - K \cdot e^{-r \cdot T} \cdot N(d_2)} \quad \text{and}$$

$$\mu_D = \mu \cdot \frac{A}{D} \cdot N(-d_1) + r \cdot \frac{K \cdot e^{-r \cdot T}}{D} \cdot N(d_2) = \frac{\mu \cdot A \cdot N(-d_1) + r \cdot K \cdot e^{-r \cdot T} \cdot N(d_2)}{A \cdot N(-d_1) + K \cdot e^{-r \cdot T} \cdot N(d_2)}.$$

The interpretation of (the first equation in) formula (20) is straightforward. The instantaneous expected return on equity $\mu_E \cdot E$ equals the instantaneous expected return of the duplicating portfolio that consists of $N(d_1)$ assets and $-N(d_2)$ risk-free zero-coupon bonds with face value K and time to maturity T. An analogous interpretation holds for the instantaneous expected return on debt. Besides the expected rate of return on assets, their volatility, the risk-free rate, and the lent term of debt, the costs of equity and debt are determined by the ratio of assets value and debt redemption amount. Appendix A provides an alternative proof

of formula (20) based on option elasticities and – since we obtained closed-form expressions for the costs of equity and debt – contains partial derivations with respect to the debt-equity ratio.

Inserting the instantaneous expected rates of return of formula (20) into WACC formula (1) proves that the WACC equals the instantaneous expected rate of return on assets:

$$(21) \quad \text{WACC}^{\text{instant}} = \mu_E \cdot \frac{E}{A} + \mu_D \cdot \frac{D}{A} = \mu.$$

A similar relation holds for the local volatility. Applying equity and debt deltas to formula (17), the weighted average volatility equals the assets volatility because the instantaneous correlation coefficient between equity and debt rates of return equals one since they are driven by the same risk factor (Gheno, 2007):

$$(22) \quad \sigma_E = \frac{\text{N}(d_1) \cdot \sigma \cdot A}{E} \quad \text{and} \quad \sigma_D = \frac{\text{N}(-d_1) \cdot \sigma \cdot A}{D} \Rightarrow \sigma_E \cdot \frac{E}{A} + \sigma_D \cdot \frac{D}{A} = \sigma.$$

To illustrate shapes of cost of capital curves depending on the debt-equity ratio, we assume the following sample data. The value of assets A equals one monetary unit, their instantaneous expected rate of return μ amounts to ten percent, and their local volatility σ is 20 percent. The risk-free rate r equals five percent and the lent term of debt T is one period. To show the influence of the debt-equity ratio on costs of capital, we vary the face value of debt K so that debt-equity ratios lay between 0 and 20. We use formula (20) to calculate the expected costs of equity and debt. For this, the value of equity E is computed using Black-Scholes formula (12) and the value of debt D is obtained as the difference between assets value A and equity value E.

Figure 3 plots costs of capital against the debt-equity ratio. As Modigliani-Miller proposition I holds, the WACC is constant. However, the cost of equity and the cost of debt increase non-linearly. The figure substantiates the sketched courses of costs of capital in Merton (1974, figure 9). With given volatility of assets, the cost of debt increases with the debt-equity ratio as the debt holders bear higher default risk. The cost of debt is limited upwards by the WACC because with an infinitely high debt-equity ratio, the debt holders bear all the assets risk and act like equity holders in an unlevered company from an economic point of view. Compared with the situation without credit risk, the risk of equity holders is lower. Therefore, the cost of equity under credit risk is reduced.

Figure 3: Costs of capital depending on the debt-equity ratio
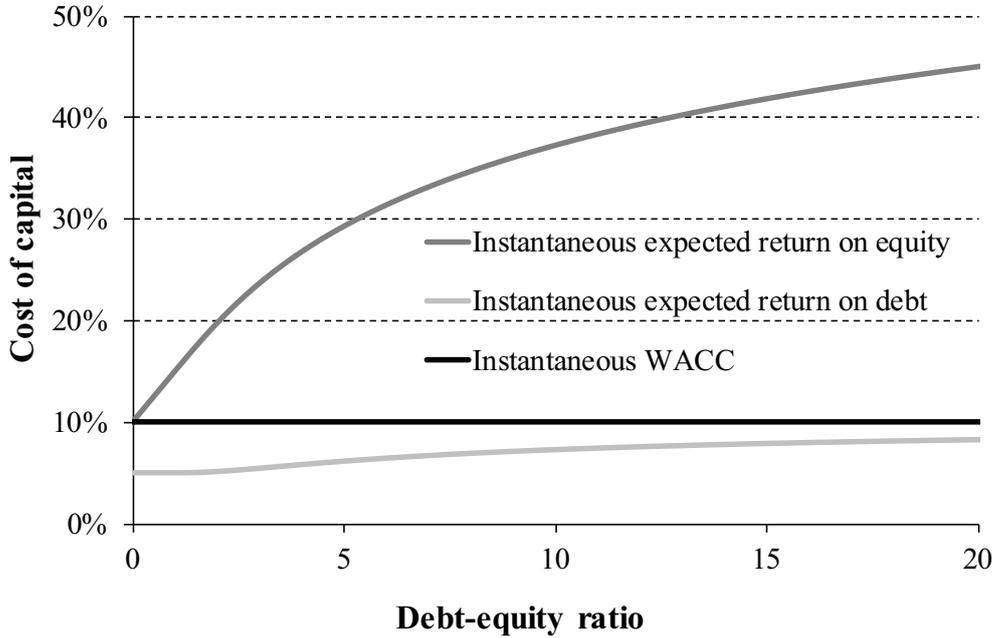
To apply our proposed instantaneous expected return approach to a non-publicly traded company, the WACC of the particular company is determined via peer group analysis: $\mu = \mu^{\text{peer}}$. According to the first equation in formula (20), μ^{peer} implicitly depends on the expected return on equity μ_E^{peer}, assets value A^{peer}, equity value E^{peer}, and assets volatility σ^{peer}. The face value of peer group companies' debt K^{peer} can be collected from financial statements. The risk-free rate r can be obtained from government bond yields. Because a real company's debt does not consist only of a single zero-coupon bond, it is suggested to use the Macauly duration of peer group companies' liabilities as a proxy for T^{peer} (Damodaran, 2012, 833).[5]

The market capitalization of peer group equity E^{peer} is known and numerical peer group values for μ_E^{peer} and σ_E^{peer} can be estimated based on historical price data. We can

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[5] We are aware that employing the Macauly duration as a proxy for T does not reflect that loans with periodic interest payments parallel straight bonds and, therefore, correspond to compound options (Geske, 1977).

refer from E^{peer} and σ_E^{peer} to A^{peer} and σ^{peer} by simultaneously solving Black-Scholes equity formula (12) and its corresponding volatility formula (22) for the value of assets and its volatility (Cooper and Davydenko, 2007; Bharath and Shumway, 2008). Thereby, the required data set is complete. Note that for relevering, the debt market value of the company to be valued is not needed to compute single costs of capital since μ_E and μ_D according to formula (20) do not depend on D.

Table 1 compares data requirements for peer group analysis according to the CAPM approaches and our option-based approach. The table confirms that data requirements for our approach are at the same level as the classical CAPM approach without credit risk. In contrast, the debt beta approach additionally requires a debt risk measure and the market debt value of the company to be valued. Both are endogenous in the option-based approach due to put-call parity.

Besides the quantity of required parameters, their quality to explain differences in costs of capital is important. This, in particular, affects the risk measures. At least since Fama and French's (1992) study, the explanatory power of stock index betas for cross-sectional stock returns is questioned. In their study, the relation between average stock rates of return and stock index betas is flat which might be attributed to return-risk inefficiency of the used stock index (Roll and Ross, 1994).[6]

A flat return-beta relation is crucial for CAPM-based company valuation since the basis for risk-adjusting the cost of equity by a beta-proportional amount of the equity risk premium is lost. Also in this regard, our option-based approach proves to be superior because formula (16) justifies the assumption that costs of capital strictly increase with volatility in the vast majority of company valuation cases. Note that formula (16) does not require continuous trading of the underlying assets for duplicating the derivative contract. However, this is assumed to price the derivative contract.

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[6] Multi-factor models possess higher explanatory power for the variance of cross-sectional returns (Fama and French, 1993 and 2015; Carhart, 1997; Hou, Xue, and Zhang, 2015). The passive strategy in terms of alternative investment to determine opportunity cost in case of applying Fama-French-style factor models includes some (self-financing) factor investing transactions.

Table 1: Required data for peer group analysis

	CAPM approach without credit risk	**Debt beta approach**	**Option-based approach**
Risk-free rate r	+	+	+
Value of equity E^{peer}	Market value	Market value	Market value
Value of debt D^{peer}	Book value	Market value	–
Comment	*By assumption of no credit risk*	*Unknown debt market value of the company to be valued is needed for relevering*	*Endogenous; only debt face value K^{peer} and time to maturity T^{peer} are required*
Equity risk measure	Equity beta β_E^{peer}	Equity beta β_E^{peer}	Equity volatility σ_E^{peer}
Comment	*Regression on stock market index I*	*Regression on stock-bond market index M (data not available)*	
Debt risk measure	–	Debt beta β_D^{peer}	–
Comment	*By assumption of no credit risk*	*Regression on stock-bond market index M (data not available)*	σ_D^{peer} *is endogenous*
Risk premium	Equity premium $\left(\text{E}(R_I)-r\right)$	Stock-bond market premium $\left(\text{E}(R_M)-r\right)$	Expected equity return μ_E^{peer}
Comment	*By assumption of no credit risk*	*Data not available*	*See appendix A for* $\left(\mu_E^{\text{peer}}-r\right)$

5. Per-period costs of capital in the option-based approach

Instantaneous expected returns are important in continuous-time finance. However, in practical applications of company valuation, it is usually assumed that end-of-period cash flows have to be discounted. Expected costs of capital per period have to be determined in this sit-

uation. We (implicitly) define continuously compounded expected rates of return per period on equity μ_E^{periodic} and debt μ_D^{periodic} by the following equations:[7]

(23) $\quad E \cdot e^{\mu_E^{\text{periodic}} \cdot T} = \mathrm{E}(E_T) \quad \text{and} \quad D \cdot e^{\mu_D^{\text{periodic}} \cdot T} = \mathrm{E}(D_T).$

In appendix B, we show that expected costs on equity and debt per period can be computed as follows:

$$e^{\mu_E^{\text{periodic}} \cdot T} = \frac{A \cdot e^{\mu \cdot T} \cdot \mathrm{N}\left(\tilde{d}_1\right) - K \cdot \mathrm{N}(\tilde{d}_2)}{A \cdot \mathrm{N}(d_1) - K \cdot e^{-r \cdot T} \cdot \mathrm{N}(d_2)}$$

$$\Rightarrow \mu_E^{\text{periodic}} = \frac{1}{T} \cdot \ln\left(\frac{A \cdot e^{\mu \cdot T}}{E} \cdot \mathrm{N}\left(\tilde{d}_1\right) - \frac{K}{E} \cdot \mathrm{N}\left(\tilde{d}_2\right)\right) \quad \text{and}$$

(24)

$$e^{\mu_D^{\text{periodic}} \cdot T} = \frac{A \cdot e^{\mu \cdot T} \cdot \mathrm{N}\left(-\tilde{d}_1\right) + K \cdot \mathrm{N}(\tilde{d}_2)}{A \cdot \mathrm{N}(-d_1) - K \cdot e^{-r \cdot T} \cdot \mathrm{N}(d_2)}$$

$$\Rightarrow \mu_D^{\text{periodic}} = \frac{1}{T} \cdot \ln\left(\frac{A \cdot e^{\mu \cdot T}}{D} \cdot \mathrm{N}\left(-\tilde{d}_1\right) + \frac{K}{D} \cdot \mathrm{N}\left(\tilde{d}_2\right)\right)$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are based on the physical distribution of the value of assets and differ from the risk-neutral quantities d_1 and d_2 of the Black-Scholes formula in that the expected rate of return on assets μ replaces the risk-free rate r. The quantities d_1 and d_2 are expressed in terms of present values, whereas $\tilde{d}_1$ and $\tilde{d}_2$ are in terms of expected future values:

$$d_1 = \frac{\ln\dfrac{A}{K \cdot e^{-r \cdot T}} + \dfrac{\sigma^2}{2} \cdot T}{\sigma \cdot \sqrt{T}} \quad \text{and} \quad \tilde{d}_1 = \frac{\ln\dfrac{\mathrm{E}(A_T)}{K} + \dfrac{\sigma^2}{2} \cdot T}{\sigma \cdot \sqrt{T}} \quad \text{since } \mathrm{E}(A_T) = A \cdot e^{\mu \cdot T}.$$

Hence, $\mathrm{N}\left(\tilde{d}_1\right)$ and $-\mathrm{N}\left(\tilde{d}_2\right)$ can be interpreted as expected amounts of assets and risk-free zero-coupon bonds in the equity duplicating portfolio. Analogously to our interpretation of the expected instantaneous return on equity in the previous section (formula (20)), the expected value of equity $\mathrm{E}(E_T) = \mathrm{E}(A_T) \cdot \mathrm{N}\left(\tilde{d}_1\right) - K \cdot \mathrm{N}(\tilde{d}_2)$ – which is in line with Rubinstein (1984) – corresponds to the expected composition of the duplicating portfolio. In this

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[7] Formula (23) requires equity and debt values. Employing Black-Scholes values in formula (24) below presupposes that the Black-Scholes formula developed in the framework of a complete (continuous-time) capital market is also valid on an incomplete (discrete-time) market.

context, $N\left(-\tilde{d}_2\right)$ equals the physical probability that the put option inherent in debt ends in the money. Thus, this term represents the company's probability of default (e.g., Sundaresan, 2013, 24).

The total period expected WACC, in the form of a discount factor, results from weighting and adding up single expected costs of capital. The per-period expected rates of return on equity and debt according to formula (24) fulfill Modigliani-Miller proposition I:

$$(25) \quad e^{\text{WACC}_T^{\text{compound}}} = e^{\mu_E^{\text{periodic}} \cdot T} \cdot \frac{E}{A} + e^{\mu_D^{\text{periodic}} \cdot T} \cdot \frac{D}{A} = \frac{\text{E}\left(E_T\right)}{A} + \frac{\text{E}\left(D_T\right)}{A} = e^{\mu \cdot T} .$$

To illustrate the curves of the company's default probability $N\left(-\tilde{d}_2\right)$ and corresponding expected loss of the debt holders $\left(K - \text{E}\left(D_T\right)\right)$ depending on the debt-equity ratio, we revisit our numerical example of the previous section. Figure 4 visualizes that, of course, both measures of credit risk increase with the debt-equity ratio. In our sample setting, a debt-equity ratio of 20 induces a probability of default of 60 percent. At the same time, the expected loss amounts to nearly 10 percent so that the recovery rate is slightly above 90 percent. This supports our reasoning within section 3 that the expected loss is meaningful for the implied debt beta.

Figure 4: Probability of default and expected loss depending on the debt-equity ratio
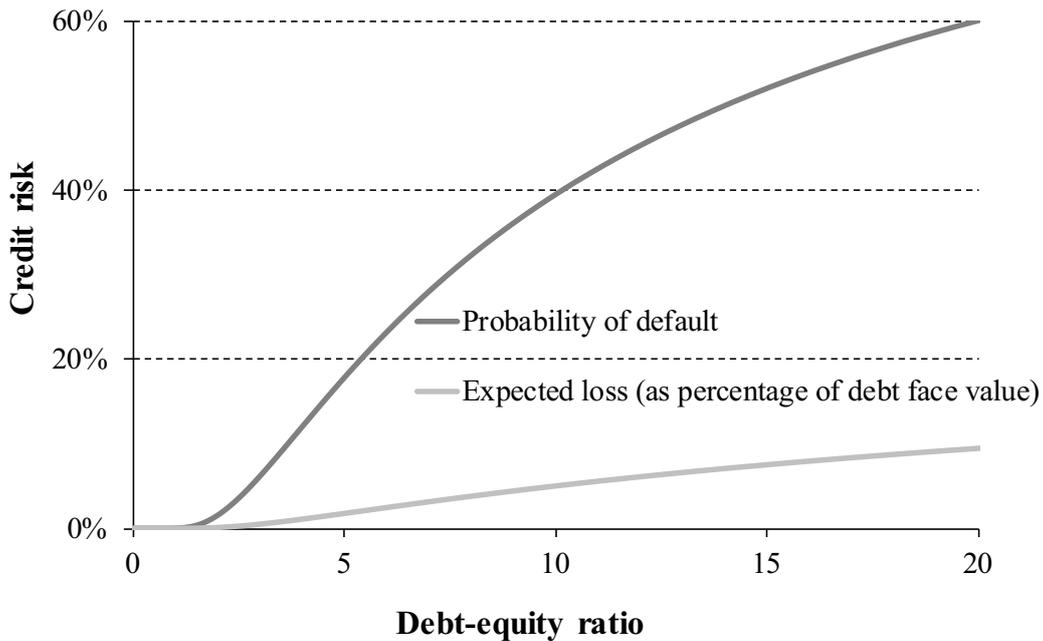
Completing our discussion of per-period expected returns, we turn to discretely compounded (simple) rates of return as an alternative way of computing costs of capital. Clearly, the use of simple returns instead of continuously compounded returns will not change the general results of cost of capital computations. Nevertheless, the definition of simple returns has to be consistent. Let μ_T^{discrete} denote the expected simple total period rate of return on assets

(26) $\quad \mu_T^{\text{discrete}} = \left(1 + \mu^{\text{discrete}}\right)^T - 1 \quad$ with $\quad \mu^{\text{discrete}} = e^{\mu} - 1$

where μ^{discrete} denotes the expected periodic rate of return on assets. Thereby, $\left(1 + \mu_T^{\text{discrete}}\right)$ represents the appropriate factor to discount $\text{E}(A_T)$. Now, define μ_E^{discrete} and μ_D^{discrete} as the corresponding expected periodic rate of return on equity and debt, respectively:

(27) $\quad \left(1 + \mu_E^{\text{discrete}}\right)^T = \dfrac{\text{E}(E_T)}{E} \quad$ and $\quad \left(1 + \mu_D^{\text{discrete}}\right)^T = \dfrac{\text{E}(D_T)}{D}.$

Then, the total period simple returns $\mu_{E,T}^{\text{discrete}} = \left(1 + \mu_E^{\text{discrete}}\right)^T - 1$ and $\mu_{D,T}^{\text{discrete}} = \left(1 + \mu_D^{\text{discrete}}\right)^T - 1$ fulfill Modigliani-Miller proposition I:

(28)
$$\begin{aligned}
\text{WACC}_T^{\text{discrete}} &= \mu_{E,T}^{\text{discrete}} \cdot \frac{E}{A} + \mu_{D,T}^{\text{discrete}} \cdot \frac{D}{A} \\
&= \frac{\text{E}(E_T)}{A} + \frac{\text{E}(D_T)}{A} - 1 = \frac{\text{E}(A_T)}{A} - 1 = \mu_T^{\text{discrete}}.
\end{aligned}$$

As a result, using simple returns in company valuation will not change the property that the WACC equals the expected return on assets, i.e., expected return on equity in the as-if unlevered company.

6.   Combing the option-based and the debt beta approach

The debt beta approach appears to be state-of-the-art in the industry. Although we prefer the pure option-based approach to calculate costs of capital, it is worth analyzing if integrating the option-based approach into the debt beta framework helps to overcome some difficulties discussed in section 3. For this, we compute instantaneous option betas based on formula (14) (Black and Scholes, 1973, formula (15); Smith, 1976, formula (89); Coval and Shumway, 2001, formula (11)):

$$(29) \quad \beta_f^{instant} = \frac{\text{Cov}\left(\dfrac{df}{f}, \dfrac{dM}{M}\right)}{\text{Var}\left(\dfrac{dM}{M}\right)} = \frac{\text{Cov}\left(\dfrac{1}{f}\cdot\Delta_f\cdot dA, \dfrac{dM}{M}\right)}{\text{Var}\left(\dfrac{dM}{M}\right)}$$

$$= \Delta_f \cdot \frac{A}{f} \cdot \frac{\text{Cov}\left(\dfrac{dA}{A}, \dfrac{dM}{M}\right)}{\text{Var}\left(\dfrac{dM}{M}\right)} = \Delta_f \cdot \frac{A}{f} \cdot \beta_A^{instant} \quad \text{where}$$

$\dfrac{dM}{M} = $ instantaneous market rate of return,

$\beta^{instant} = $ instantaneous beta.

Analogously to the computation of instantaneous expected rates of return in section 4 and following Galai and Masulis (1976, formula (8) and footnote 15), we insert the delta of equity and debt, respectively, into formula (29). The resulting instantaneous betas of equity and debt are

$$(30) \quad \beta_E^{instant} = \underbrace{\Delta_E \cdot \frac{A}{E}}_{=\Omega_E \text{ (equity omega)}} \cdot \beta_A^{instant} = N(d_1)\cdot\left(1+\frac{D}{E}\right)\cdot\beta_A^{instant} \quad \text{and}$$

$$\beta_D^{instant} = \underbrace{\Delta_D \cdot \frac{A}{D}}_{=\Omega_D \text{ (debt omega)}} \cdot \beta_A^{instant} = N(-d_1)\cdot\left(1+\frac{E}{D}\right)\cdot\beta_A^{instant}.$$

Therefore, the betas of equity and debt are proportional to their elasticities (omegas) with respect to the value of assets. Inserting the betas from formula (30) into CAPM formulas (2) and (9) leads to expected rates of return on equity and debt. Weighting single costs of capital by the corresponding equity and debt ratios proves that Modigliani-Miller proposition I is also valid in this approach:

$$(31) \quad \text{WACC}^{option\ beta} = \underbrace{\left(r + N(d_1)\cdot\left(1+\frac{D}{E}\right)\cdot\beta_A^{instant}\cdot\left(E(R_M)-r\right)\right)}_{=E\left(R_E^l\right)}\cdot\frac{E}{E+D}$$

$$+ \underbrace{\left(r + \left(1-N(d_1)\right)\cdot\left(1+\frac{E}{D}\right)\cdot\beta_A^{instant}\cdot\left(E(R_M)-r\right)\right)}_{=E(R_D)}\cdot\frac{D}{E+D}$$

$$= r + \beta_A^{instant}\cdot\left(E(R_M)-r\right) = E\left(R_E^u\right).$$

The option beta framework, compared with the debt beta approach, exhibits the advantage that the debt beta can be endogenously computed with the help of the assets beta. A separate estimation of the debt beta is not needed. Nevertheless, the debt market value of the

company to be valued is required in the relevering step of peer group analysis. In addition, a consistent equity beta still results only from an estimation with respect to a (preferably return-risk efficient) stock-corporate bond index. Finally, practical use of the option beta approach is limited since it mixes instantaneous betas with the expected per-period market risk premium.

7. Conclusions

This paper develops and discusses (quasi-) analytic formulas of single costs of capital and WACC in two different frameworks under credit risk. We show that the debt beta approach – as state-of-the-art in CAPM-style cost of capital computations – leads to distorted results even if betas are computed with respect to a return-risk efficient stock index in the stock universe. Instead, an appropriate index has to consist of both stocks and corporate bonds to represent (systematic) equity and debt risk. Since historical data of a combined stock-corporate bond index is not provided by exchanges, a time series analysis to estimate valid betas is not feasible.

Our results show that the implied debt beta approach is not able to solve this difficulty since this method indeed leads to a consistent overall WACC – in terms of fulfilling Modigliani-Miller proposition I – but implies an inconsistent debt-equity ratio with distorted single costs of capital as a consequence. We attribute this to the difference between the debt interest rate – which is used in the implied debt beta approach – and the required rate of return of debt holders. In sum, the debt beta approach suffers from both theoretical shortcomings and application barriers.

Therefore, this paper applies the classical Black-Scholes-Merton model of credit valuation to determine single costs of capital and WACC. In a continuous-time setting, we develop instantaneous costs of capital by bringing together partial derivations of equity and debt pricing formulas and their corresponding stochastic processes. This enables us to analyze shapes of costs of capital with varying debt-equity ratios. Moreover, our approach proves to be superior to CAPM-based frameworks regarding the quantity and quality of required data in peer group analysis.

Company valuation usually assumes a discrete-time setting. Therefore, we also provide per-period cost of capital formulas in terms of continuously and discretely compounded required rates of return of the capital holders. Our results are based on expected option pay-offs assuming a log-normal distribution of the value of assets which is consistent with the

Black-Scholes framework. In this setting, we are able to analyze the influence of the debt-equity ratio on the probability of default of the particular company and the expected loss of the company's debt holders. Moreover, we prove that our results for instantaneous and per-period costs of capital are consistent with Modigliani-Miller proposition I.

Finally, not as a recommendation but to complete our analysis, we show that combining our option-based approach with the debt beta framework reduces some shortcomings of the latter method. Nevertheless, the pure option-based approach is still advantageous.

Appendix A

Inserting the well-known Black-Scholes partial differential equation for a European-style derivate contract $f(A,t)$

(A1) $\quad \dfrac{\sigma^2}{2} \cdot A^2 \cdot \Gamma_f = r \cdot f - \Theta_f - r \cdot A \cdot \Delta_f$

into Itô formula (14) of the body text leads to

(A2) $\quad df = r \cdot f \ dt + \Delta_f \cdot (dA - r \cdot A \ dt)$.

Thereby, the drift rate μ_f and the local volatility σ_f of the derivative contract can be written as (Kraft, 2003)

(A3) $\quad \dfrac{df}{f} = r \ dt + \underbrace{\Delta_f \cdot \dfrac{A}{f}}_{=\Omega_f \ (\text{omega})} \cdot \left(\dfrac{dA}{A} - r \ dt \right) = \underbrace{\left(r + \Omega_f \cdot (\mu - r) \right)}_{=\mu_f} dt + \underbrace{\Omega_f \cdot \sigma}_{=\sigma_f} dW$

where Ω_f represents the derivative contract's elasticity with respect to the value of the underlying.

We apply μ_f of formula (A3) for equity and debt in Merton's model and find

(A4)
$$\mu_E = r + \Omega_E \cdot (\mu - r) = r + N(d_1) \cdot \dfrac{A}{E} \cdot (\mu - r) \Leftrightarrow$$
$$\mu_E \cdot E = \mu \cdot A \cdot N(d_1) - r \cdot \left(A \cdot N(d_1) - E \right) = \mu \cdot A \cdot N(d_1) - r \cdot K \cdot e^{-r \cdot T} \cdot N(d_2) \quad \text{and}$$
$$\mu_D = r + \Omega_D \cdot (\mu - r) = r + N(-d_1) \cdot \dfrac{A}{D} \cdot (\mu - r) \Leftrightarrow$$
$$\mu_D \cdot D = \mu \cdot A \cdot N(-d_1) - r \cdot \left(A \cdot N(-d_1) - D \right) = \mu \cdot A \cdot N(-d_1) + r \cdot K \cdot e^{-r \cdot T} \cdot N(d_2)$$

which is equivalent to formula (20). The first equation for μ_E in formula (A4) corresponds to Galai and Masulis (1976, formula (13)) which was developed in a combined option pricing-

CAPM framework. Formula (22) regarding equity volatility follows immediately from σ_f in formula (A3).

Following the idea of Branger and Schlag (2007, appendix A) to apply L'Hôpital's rule when taking limits for option elasticities, it holds

(A5)
$$\lim_{K \to 0} \mu_E = \mu \quad \text{and} \quad \lim_{K \to \infty} \mu_E = \infty,$$
$$\lim_{K \to 0} \mu_D = r \quad \text{and} \quad \lim_{K \to \infty} \mu_D = \mu.$$

To compute the derivations of costs of equity and debt with respect to the debt-equity ratio, we proceed as follows. Firstly, the derivations of μ_E and μ_D with respect to the face value of debt K are (Galai and Masulis, 1976, appendix I)

(A6)
$$\frac{\partial \mu_E}{\partial K} = (\mu - r) \cdot \frac{A \cdot N(d_1) \cdot e^{-r \cdot T} \cdot N(d_2)}{E^2 \cdot \sigma \cdot \sqrt{T}} \cdot \left[\frac{n(d_1)}{N(d_1)} - \frac{n(d_2)}{N(d_2)} + \sigma \cdot \sqrt{T} \right] \quad \text{and}$$

$$\frac{\partial \mu_D}{\partial K} = (\mu - r) \cdot \frac{A \cdot N(-d_1) \cdot e^{-r \cdot T} \cdot N(d_2)}{D^2 \cdot \sigma \cdot \sqrt{T}} \cdot \left[\frac{n(-d_1)}{N(-d_1)} + \frac{n(d_2)}{N(d_2)} - \sigma \cdot \sqrt{T} \right].$$

Galai and Masulis (1976, appendix I (D)) show that both terms in square brackets in formula (A6) are positive, i.e., the cost of equity and the cost of debt are increasing in K.

Secondly, since the derivation of the debt-equity ratio with respect to K amounts to

$$\frac{\partial \frac{D}{E}}{\partial K} = e^{-r \cdot T} \cdot N(d_2) \cdot \frac{A}{E^2} > 0,$$ the derivations of costs of equity and debt with respect to the debt-equity ratio are

(A7)
$$\frac{\partial \mu_E}{\partial \frac{D}{E}} = (\mu - r) \cdot \frac{N(d_1)}{\sigma \cdot \sqrt{T}} \cdot \left[\frac{n(d_1)}{N(d_1)} - \frac{n(d_2)}{N(d_2)} + \sigma \cdot \sqrt{T} \right] > 0 \quad \text{and}$$

$$\frac{\partial \mu_D}{\partial \frac{D}{E}} = (\mu - r) \cdot \left(\frac{E}{D} \right)^2 \cdot \frac{N(-d_1)}{\sigma \cdot \sqrt{T}} \cdot \left[\frac{n(-d_1)}{N(-d_1)} + \frac{n(d_2)}{N(d_2)} - \sigma \cdot \sqrt{T} \right] > 0.$$

As a result – besides μ, σ, r, and T – the ratio of value of assets A and debt redemption amount K determines the slopes of costs of equity and debt with regard to the debt-equity ratio.

Appendix B

To determine continuously compounded per-period expected rates of return on equity and debt, we extend the theorem of Smith (1976) for call option-style pay-offs to put option-style pay-offs. Smith's theorem reads as follows:

If the value of assets A_T follows a log-normal distribution and X_T shows a call option-style pay-off at maturity T:

(B1) $\quad X_T = \begin{cases} a \cdot A_T - b \cdot K & \text{if} \quad A_T - c \cdot K \geq 0 \\ 0 & \text{if} \quad A_T - c \cdot K < 0 \end{cases}$

where a, b, and c are constants, then the expected value of X_T is

(B2) $\quad \mathrm{E}(X_T) = a \cdot A \cdot e^{\mu \cdot T} \cdot \mathrm{N}\left(\dfrac{\ln \dfrac{A}{c \cdot K} + \left(\mu + \dfrac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}} \right) - b \cdot K \cdot \mathrm{N}\left(\dfrac{\ln \dfrac{A}{c \cdot K} + \left(\mu - \dfrac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}} \right).$

Now, let Y_T be a put option-style pay-off:

(B3) $\quad Y_T = \begin{cases} 0 & \text{if} \quad A_T - c \cdot K \geq 0 \\ b \cdot K - a \cdot A_T & \text{if} \quad A_T - c \cdot K < 0 \end{cases}.$

With this, we can set up the following put-call parity:

(B4) $\quad X_T + b \cdot K = \begin{cases} a \cdot A_T & \text{if} \quad A_T - c \cdot K \geq 0 \\ b \cdot K & \text{if} \quad A_T - c \cdot K < 0 \end{cases}$

$\qquad\qquad\quad = Y_T + a \cdot A_T.$

Thus, the expected value of Y_T is

(B5) $\quad \mathrm{E}(Y_T) = \mathrm{E}(X_T) + b \cdot K - a \cdot \mathrm{E}(A_T)$

$\qquad = b \cdot K \cdot \mathrm{N}\left(-\dfrac{\ln \dfrac{A}{c \cdot K} + \left(\mu - \dfrac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}} \right) - a \cdot A \cdot e^{\mu \cdot T} \cdot \mathrm{N}\left(-\dfrac{\ln \dfrac{A}{c \cdot K} + \left(\mu + \dfrac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}} \right)$

since $\mathrm{E}(A_T) = A \cdot e^{\mu \cdot T}$.

For $a = b = c = 1$, $X_T = E_T$ represents the equity pay-off with expected value (Smith, 1976, formula (24)):

(B6) $\quad E\left(E_T\right) = A \cdot e^{\mu \cdot T} \cdot N\left(\tilde{d}_1\right) - K \cdot N\left(\tilde{d}_2\right)$

where $\tilde{d}_1 = \dfrac{\ln \dfrac{A}{K} + \left(\mu + \dfrac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}}$ and $\tilde{d}_2 = \tilde{d}_1 - \sigma \cdot \sqrt{T}$. At the same time, the expected value

of the debt pay-off $\left(D_T = K - Y_T\right)$ is

(B7) $\quad E\left(D_T\right) = K \cdot N\left(\tilde{d}_2\right) + A \cdot e^{\mu \cdot T} \cdot N\left(-\tilde{d}_1\right)$.

Inserting formulas (B6) and (B7) into formula (23) of the body text and solving for the per-period expected rates of return yields formula (24).

In addition, formula (B5) shows that the expected loss $\left(EL = E\left(Y_T\right)\right)$ reflects the put component of debt (Vasicek, 1984):

(B8) $\quad EL = K \cdot N\left(-\tilde{d}_2\right) - A \cdot e^{\mu \cdot T} \cdot N\left(-\tilde{d}_1\right)$.

Furthermore, $Y_T = \mathbf{1}_{A_T < K}$ for $a = 0$, $b = \dfrac{1}{K}$ and $c = 1$. Therefore, the probability of default PD amounts to

(B9) $\quad PD = N\left(-\tilde{d}_2\right)$.

Cooper and Davydenko (2007) focus on the difference between the expected per-period rate of return on debt $\left(\mu_D^{\text{periodic}} = \dfrac{1}{T} \cdot \ln \dfrac{E(D_T)}{D}\right)$ and debt interest rate $\left(i_D = \dfrac{1}{T} \cdot \ln \dfrac{K}{D}\right)$ according to Merton (1974, 454). Using formula (B7), it becomes clear that $\left(\mu_D^{\text{periodic}} - i_D\right)$ does not depend either on the value of equity, the value of debt, or the risk-free rate:

(B10) $\quad \mu_D^{\text{periodic}} - i_D = \dfrac{1}{T} \cdot \ln \left(\dfrac{A \cdot e^{\mu \cdot T}}{K} \cdot N\left(-\tilde{d}_1\right) + N\left(\tilde{d}_2\right)\right)$.

Finally, the difference between the pay-offs of underlying and corresponding call option is known as the pay-off of a covered call in the field of option strategies. Ferguson (1993) applies log returns $\left(\mu^{\log} = \dfrac{1}{T} \cdot E\left(\ln \dfrac{A_T}{A}\right) = \mu - \dfrac{\sigma^2}{2}\right)$ to compute the expected per-period return of a covered call. Since the expected pay-off of a covered call mirrors the pay-off of debt, the expected debt pay-off in terms of log returns reads

$$\text{(B11)} \quad \mathrm{E}(D_T) = A \cdot e^{\left(\mu^{\log} + \frac{\sigma^2}{2}\right) \cdot T} \cdot \mathrm{N}\left(-\frac{\ln\frac{A}{K} + (\mu^{\log} + \sigma^2) \cdot T}{\sigma \cdot \sqrt{T}}\right) + K \cdot \mathrm{N}\left(\frac{\ln\frac{A}{K} + \mu^{\log} \cdot T}{\sigma \cdot \sqrt{T}}\right).$$

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