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Stefan Hlawatsch • Peter Reichling

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A Framework for LGD Validation of Retail Portfolios

Stefan Hlawatsch and Peter Reichling*

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Modeling and estimating the loss given default (LGD) is necessary for banks which apply for the Internal-Ratings Based Approach for retail portfolios. To validate LGD estimations there are only very few approaches discussed in the literature. In this paper, two models for validating relative LGDs and absolute losses are developed. The validation of relative LGDs is important for risk-adjusted credit pricing and interest rate calculations. The validation of absolute losses is important to meet the capital requirements of Basel II. Both models are tested with real data of a bank. Estimations are tested for robustness with in-sample and out-of-sample tests.

Keywords: Loss Given Default, Validation, Retail Portfolio

J.E.L. classification: C52, G21, G28, G32, G38

*Both authors are from Otto-von-Guericke-University Magdeburg, Faculty of Economics and Management, Department of Banking and Finance, Postfach 4120, 39106 Magdeburg, Germany, e-mail Stefan.Hlawatsch@ovgu.de, phone +49.391.67.12256 (corresponding author). We thank Hendrik Ritter for helpful comments.

1 Introduction

According to Basel II, banks can choose between two approaches to measure their credit risk, the Standardised Approach (STA) and the Internal-Ratings Based Approach (IRBA). So-called IRBA banks have to underlay credits with equity depending on the unexpected loss (UL) according to Equation (1):[1]

$$\mathrm{UL} = \overbrace{\left[\underbrace{\mathrm{N}\left(\frac{\mathrm{N}^{-1}(\mathrm{PD}) + \sqrt{R} \cdot \mathrm{N}^{-1}(0{,}999)}{\sqrt{1-R}} \right) \cdot \mathrm{LGD}}_{\text{maximum loss}} - \underbrace{\mathrm{PD} \cdot \mathrm{LGD}}_{\text{expected loss}} \right]}^{\text{relative unexpected loss}} \cdot \mathrm{EAD}. \qquad (1)$$

The unexpected loss equals the difference between the so-called maximum loss, which is computed as the value at risk of the loss, and the expected loss, which is computed as the product of probability of default (PD) and loss given default (LGD). Here, R denotes the correlation coefficient of the PD with the systematic risk factor. The product of the so computed relative unexpected loss and the exposure at default (EAD) results in the unexpected loss in monetary units.

According to Article 87 No. 6 and 7 of the Capital Requirement Directive (CRD),[2] banks have to estimate PD and LGD for retail claims or contingent retail claims on their own. Furthermore, the estimation procedures have to be validated for robustness and accuracy of the models. This validation should transcend the simple comparison of historical data with estimated parameters as it is mentioned in Annex VII, Part 2 No. 112 CRD. While validation techniques for PD estimations are discussed extensively in the literature, research on quantitative validation instruments for LGD estimation models is rare.

Validating LGD estimations is crucial because the required capital reacts more sensitive to changes in LGD as to changes in PD. For demonstration, the LGD and the PD elasticities of UL are shown in Figure 1.[3] The LGD elasticity is constant and amounts to one. However, the PD elasticities for the shown subcategories for retail claims (up to a PD of 50 percent) are absolutely smaller than the LGD elasticity. Therefore, the risk weighted assets react more sensitive on changes in LGD.[4] Thus, the high sensitivity of the risk weight function with respect to the LGD (in the relevant PD range up to 50 percent)

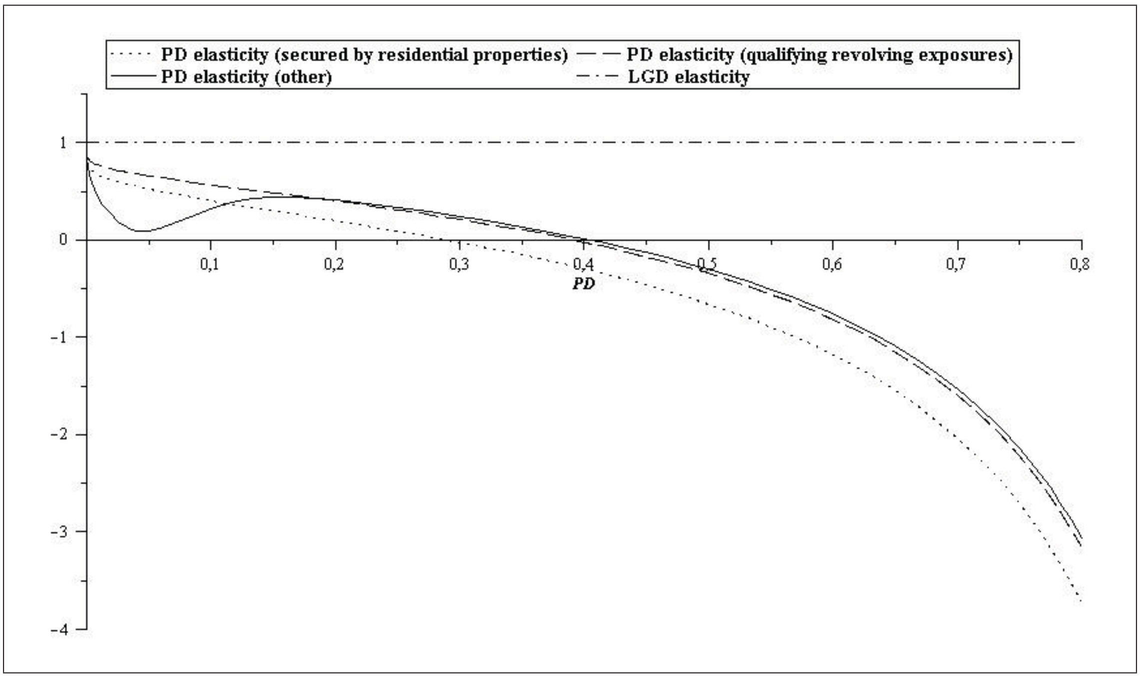
[1] N indicates the standard normal distribution function and N^{-1} indicates the inverse of N.

[2] Directive 2006/48/EC of the european Parliament and of the Council.

[3] The right hand side of Equation (1) is multiplied by factor 12.5 in Figure 1, so that the UL equals the risk weighted assets.

[4] The partly negative PD elasticity shown in Figure 1 results from the fact that with an increasing PD the unexpected loss becomes smaller and the expected loss, which reduces the capital requirement,

Figure 1: PD and LGD Elasticity of the Risk Weighted Assets for the Retail IRB Approach



implies the necessity for precise estimations of LGDs.[5] An evaluation of the accuracy of LGD estimations can be done by the technique of validation.

Our paper is organized as follows: After a brief review of the literature of different LGD estimation models, two validation models are developed in Section 2. For an empirical analysis, real data from a bank are used. Section 3.1 describes the data and empirical analysis and Section 3.2 reports the results. Section 4 concludes.

2 LGD-Validation

2.1 Literature Review

There are four different approaches to compute the LGD: Workout LGD, Market LGD, Implied Market LGD, and Implied Historical LGD.[6]

The Workout LGD belongs to the so-called explicit methods of LGD estimation. "Explicit" here refers to the used data. Explicit methods use historical LGDs of defaulted

rises. Expected losses are considered by depreciations, provisions or write-offs and, therfore, no capital underlay is required.

[5]Despite the higher LGD sensitivity within the relevant range, this does not imply a less precise or less robust estimation of PD.

[6]See Basel Committee on Banking Supervision (2005), pp. 61-62.

credits in order to derive prognoses for future LGDs. The Workout LGD is cash flow-oriented.[7] To compute the Workout LGD, all recoveries as well as all costs are considered in the period from the day of the credit event up to final recovery. In order to consider different points in time where costs and recoveries emerge, payments have to be discounted to the day of the credit event. Therefore, the Workout LGD is computed as follows:[8]

$$\begin{aligned}
\text{LGD}_i &= \frac{\text{EAD}_i - \sum_{j=1}^{n} E_{i,j}(r) + \sum_{k=1}^{m} K_{i,k}(r)}{\text{EAD}_i} \\
&= 1 - \frac{\sum_{j=1}^{n} E_{i,j}(r) - \sum_{k=1}^{m} K_{i,k}(r)}{\text{EAD}_i} \\
E_{i,j}(r) &= \text{discounted recoveries } j \text{ of credit } i \\
K_{i,k}(r) &= \text{discounted costs or losses } k \text{ of credit } i \\
r &= \text{discount rate} \qquad (2)
\end{aligned}$$

Typical recoveries are collaterals or securities.[9] Examples for costs and losses are a loss on interest payments, opportunity costs for equity, handling costs, and workout costs like overhead costs of the recovery departement.[10]

The discount rate to determine the economic loss has to be risk-adjusted. In particular, for parameters such as collaterals or workout costs no market exists. Therefore, the determination of the discount rate is difficult. If historical interest rates are used, the risk-free interest rate plus a loss impact or the initially agreed interest rate can be used.[11]

After computing the Workout LGD, the estimation model can be developed using regressions for example.[12] The independent variables that should be used here depend on the institute and branch. Commonly accepted variables are for example provisions of security,

[7]See Kaltofen (2006), pp. 38-39.

[8]See Basel Committee on Banking Supervision (2005), p. 66.

[9]Especially for tangible fixed assets, for example real estates or machines, market prices can change until the collateral utilization. Therefore, haircuts should be computed to consider this loss in value. See also Basel Committee on Banking Supervision (2005), p. 67.

[10]See Oesterreichische Nationalbank/Finanzmarktaufsicht (2004), p. 151, 165.

[11]Brady, Chang, Miu, Ozdemir & Schwartz (2006) empirically show that discount rates significantly differ for different branches of industry and ratings. The determined discount rate ranges from 0.9 to 29.3 percent.

[12]Hamerle, Knapp & Wildenauer (2006) use a two stage regression. Siddiqi & Zhang (2004) assume that the LGD is beta-distributed and transform the LGD into a normal-distributed variable before estimation.

repayment priority, industry affiliation, macro-economic factors like economic growth or ratings.[13]

A further explicit method to determine the LGD is the so-called Market LGD method. In this approach, market prices of publicly traded defaulted loans or securitized credits are used. After a default, the recovery rate can be determined by the market value of the loan because investors anticipate possible proceeds from realizations as well as possible costs. Thus, loss results as the difference between the par offering price and the market price after default. For publicly traded loans, these data are collected by rating agencies. The charm of this concept is made up of the fact that only the recovery rate is needed for the LGD computation. However, this recovery rate corresponds to the market price after default for initially priced at par loans.

In this approach, it is critical that several parameters are based on subjective estimates. It remains doubtful, which time horizon should have been taken after the point in time of default of the loan, at that all investors anticipate possible earnings and costs.[14] In addition, internal workout costs of the bank are not reflected by the market price.[15] Moreover, market prices are not only influenced by supply and demand. Therfore, on illiquid markets, the use of market prices can lead to false estimates of the LGDs.

After computation of the Market LGD, the development of the estimation model follows similarly to the Workout LGD. A well-known model for the LGD estimation based on Market LGDs is LossCalc 2.0 from Moody's KMV.[16]

The Market LGD is only suitable for securitized loans or credits due to the need of market data. Not considered workout costs have to be integrated by an adjustment of the LGDs. Therefore, internal institute data are required. But then again, the Workout LGD can be used.

A further possibility to determine the LGD is the Implied Market LGD, which belongs to the implicit methods. Non-defaulted securitized loans or credits form the data base for this approach. Here, it is assumed that the spread between a loan-specific interest rate and the risk-free interest rate equals the expected loss in percent. If the spread is known the LGD results from the ratio of spread and default probability.[17] This concept is based on the model of Jarrow, Lando & Turnbull (1997). The value of a loan V equals the value of a loan without risk V_{r_f} multiplied with the probability of a non-default plus the value

[13]For more examples and analyses of influencing factors see Schuermann (2005) and Böttger, Guthoff & Heidorn (2008).

[14]Moody's for example uses a time horizon of one month after default. See also Gupton (2005).

[15]See Oesterreichische Nationalbank/Finanzmarktaufsicht (2004), p. 160.

[16]Gupton (2005) gives an overview of the modeling and procedure of LossCalc 2.0.

[17]See Böttger, Guthoff & Heidorn (2008).

of a loan without risk multiplied by the recovery rate (RR) and the probability of default. Then, the following valuation equation holds for risky bonds:[18]

$$V \quad = \quad V_{r_f} \cdot (1 - \mathrm{PD}) + V_{r_f} \cdot \mathrm{PD} \cdot \mathrm{RR}. \tag{3}$$

Implied Market LGD models differ only in the statistic modeling of the parameters of Equation (3) and different interpretations of the recovery rate. In general, there are three possible interpretations: the recovery rate is defined as a portion of the issue price, a portion of the current present value and a portion of the value of the loan shortly before default.[19]

Madan & Unal (2000) and Bakshi, Madan & Zhang (2006) use a hazard process in order to model the default probability. The recovery rate as well as the process of the risk-free interest rate term structure are modeled by stochastic processes.[20] Also, the use of alternative interest rate spreads is discussed.[21]

For the Implied Market LGD, the decomposition of the interest rate spread into its components is crucial. Since the interest rate spread may contain a liquidity premium as well as a risk premium for the unexpected loss, the applied asset pricing models must be able to determine the single components separately.[22]

The Implied Historical LGD is also a concept of the implicit LGD determination. According to the Basel Committee on Banking Supervision, the use of the Implied Historical LGD is only allowed for retail portfolios.[23] According to this approach, banks are allowed to determine their LGDs on the basis of PD estimations. The data base consists of historical loss data of retail portfolios. Here, the LGD of a retail credit is computed similarly as for the case of the Implied Market LGD: EL = PD · LGD.

2.2 Proportional Decomposition of the Credits

As seen in the previous section, especially for retail claims, the discussed concepts of LGD computations are developed on already defaulted credits. If realized LGDs are known a validation model should use those realized LGD as a benchmark for LGD estimation

[18]See Jarrow, Lando & Turnbull (1997).

[19]See Bakshi, Madan & Zhang (2006).

[20]For a detailed overview of the modeling of the single parameters see Ong (1999), pp. 61-92.

[21]For the use of credit default swap spreads to determine implicit recovery rates see Pan & Singleton (2008) and Das & Hanouna (2007).

[22]See also Böttger, Guthoff & Heidorn (2008).

[23]See Basel Committee on Banking Supervision (2004), paragragh 465.

models. This idea is also used in PD validation.[24] Therefore, the basic idea of the following validation models is to use well known PD validation techniques for LGD validation purposes. Firstly, transformed ratios of the PD validation, e.g. Area under Curve (AUC) or Accuracy Ratio (AR), are computed for realized LGDs. Since the interpretation of these ratios is different as for the PD validation, one has to compare the ratios generated on realized LGDs with those ratios generated on estimated LGDs. As a result, the quality of the LGD estimation model does not depend on the ratio itself but on the comparison of the ratios of realized and estimated LGDs. The estimation model fits better, the more equal the compared ratios are.

In order to simplify the interpretation of the following model, an exemplary retail portfolio of a bank is analyzed. At first, the technique of proportional decomposition is developed on this exemplary retail portfolio. Subsequently, the model is applied to real data. Assume, the portfolio consists of 100 credits with individual EADs and losses. The number of defaulted credits amounts to 54 with a total loss of 600,000 euros. The total EAD adds up to three million euros, which results in an average LGD of 20 percent according Appendix E.

Each EAD is divided into n portions of equal size.[25] The number of portions should realign to the EAD amount.[26] For every portion $i = 1,...,n$ of credit $k = 1,...,K$, a binary variable $\mathrm{d}_{i,k}$ is defined as follows:

$$\mathrm{d}_{i,k} = \begin{cases} 1 & \text{if portion } i \text{ of the } \mathrm{EAD}_k \text{ is defaulted} \\ 0 & \text{if portion } i \text{ of the } \mathrm{EAD}_k \text{ is not defaulted.} \end{cases} \tag{4}$$

Furthermore, a second binary variable $\mathrm{nd}_{i,k}$ is defined by $\mathrm{nd}_{i,k} \equiv 1 - \mathrm{d}_{i,k}$. Credit 1 of our exemplary retail portfolio is not defaulted. Therefore, all d_i possess a value of zero and all nd_i possess a value of one. For credit 12, the variables d_1 to d_{462} possess a value of one and the variables d_{463} to d_{1000} possess a value of zero.[27] After computing the variables d_i and nd_i for all credits K, the variables are added up for each portion. Therefore, $\mathrm{D}_i \equiv \sum_{k=1}^{K} \mathrm{d}_{i,k}$ represents the number of credits where portion i is defaulted. Similarly, ND_i is defined as

[24] See Engelmann et al. (2003*a*) and Keenan & Sobehart (1999) for a detailed overview of PD validation techniques.

[25] If n = 100 is selected the decomposition corresponds to a percental decomposition, for n = 1,000 it corresponds to an one-tenth of a percent decomposition.

[26] If the portfolio consists of credits with EADs below 1,000 euros, a more precise decomposition than n = 1,000 makes little sense, since every portion then corresponds to an amount of less than one euro. For our exemplary retail portfolio, n = 1,000 was selected. Here, the resulting portion amounts to 51.60 euros in maximum.

[27] The number 462 was rounded. For a more precise decomposition, the deviation converges to zero. In our case, the deviation of the loss due to rounding amounts to 2.80 euros and lies thereby in a negligible range.

$\mathrm{ND}_i \equiv \sum_{k=1}^{K} \mathrm{nd}_{i,k}$ and corresponds to the number of credits where portion i is not defaulted. As a consequence, the sum of D_i and ND_i for each portion must be equal to the number of credits. The results for our exemplary retail portfolio are shown in extracts in Table 1.

Table 1: Proportional Decomposition of our Exemplary Retail Portfolio

portion i in ‰	D_i	ND_i
1	54	46
2	54	46
3	54	46
4	54	46
5	54	46
⋮	⋮	⋮
450	12	88
451	11	89
452	11	89
453	9	91
454	8	92
455	8	92
456	8	92
457	8	92
458	8	92
459	8	92
⋮	⋮	⋮
996	3	97
997	3	97
998	3	97
999	3	97
1000	3	97

The decomposition in Table 1 is based on the assumption that all credits exhibit an LGD smaller or equal to one. However, this is not always ensured. For credits with high workout costs and small EADs, LGDs above one are possible. In order to prevent this, the variables d_i and nd_i can be redefined as follows:

$$\mathrm{d}_{i,k} = \begin{cases} 1 & \text{if portion } i \text{ of the double } \mathrm{EAD}_k \text{ is defaulted} \\ 0 & \text{if portion } i \text{ of the double } \mathrm{EAD}_k \text{ is not defaulted.} \end{cases} \tag{5}$$

With this redefinition, credits with LGDs smaller or equal to 200 percent are possible.[28] The further computations take place similarly to the model with simple EADs.

[28] To reduce rounding errors, the number of portions should be increased when increasing the permitted LGD.

Analogously to measures of the accuracy of rating functions, hit rates and false alarm rates can be computed for each portion. The interpretation of these rates are, however, not equal to those of rating functions. The hit rate hr_i and the false alarm rate far_i are computed as follows:

$$\mathrm{hr}_i = \frac{\mathrm{D}_i}{\sum_{i=1}^{n} \mathrm{D}_i} \qquad \mathrm{far}_i = \frac{\mathrm{ND}_i}{\sum_{i=1}^{n} \mathrm{ND}_i}. \tag{6}$$

The hit rate hr_i is the fraction of portion i of all defaulted portions. The false alarm rate far_i is the fraction of portion i of all portions, which are not defaulted. If the hit rates and the false alarm rates are summed up the cumulated hit rate HR_j and the cumulated false alarm rate FAR_j for portion j, respectively, result:[29]

$$\begin{aligned} \mathrm{HR}_j &= \sum_{i=1}^{j} \mathrm{hr}_i & \mathrm{FAR}_j &= \sum_{i=1}^{j} \mathrm{far}_i \\ &= \frac{j}{n} \cdot \frac{\overline{\mathrm{LGD}_j}}{\overline{\mathrm{LGD}}} & &= \frac{j}{n} \cdot \frac{\overline{\mathrm{RR}_j}}{\overline{\mathrm{RR}}} \end{aligned}$$

where $\qquad\qquad$ where

$$\begin{aligned} \overline{\mathrm{LGD}_j} &= \frac{\sum_{i=1}^{j} \mathrm{D}_i}{j \cdot K} & \overline{\mathrm{RR}_j} &= \frac{\sum_{i=1}^{j} \mathrm{ND}_i}{j \cdot K} \\ \overline{\mathrm{LGD}} &= \frac{\sum_{i=1}^{n} \mathrm{D}_i}{n \cdot K} & \overline{\mathrm{RR}} &= \frac{\sum_{i=1}^{n} \mathrm{ND}_i}{n \cdot K}\ . \end{aligned} \tag{7}$$

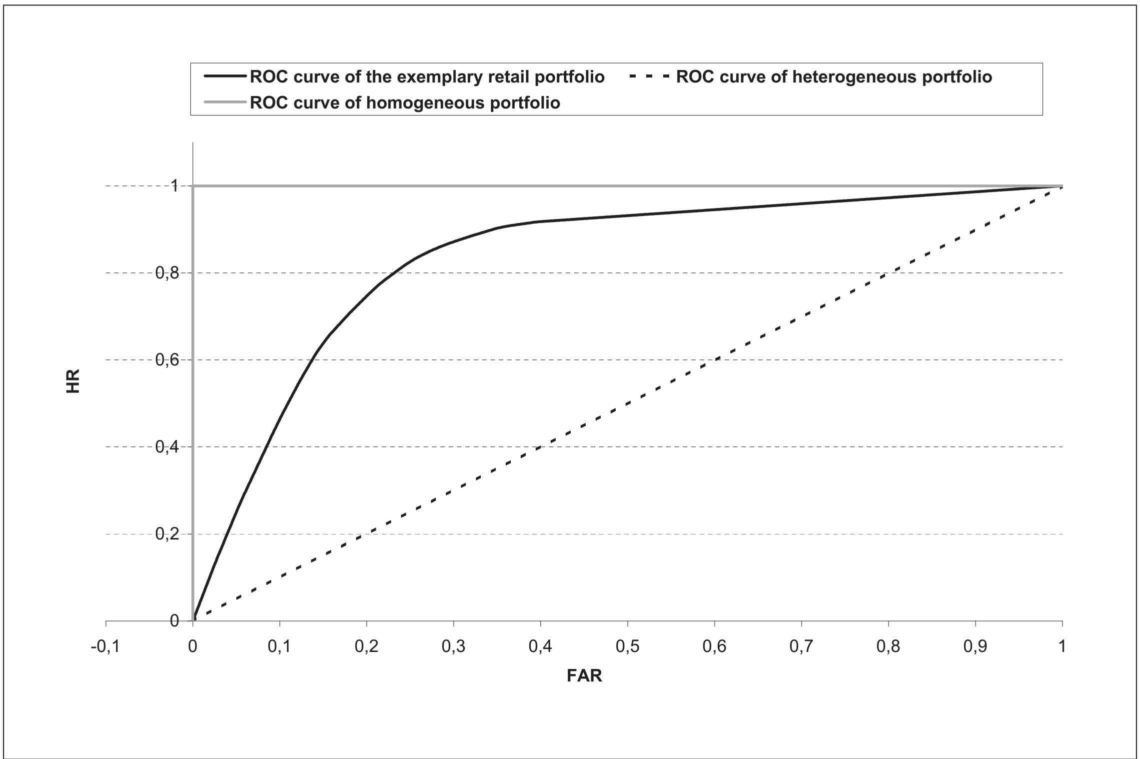
The cumulated hit rate, thus, corresponds to the ratio of the average LGD of the first j portions (denoted by $\overline{\mathrm{LGD}_j}$) to the average LGD over all portions (denoted by $\overline{\mathrm{LGD}}$), multiplied by a weighting factor. The cumulated false alarm rate corresponds to the ratio of the average RR of the first j portions (denoted by $\overline{\mathrm{RR}_j}$) to the average RR over all portions (denoted by $\overline{\mathrm{RR}}$), multiplied by the same weighting factor. The average Loss Given Defaults $\overline{\mathrm{LGD}_j}$ and average Recovery Rates $\overline{\mathrm{RR}_j}$ are unweighted. This implies the advantage that LGDs can be validated without any influence of the size of the EADs. Thus, it can be ruled out that banks arrange their models so that LGDs of credits with high EADs are estimated more precisely while estimations for credits with smaller EADs are imprecise. The $\overline{\mathrm{LGD}}$ for our exemplary retail portfolio equals 18.423 percent.

The Receiver Operating Characteristic (ROC) curve evolves by plotting the cumulated hit rates against the cumulated false alarm rates. This curve shows the homogeneity of the credit portfolio with respect to the LGDs. The more steeply it runs, the more homo-

[29]For the derivation see Appendix A.

geneous are the LGDs of single credits. If all credits have the same LGD, the portfolio is perfectly homogeneous concerning the LGDs. Thus, the ROC curve runs vertically along the y axis and then horizontally at the level of 1. If the credit portfolio consists of two disjunct quantities, one with credits, that have an LGD of 100 percent and the other with only non defaulted credits, the portfolio is perfectly heterogeneous and the ROC curve possesses a slope of one. Since the latter portion can only default if the first one also defaults, D_i cannot be larger than D_{i-1}. Therefore, the ROC curve is linear in sections but overall concave. For our example, a ROC curve arises according to Figure 2.

Figure 2: ROC Curve of the Exemplary Retail Portfolio



The Area Under Curve (AUC) provides similar information like the ROC curve. It corresponds to the probability that the rank of a defaulted portion $\left(\mathrm{Rank}^{\mathrm{d}}_{\mathrm{Por}}\right)$ is higher than the rank of a non-defaulted portion $\left(\mathrm{Rank}^{\mathrm{nd}}_{\mathrm{Por}}\right)$ plus half of the probability that the rank of a defaulted portion is the same as a non-defaulted:[30]

$$\begin{aligned} \mathrm{AUC} &= \sum_{i=1}^{n} \left((\mathrm{FAR}_i - \mathrm{FAR}_{i-1}) \cdot \frac{\mathrm{HR}_i + \mathrm{HR}_{i-1}}{2} \right) \\ &= \sum_{i=1}^{n} \left(\mathrm{far}_i \cdot \frac{\mathrm{HR}_i + \mathrm{HR}_{i-1}}{2} \right). \end{aligned} \tag{8}$$

[30]See Bamber (1975) for a derivation in the context of the accuracy of discriminative power.

In probabilistic terms, AUC reads as:

$$\begin{aligned}\text{AUC} &= \sum_{i=1}^{n}\left(\text{Prob}(\text{Rank}^{\text{nd}}_{\text{Por}} = i)\cdot\frac{\text{Prob}(\text{Rank}^{\text{d}}_{\text{Por}} \leq i) + \text{Prob}(\text{Rank}^{\text{d}}_{\text{Por}} \leq i-1)}{2}\right)\\ &= \text{Prob}(\text{Rank}^{\text{d}}_{\text{Por}} < \text{Rank}^{\text{nd}}_{\text{Por}}) + 0.5\cdot\text{Prob}(\text{Rank}^{\text{d}}_{\text{Por}} = \text{Rank}^{\text{nd}}_{\text{Por}}). \qquad (9)\end{aligned}$$

AUC equals one for a perfectly homogeneous portfolio. For a perfectly heterogeneous portfolio AUC becomes 0.5. The AUC of our exemplary retail portfolio shows a value of 0.8342.

Another well-known validation measure is the Cumulative Accuracy Profile (CAP) curve. The CAP curve measures, analogously but not equal to the Lorenz curve, the degree of inequality, i.e. how the hit rates are distributed over all portions. Therefore, the cumulated hit rates are plotted against the cumulated portions. The CAP curve, like the ROC curve, is linear in sections but overall concave. If the hit rates are equally distributed over all portions, the CAP curve possesses a slope of one and the portfolio is perfectly heterogeneous. If all credits have the same LGD, the portfolio is perfectly homogeneous and the CAP curve runs linearly rising up to the unweighted average LGD and then horizontally. For our exemplary retail portfolio, a CAP curve results in accordance with Figure 3.[31] Additionally, an inequality coefficient can be formed analogously to the Gini coefficient. In order to retain the notation of the accuracy measures for rating functions, the coefficient is called Accuracy Ratio (AR) and is computed as follows:[32]

$$\begin{aligned}\text{AR} &= \frac{\sum_{i=1}^{n}\left(\frac{1}{n}\cdot\frac{\text{HR}_i+\text{HR}_{i-1}}{2}\right) - 0.5}{1 - 0.5 - \frac{\overline{\text{LGD}}}{2}}\\ &= \frac{2\cdot\sum_{i=1}^{n}\text{HR}_i - 1 - n}{n\cdot(1-\overline{\text{LGD}})}. \qquad (10)\end{aligned}$$

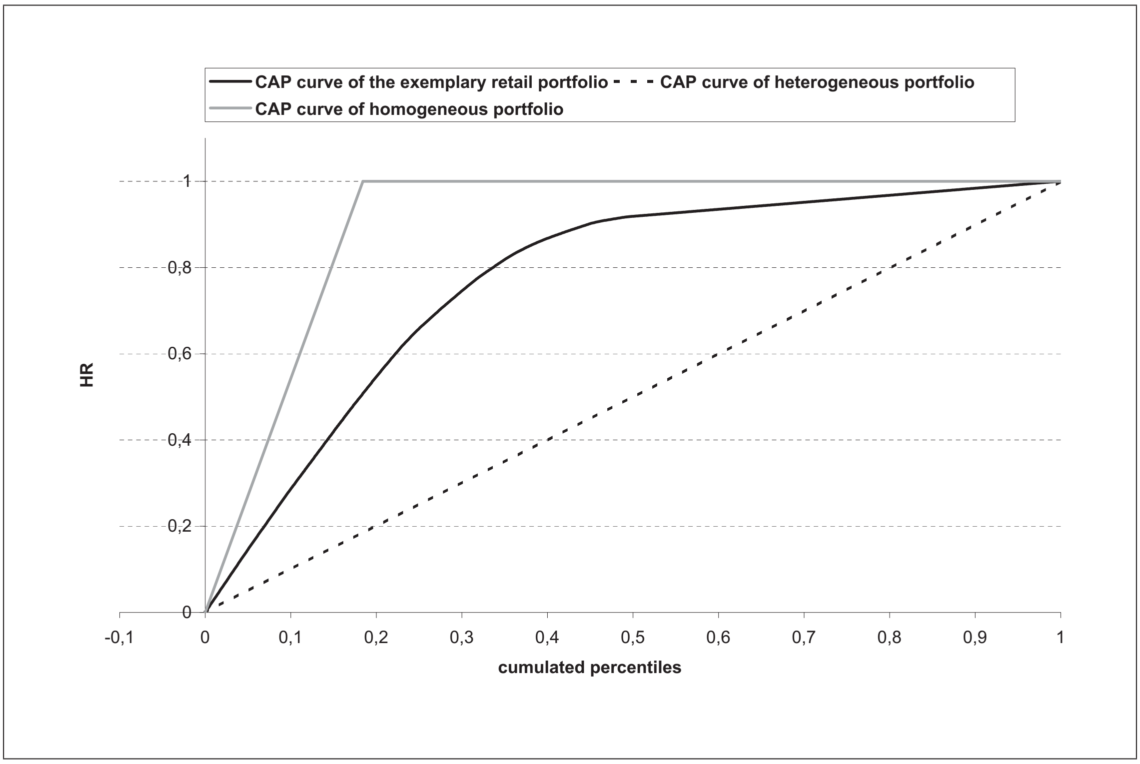
AR increases the more unequal the hit rates are distributed over all portions, i.e. the more homogeneous the portfolio is with respect to its LGDs. For a perfectly heterogeneous portfolio the AR becomes zero, for a perfectly homogeneous portfolio AR equals one. Furthermore, the relationship between AUC and AR is $\text{AR} = 2\cdot\text{AUC}-1$.[33] Our exemplary retail portfolio possesses an AR of 0.6683.

[31] The CAP curve of the perfectly homogeneous portfolio corresponds to a portfolio with an unweighted average LGD ($\overline{\text{LGD}}$) of 18.423 percent.

[32] See Appendix B for the derivation. For the functional relationship between Gini coefficient and AR for rating functions see Rauhmeier (2004), pp. 129-130.

[33] This relation is well-known for the accuracy measure of rating functions. For a proof with respect to our LGD framework see Appendix C.

Figure 3: CAP Curve of the Exemplary Retail Portfolio



The ROC curve, the CAP curve, AUC and AR of the realized LGDs provide a benchmark for the parameters and curves of the data set of the LGD estimation model, which have to be computed similarly. Afterwards, the results of both data sets can be compared with each other. Nevertheless, it has to be considered that the interpretations of ROC, CAP, AUC and AR in our LGD framework differ from the interpretation of these measures in the framework of rating accuracy. There are no perfect curves. Rather the parameters and curves of the LGD estimation model should deviate as few as possible from those of the historical data set. If the AUC and AR of the realized and estimated LGDs are almost equal, then the LGD estimation model can be seen as a good forecast tool for future LGDs.

ROC curves can intersect each other. In this case, area differences might be compensated. It can happen that the data sets of the historical, already realized, losses and of the LGD estimation model possess the same AUC and AR but the results of the LGD estimation model are not related with the historical loss data set. Therefore, a modified AUC, that we call MAUC, should be computed additionally, where MAUC equals the sum of the absolute values of the area differences for each portion i:

$$\begin{aligned}\text{MAUC} &= \sum_{i=1}^{n}\Bigg|\left(\left(\text{FAR}_i^{\text{hist}}-\text{FAR}_{i-1}^{\text{hist}}\right)\cdot\frac{\text{HR}_i^{\text{hist}}+\text{HR}_{i-1}^{\text{hist}}}{2}\right)\\ &\quad -\left(\left(\text{FAR}_i^{\text{est}}-\text{FAR}_{i-1}^{\text{est}}\right)\cdot\frac{\text{HR}_i^{\text{est}}+\text{HR}_{i-1}^{\text{est}}}{2}\right)\Bigg|\\ &\text{where}\\ \text{FAR}_i^{\text{hist}} &= \text{false alarm rate of portion } i \text{ of the historical loss distribution}\\ \text{FAR}_i^{\text{est}} &= \text{false alarm rate of portion } i \text{ of the estimation model}\\ \text{HR}_i^{\text{hist}} &= \text{hit rate of portion } i \text{ of the historical loss distribution}\\ \text{HR}_i^{\text{est}} &= \text{hit rate of portion } i \text{ of the estimation model.}\end{aligned} \tag{11}$$

A goal within validation should be to develop LGD estimations that minimize the value of MAUC.

In order to create a figure that measures AUC differences of the two ROC curves of the historical and estimation data set, we compute the following single AUC_i:

$$\begin{aligned}\text{AUC}_i &= (\text{FAR}_i-\text{FAR}_{i-1})\cdot\frac{\text{HR}_i+\text{HR}_{i-1}}{2}\\ &= \text{far}_i\cdot\frac{\text{HR}_i+\text{HR}_{i-1}}{2}.\end{aligned} \tag{12}$$

Afterwards, we suggest the following linear regression:

$$\begin{aligned}\text{AUC}_i^{\text{hist}} &= \alpha+\beta\cdot\text{AUC}_i^{\text{est}}\\ &\text{where}\\ \text{AUC}_i^{\text{hist}} &= \text{AUC}_i \text{ of the historical loss distribution}\\ \text{AUC}_i^{\text{est}} &= \text{AUC}_i \text{ of the estimation model.}\end{aligned} \tag{13}$$

If the LGD estimation model perfectly forecasts future LGDs, α should be zero and β should be one. If α is signicantly different from zero, there is a bias in the LGD estimation. A further linear regression with suppression of the location parameter shows whether the AUC of the historical data set is, on average, systematically underestimated ($\beta > 1$) or overestimated ($\beta < 1$). A measure of the quality of the estimation model is the coefficient of determination $R^2(45°)$. It is computed as follows:

$$R^2(45°) = 1-\frac{\sum_{i=1}^{n}\left(\text{AUC}_i^{\text{hist}}-\text{AUC}_i^{\text{est}}\right)^2}{\sum_{i=1}^{n}\left(\text{AUC}_i^{\text{hist}}-\overline{\text{AUC}^{\text{hist}}}\right)^2}. \tag{14}$$

As mentioned before, the proportional decomposition of credits has the charm that it validates LGD estimations without size considerations of the EADs. This is reasonable if the LGDs should be estimated as exactly as possible to price credits and calculate interest rates. Imprecise estimations of LGDs can lead to wrong credit interest rates of new contracts.

If the LGDs are used to compute losses in euros, additional validation instruments should be implemented to guarantee that losses of large credits are estimated precisely.[34] For this purposes, a marginal decomposition of the credit should be used. This is topic of the following subsection.

2.3 Marginal Decomposition of the Credit

In the framework of a marginal decomposition of credits, each credit is divided into single euros.[35] Here, the number of euros differs for each credit because each credit amount varies. For each single euro $i = 1,...,\mathrm{EAD}_k$ of credit $k = 1,...,K$, now, a binary variable $\mathrm{d}^e_{i,k}$ is defined as follows:[36]

$$\mathrm{d}^e_{i,k} = \begin{cases} 1 & \text{if single euro } i \text{ of the } \mathrm{EAD}_k \text{ is defaulted} \\ 0 & \text{if single euro } i \text{ of the } \mathrm{EAD}_k \text{ is not defaulted.} \end{cases} \tag{15}$$

Corresponding to the proportional decomposition, again a second binary variable $\mathrm{nd}^e_{i,k} \equiv 1 - \mathrm{d}^e_{i,k}$ is defined.[37] Subsequently, the sum of $\mathrm{d}^e_{i,k}$ and $\mathrm{nd}^e_{i,k}$ over all credits for each single euro, denoted by D^e_i and ND^e_i, is computed. In contrast to the proportional decomposition, the sum of D^e_i and ND^e_i is not equal for every single euro. For the marginal decomposition, the same exemplary retail portfolio as for the proportional decomposition is used in order to simplify the interpretation. The results of the marginal decomposition for our exemplary retail portfolio are shown in Table 2.

The hit rate hr^e_i and the false alarm rate far^e_i are again computed according to Equation (6). The hit rate hr^e_i equals the fraction of the single euro i over all defaulted euros. The false alarm rate far^e_i equals the fraction of the single euro i over all euros, which are

[34] Clearly, for a credit with an EAD of 100 euros, a stronger deviation of the relative LGD estimation is less problematic than for a credit with an EAD of 10,000 euros.

[35] It is also possible to define a different decomposition. A decomposition for example into single hundred euros is more reasonable for large credit amounts.

[36] The superscript e denotes equations and parameters for the marginal decomposition. For a decomposition in single hundred euros, i lies between one and $\arg\max_k(\mathrm{EAD}_k/100)$.

[37] For losses larger than the EAD, a modification of the variables $\mathrm{d}^e_{i,k}$ and $\mathrm{nd}^e_{i,k}$ like for the proportional decomposition is also possible.

Table 2: Marginal Decomposition of our Exemplary Retail Portfolio

single euro i	D_i^e	ND_i^e
1st	54	46
2nd	54	46
3rd	54	46
4th	54	46
5th	54	46
⋮	⋮	⋮
20,595th	6	58
20,596th	6	58
20,597th	6	58
20,598th	6	58
20,599th	6	58
20,600th	6	58
20,601st	6	57
20,602nd	6	57
20,603rd	6	57
20,604th	6	57
⋮	⋮	⋮
51,596th	0	1
51,597th	0	1
51,598th	0	1
51,599th	0	1
51,600th	0	1

not defaulted. The function of the hit rates is monotonously decreasing in i because the second single euro can only default if the first single euro also defaults. The function of the false alarm rates can increase or decrease in i because of the different amounts of EADs of every credit. The sum of hit rate and false alarm rate is again a decreasing function in i. The cumulated hit rate HR_j^e and the cumulated false alarm rate FAR_j^e are computed as follows:[38]

$$\mathrm{HR}_j^e = \sum_{i=1}^{j} \mathrm{hr}_i^e = \frac{\sum\limits_{i=1}^{j} \mathrm{EAD}_i}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i} \cdot \frac{\overline{\mathrm{LGD}_j^e}}{\overline{\mathrm{LGD}^e}}$$

where

$$\overline{\mathrm{LGD}_j^e} = \frac{\sum\limits_{i=1}^{j} \mathrm{D}_i^e}{\sum\limits_{i=1}^{j} \mathrm{EAD}_i}$$

$$\overline{\mathrm{LGD}^e} = \frac{\sum\limits_{i=1}^{\max} \mathrm{D}_i^e}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i}$$

$$\mathrm{FAR}_j^e = \sum_{i=1}^{j} \mathrm{far}_i^e = \frac{\sum\limits_{i=1}^{j} \mathrm{EAD}_i}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i} \cdot \frac{\overline{\mathrm{RR}_j^e}}{\overline{\mathrm{RR}^e}}$$

where

$$\overline{\mathrm{RR}_j^e} = \frac{\sum\limits_{i=1}^{j} \mathrm{ND}_i^e}{\sum\limits_{i=1}^{j} \mathrm{EAD}_i}$$

$$\overline{\mathrm{RR}^e} = \frac{\sum\limits_{i=1}^{\max} \mathrm{ND}_i^e}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i}. \tag{16}$$

The variable max is computed as $\arg\max_k(\mathrm{EAD}_k)$ and equals the largest EAD amount of all credits of the portfolio, when assuming a decomposition into single euros. Here, $\overline{\mathrm{LGD}_j^e}$, $\overline{\mathrm{LGD}^e}$, $\overline{\mathrm{RR}_j^e}$ and $\overline{\mathrm{RR}^e}$ are credit weighted averages. The credit weighted LGD of our exemplary retail portfolio is 0.2.

The ROC^e curve, the CAP^e curve, AUC^e and AR^e are computed similarly to the proportional decomposition. AUC^e can be interpreted as the probability that the rank of a defaulted single euro is higher than the rank of a non-defaulted single euro plus half of the probability that a defaulted single euro ranks on the same position like a non-defaulted single euro. The formula for AUC^e reads as follows:

$$\begin{aligned}\mathrm{AUC}^e &= \sum_{i=1}^{\max} \left(\mathrm{FAR}_i^e - \mathrm{FAR}_{i-1}^e \cdot \frac{\mathrm{HR}_i^e + \mathrm{HR}_{i-1}^e}{2} \right) \\ &= \sum_{i=1}^{\max} \left(\mathrm{far}_i^e \cdot \frac{\mathrm{HR}_i^e + \mathrm{HR}_{i-1}^e}{2} \right).\end{aligned} \tag{17}$$

In probabilistic terms, AUC^e reads as:

$$\begin{aligned}\mathrm{AUC}^e &= \sum_{i=1}^{\max} \left(\mathrm{Prob}(\mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{nd}} = i) \cdot \frac{\mathrm{Prob}(\mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{d}} \le i) + \mathrm{Prob}(\mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{d}} \le i-1)}{2} \right) \\ &= \mathrm{Prob}(\mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{d}} < \mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{nd}}) + 0.5 \cdot \mathrm{Prob}(\mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{d}} = \mathrm{Rank}_{\mathrm{Eur}}^{\mathrm{nd}}).\end{aligned} \tag{18}$$

[38]For the derivation see Appendix D.

If all credits show the same amount of loss, AUC^e assumes a value of one. If the ratio of defaulted to non-defaulted euros is equal for every single euro i, AUC^e assumes a value of a half. In contrast to the proportional decomposition, AUC^e and AR^e can assume values below 0.5.

The validation procedure is similar to the procedure for the proportional decomposition. After computing the validation ratios for realized and estimated losses, one has to compare them with each other. If the ratios are almost equal, again the estimation model can be seen as a good forecast instrument for future losses.

In order to avoid mistakes in the interpretation caused by intersections of ROC curves, again a modified AUC^e, denoted as MAUC^e, should be computed corresponding to Equation (11). For statistical validation of the results, $R^2(45°)^e$ can be computed analogously to Equation (14).

3 Empirical Analysis

3.1 Data

For our empirical analysis, real loss data of a retail portfolio of a commercial bank are used. The LGD estimation model is based on Workout LGDs. It consists of two parts. At first, a logistic regression for estimating the probability of a recovery or a write-off is carried out and subsequently, a linear regression for estimating LGDs for each case is run. Afterwards, the sum of both LGDs weighted by the rates of a recovery and a write-off is computed and is used as the LGD for the regarded credit.[39] The retail portfolio is split up into four subportfolios, distinct into private and commercial clients and collateralized and uncollateralized credits at default date.

The four subportfolios are analyzed with the proportional and marginal decomposition models of Section 2. Both in-sample and out-of-sample tests for the robustness of the estimation model are implemented. For the in-sample test the complete modeling data base is used for validation. Therefore, the AUCs of the realized LGDs, estimated LGDs, realized losses and estimated losses are computed. Afterwards, the MAUC and the $R^2(45°)$ are calculated.

For the out-of-sample test, a rejection level is computed for the MAUC at a 90 percent, 95 percent and 99 percent confidence level and for the $R^2(45°)$ at a 10 percent, 5 percent and 1 percent confidence level using the bootstrapping method. For bootstrapping a

[39] A similar model is presented by Appasamy, Dörr, Ebel & Stützle (2008). Also Peter (2006) and Hamerle, Knapp & Wildenauer (2006) divide the LGD estimation into two parts.

random subset of the modeling data base, reduced to each single subportfolio, is drawn 100 times. Afterwards, a validation data base is used, which was not in the modeling set.[40] The validation data time period follows the modeling data time period. Therefore, the out-of-sample test is also an out-of-universe test. The computed MAUC and $R^2(45°)$ of the validation data base are compared with the confidence levels of the modeling data base. If MAUC and/or $R^2(45°)$ of the validation data base exceed the rejection levels, the analyzed estimation model is not robust with respect to time or sample changes.

3.2 Results

Firstly, relative LGDs are validated using the proportional decomposition. Decompostions into 100, 500 and 1,000 portions are chosen. Because some contracts of the subportfolios possess LGDs larger than one, a modification according to Equation (5) is used. For each subportfolio an EAD multiple is chosen, so that at least 95 percent of the data base do not have to be modified.[41] After computing the validation ratios for the realized and estimated LGDs for all portfolios, MAUC and $R^2(45°)$ are calculated for further analysis and are shown in Table 3.

Table 3: Results for the Proportional Decomposition of the Modeling Data Base

Subportfolio	Portions	MAUC	$R^2(45°)$
private clients, collateralized	100	0.2751	0.1658
private clients, uncollateralized	100	0.2900	0.5721
commercial clients, collateralized	100	0.2314	0.4879
commercial clients, uncollateralized	100	0.2325	0.7785
private clients, collateralized	500	0.2749	0.1666
private clients, uncollateralized	500	0.2898	0.5723
commercial clients, collateralized	500	0.2313	0.4883
commercial clients, uncollateralized	500	0.2331	0.7785
private clients, collateralized	1,000	0.2749	0.1668
private clients, uncollateralized	1,000	0.2898	0.5723
commercial clients, collateralized	1,000	0.2313	0.4883
commercial clients, uncollateralized	1,000	0.2331	0.7782

The LGD estimation model, which is analyzed in our paper, is mainly designed to meet the Basel II capital requirements, therefore, we are interested in precise estimations of absolute losses. However, for risk-adjusted credit pricing, a different approach will be used. So, the validation for the relative LGDs is only of secondary importance in the

[40] The size of the random subset for bootstrapping equals the portion of the validation data base on the modeling data base, computed for each subportfolio.

[41] The 95 percent level was chosen to avoid data shortening due to outliers.

latter case. Nevertheless, three of the subportfolios possess an $R^2(45°)$ larger than 0.48. Only the LGD estimation of the subportfolio "private clients, collateralized" exhibits an inadequate accuracy.

The LGD estimation model is robust with respect to changes of the decomposition. Therefore, for the rejection levels of MAUC and $R^2(45°)$ a proportional decompositon of 100 for bootstrapping is used. The rejection levels for the out-of-universe test are presented in Table 4. For MAUC, the 90 percent, 95 percent and 99 percent confidence level are computed. However, a lower MAUC of the validation data base in contrast to the modeling data base do not harm, since then, the estimation model works even better for the validation data base than for the modeling data base. For the $R^2(45°)$, the 10 percent, 5 percent and 1 percent confidence levels are computed. In case of a higher $R^2(45°)$ of the validation data base in contrast to the modeling data base, again the estimation model works even better for the validation data base.

Table 4: Confidence Levels of MAUC and $R^2(45°)$

MAUC			
Subportfolio	90% c.l.	95% c.l.	99% c.l.
private clients, collateralized	0.2777	0.2781	0.2793
private clients, uncollateralized	0.3005	0.3038	0.3099
commercial clients, collateralized	0.2354	0.2365	0.2403
commercial clients, uncollateralized	0.2795	0.2905	0.3112
$R^2(45°)$			
Subportfolio	10% c.l.	5% c.l.	1% c.l.
private clients, collateralized	0.1475	0.1417	0.1358
private clients, uncollateralized	0.5357	0.5255	0.4986
commercial clients, collateralized	0.4692	0.4619	0.4403
commercial clients, uncollateralized	0.6990	0.6751	0.6317

Note: The abbreviation c.l. means confidence level.

For the out-of-universe test, MAUC and $R^2(45°)$ for the validation data base are computed using a proportional decomposition with again 100 portions. Afterwards, these ratios are compared with the corresponding rejection levels. Table 5 presents the results for the validation data base. For every subportfolio, the $R^2(45°)$ figures of the validation data base are smaller than the confidence level. The results are similar for MAUC. Only for the subportfolio "commercial clients, collateralized", the MAUC figure would lead to no rejection. Here, it can be seen that data quality is very important. Since the validation data time period follows the modeling data time period, an increase in data computation and data collection quality can be assumed. Also changes in the portfolio structure or in debtor-specific characteristics may lead to this result.

Table 5: Results for the Proportional Decomposition of the Validation Data Base

Subportfolio	MAUC	$R^2(45°)$	Rejection
private clients, collateralized	0.3160	-0.2955	yes
private clients, uncollateralized	0.5152	-1.4947	yes
commercial clients, collateralized	0.2299	0.3581	no/yes
commercial clients, uncollateralized	0.7479	-3.9974	yes

The next step is the validation of losses in euros using the marginal decomposition. Therefore, decompositions into single 100 euros, 50 euros and 10 euros are chosen. The EAD multiples remain the same as for the proportional decompositions. Again, AUCs of realized and estimated losses are calculated at the beginning. Afterwards, the comparison of both, realized and estimated losses, is done by using MAUC and $R^2(45°)$. The results of the in-sample test for the marginal decompositions are presented in Table 6.

Table 6: Results for the Marginal Decomposition of the Modeling Data Base

Subportfolio	Marginal	MAUC	$R^2(45°)$
private clients, collateralized	100	0.2549	0.6677
private clients, uncollateralized	100	0.1029	0.9322
commercial clients, collateralized	100	0.2164	0.7788
commercial clients, uncollateralized	100	0.0955	0.9178
private clients, collateralized	50	0.2546	0.6689
private clients, uncollateralized	50	0.1032	0.9320
commercial clients, collateralized	50	0.2161	0.7797
commercial clients, uncollateralized	50	0.0950	0.9197
private clients, collateralized	10	0.2546	0.6691
private clients, uncollateralized	10	0.1030	0.9322
commercial clients, collateralized	10	0.2161	0.7797
commercial clients, uncollateralized	10	0.0948	0.9200

The estimations of losses in euros are more precise for every subportfolio than the estimations of the relative losses. Every subportfolio possesses an $R^2(45°)$ above 0.66, two subportfolios even show an $R^2(45°)$ above 90 percent. Therefore, the estimation model is an appropriate model for estimating absolute losses, which are needed to determine the capital requirements according to Basel II. Thus, the incentive of the bank, namely the developement of an estimation model for absolute losses, is achieved.

The results are robust with respect to changes in the marginal size. Therefore, for computing the rejection levels via bootstrapping, a marginal decomposition into single 100 euros is chosen. The rejection levels for the out-of-universe test are presented in Table 7.

To compare the modeling data base with the validation data base, firstly, a marginal decomposition into single 100 euros of the validation data base is done. Afterwards, MAUC

Table 7: Confidence Levels of MAUC and $R^2(45°)$

MAUC			
Subportfolio	90% c.l.	95% c.l.	99% c.l.
private clients, collateralized	0.2588	0.2607	0.2630
private clients, uncollateralized	0.1175	0.1228	0.1262
commercial clients, collateralized	0.2240	0.2267	0.2326
commercial clients, uncollateralized	0.1483	0.1578	0.1763
$R^2(45°)$			
Subportfolio	10% c.l.	5% c.l.	1% c.l.
private clients, collateralized	0.6450	0.6399	0.6332
private clients, uncollateralized	0.9059	0.9017	0.8938
commercial clients, collateralized	0.7415	0.7373	0.7209
commercial clients, uncollateralized	0.7993	0.7704	0.6900

Note: The abbreviation c.l. means confidence level.

and $R^2(45°)$ of the validation data base are compared with the corresponding rejection levels. If the validation ratios exceed the rejection levels, the estimation model is not robust with respect to changes in time or in the sample. Table 8 shows the corresponding results.

Table 8: Results for the Marginal Decomposition of the Validation Data Base

Subportfolio	MAUC	$R^2(45°)$	Rejection
private clients, collateralized	0.3048	0.5460	yes
private clients, uncollateralized	0.1110	0.9578	no
commercial clients, collateralized	0.2614	0.5281	yes
commercial clients, uncollateralized	0.2682	0.7833	yes/no

The reasons of rejections are the same as for the proportional decomposition. Again, an increase in data quality and possible changes in the portfolio structure may lead to differences. However, the magnitude of misspecification of absolute losses is smaller than the misspecification of relative LGDs. Even if two subportfolios are rejected by both measures, the $R^2(45°)$ figures of all subportfolios are above 50 percent and for two subportfolios even above 75 percent. Thus, the estimation model for absolute losses is still applicably for forecasting future losses. Therefore, the estimation model can be implemented to determine Basel II capital requirements.

4 Conclusion

The idea of this paper was to develop an LGD validation method for retail portfolios. This topic is important because banks have to estimate LGDs on their own if they want to apply for the IRB approach for retail portfolios. The Basel II regulations postulate that retail portfolios have to be homogeneous concerning at least risk drivers like borrower and transaction risk characteristics and delinquency of exposure.[42] This makes it difficult to develop LGD rating or scoring models for retail portfolios because of their similar characteristics.

In contrast, our suggested proportional and marginal decomposition methods are applicably without LGD ratings. Furthermore, there are no specific requirements on the LGD estimation model. Even for arithmetic mean estimations of the LGD for retail portfolios both methods can be used. Our methods use validation instruments, that are well-known from PD validation, where the interpretation is different. The used instruments, e.g. AUC and AR, are accepted by supervisory authorities. Because of the different interpretation, the ratio itself contains no information about the accuracy of the estimation model. In fact, they describe the composition and structure of the portfolio. To validate the estimation model, one has to compare the ratios calcultated on realized LGD or losses with those calculated on estimated LGDs or losses. Therefore, AUC and AR of the historical data base provide a benchmark for the ratios of the estimated values. If the measures are nearly similar, the estimation model can be seen as a good forecast tool for future losses.

It also turns out, that the proportional decomposition is credit size-independent. Thus, the method is also an instrument to proof the functionality of the estimation model over all single credits of the portfolio. The proportional decomposition can reveal possible weaknesses of the estimation model. Therefore, one can avoid that banks arrange their models so that LGDs of credits with high EADs are estimated more precisely while estimations for credits with smaller EADs are imprecise. This fact is important if the estimation model shall be used for credit pricing, where a precise estimation of the LGD is important for the calculation of the contract's interest rate.

Our paper also showed that the models work on real data and that out-of-sample and out-of-time tests can easily be implemented.

[42] See Basel Committee on Banking Supervision (2004), paragraph 402.

References

Appasamy, B., Dörr, U., Ebel, H. & Stützle, E. (2008), 'LGD-Schätzung im Retailgeschäft am Beispiel Automobilfinanzierung', *Zeitschrift für das gesamte Kreditwesen* **61**(5), 206–209.

Bakshi, G., Madan, D. & Zhang, F. (2006), 'Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates'. Working Paper, Federal Deposit Insurance Corporation, Center for Financial Research.

Bamber, D. (1975), 'The Area above the Ordinal Dominance Graph and the Area below the Receiver Operating Characteristic Graph', *Journal of Mathematical Psychology* **12**(4), 387–415.

Basel Committee on Banking Supervision (2004), 'International Convergence of Capital Measurements and Capital Standards - A Revised Framework'. Bank for International Settlements.

Basel Committee on Banking Supervision (2005), 'Studies on the Validation of Internal Rating Systems'. Working Paper No. 14, Bank for International Settlements.

Brady, B., Chang, P., Miu, P., Ozdemir, B. & Schwartz, D. (2006), 'Discount Rate for Workout Recoveries: An Empirical Study'. Working Paper.

Böttger, M., Guthoff, A. & Heidorn, T. (2008), 'Loss Given Default Modelle zur Schätzung von Recovery Rates'. Working Paper, Frankfurt School of Finance & Management.

Das, S. R. & Hanouna, P. (2007), 'Implied Recovery'. Working Paper.

Engelmann, B., Hayden, E. & Tasche, D. (2003*a*), 'Measuring the Disciminative Power of Rating Systems'. Working Paper, Deutsche Bundesbank.

Engelmann, B., Hayden, E. & Tasche, D. (2003*b*), 'Testing Rating Accuracy', *Risk* **16**(1), 82–86.

Gupton, G. M. (2005), 'Advancing Loss Given Default Prediction Models: How the Quiet Have Quickened', *Economic Notes* **34**(2), 185–230.

Hamerle, A., Knapp, M. & Wildenauer, N. (2006), Modelling Loss Given Default:A "'Point in Time"'-Approach, *in* B. Engelmann & R. Rauhmeier, eds, 'The Basel II Risk Parameters', Berlin, pp. 127–142.

Jarrow, R. A., Lando, D. & Turnbull, S. (1997), 'A Markov Model for the Term Structure of Credit Risk Spreads', *Review of Financial Studies* **10**(2), 481–523.

Kaltofen, D. (2006), *Kreditrisiko- und Eigenkapitalmanagement im Retailportfolio*, Frankfurt a. M.: Bankakademie-Verlag.

Keenan, S. C. & Sobehart, J. R. (1999), 'Performance Measures for Credit Risk Models'. Research Report, Moody's Risk Management Services.

Madan, D. & Unal, H. (2000), 'A Two-Factor Hazard Rate Model for Risky Debt and the Term Structure of Credit Spreads', *Journal of Financial and Quantitative Analysis* **35**(1), 43–65.

Oesterreichische Nationalbank/Finanzmarktaufsicht (2004), 'Ratingmodelle und -validierung'. Leitfadenreihe zum Kreditrisiko.

Ong, M. K. (1999), *Internal Credit Risk Models, Capital Allocation and Performance Measurement*, London: Risk Books.

Pan, J. & Singleton, K. J. (2008), 'Default and Recovery Implicit in the Term Structure of Sovereign CDS Spreads', *Journal of Finance* **63**(5), 2345–2384.

Peter, C. (2006), Estimating loss given default - experiences from banking practise, *in* B. Engelmann & R. Rauhmeier, eds, 'The Basel II Risk Parameters', Berlin, pp. 143–175.

Rauhmeier, R. (2004), *Validierung und Performancemessung bankinterner Ratingsysteme*, Bad Soden: Uhlenbruch Verlag.

Schuermann, T. (2005), What Do We Know About Loss Given Default?, *in* E. Altman, A. Resti & A. Sironi, eds, 'Recovery Risk, The Next Challenge in Credit Risk Management', London: Risk Books, pp. 3–24.

Siddiqi, N. A. & Zhang, M. (2004), 'A General Methodology for Modeling Loss Given Default', *The Risk Management Association Journal* **86**(8), 92–95.

Appendix

A Derivation of HR and FAR

$$
\begin{aligned}
\mathrm{hr}_i &= \frac{\mathrm{D}_i}{\sum_{i=1}^{n} \mathrm{D}_i} = \frac{K - \mathrm{ND}_i}{\sum_{i=1}^{n} \mathrm{D}_i} \\
&= \frac{K}{\sum_{i=1}^{n} \mathrm{D}_i} - \frac{\mathrm{ND}_i}{\sum_{i=1}^{n} \mathrm{D}_i} \\
&= \frac{1}{\overline{\mathrm{LGD}} \cdot n} - \frac{\mathrm{ND}_i}{\overline{\mathrm{LGD}} \cdot n \cdot K} \\
\mathrm{HR}_j &= \sum_{i=1}^{j} \mathrm{hr}_i \\
&= \frac{j}{\overline{\mathrm{LGD}} \cdot n} - \frac{\sum_{i=1}^{j} \mathrm{ND}_i}{\overline{\mathrm{LGD}} \cdot n \cdot K} \\
&= \frac{j}{\overline{\mathrm{LGD}} \cdot n} - \frac{(1 - \overline{\mathrm{LGD}_j}) \cdot j}{\overline{\mathrm{LGD}} \cdot n} \\
&= \frac{j - (1 - \overline{\mathrm{LGD}_j}) \cdot j}{\overline{\mathrm{LGD}} \cdot n} \\
&= \frac{j}{n} \cdot \frac{\overline{\mathrm{LGD}_j}}{\overline{\mathrm{LGD}}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{far}_i &= \frac{\mathrm{ND}_i}{\sum_{i=1}^{n} \mathrm{ND}_i} = \frac{K - \mathrm{D}_i}{\sum_{i=1}^{n} \mathrm{ND}_i} \\
&= \frac{K}{\sum_{i=1}^{n} \mathrm{ND}_i} - \frac{\mathrm{D}_i}{\sum_{i=1}^{n} \mathrm{ND}_i} \\
&= \frac{1}{\overline{\mathrm{RR}} \cdot n} - \frac{\mathrm{D}_i}{\overline{\mathrm{RR}} \cdot n \cdot K} \\
\mathrm{FAR}_j &= \sum_{i=1}^{j} \mathrm{far}_i \\
&= \frac{j}{\overline{\mathrm{RR}} \cdot n} - \frac{\sum_{i=1}^{j} \mathrm{D}_i}{\overline{\mathrm{RR}} \cdot n \cdot K} \\
&= \frac{j}{\overline{\mathrm{RR}} \cdot n} - \frac{(\overline{\mathrm{LGD}_j}) \cdot j}{\overline{\mathrm{RR}} \cdot n} \\
&= \frac{j - (1 - \overline{\mathrm{RR}_j}) \cdot j}{\overline{\mathrm{RR}} \cdot n} \\
&= \frac{j}{n} \cdot \frac{\overline{\mathrm{RR}_j}}{\overline{\mathrm{RR}}}
\end{aligned}
$$

B Derivation of AR

$$
\begin{aligned}
\text{AR} &= \frac{\sum\limits_{i=1}^{n}\left(\frac{1}{n}\cdot\frac{\text{HR}_i+\text{HR}_{i-1}}{2}\right)-0.5}{1-0.5-\frac{\overline{\text{LGD}}}{2}} \\
&= \frac{\sum\limits_{i=1}^{n}\left(\frac{1}{n}\cdot\frac{\frac{i\cdot\overline{\text{LGD}_i}}{n\cdot\overline{\text{LGD}}}+\frac{(i-1)\cdot\overline{\text{LGD}_{i-1}}}{n\cdot\overline{\text{LGD}}}}{2}\right)-0.5}{0.5-\frac{\overline{\text{LGD}}}{2}} \\
&= \frac{\frac{\sum\limits_{i=1}^{n}\left(i\cdot\overline{\text{LGD}_i}\right)-0.5\cdot n\cdot\overline{\text{LGD}}}{n^2\cdot\overline{\text{LGD}}}-0.5}{0.5-0.5\cdot\overline{\text{LGD}}} \\
&= \frac{2\cdot\sum\limits_{i=1}^{n}\left(i\cdot\overline{\text{LGD}_i}\right)-n\cdot\overline{\text{LGD}}-n^2\cdot\overline{\text{LGD}}}{n^2\cdot\overline{\text{LGD}}\cdot\left(1-\overline{\text{LGD}}\right)} \\
&= \frac{2\cdot\sum\limits_{i=1}^{n}\text{HR}_i-1-n}{n\cdot\left(1-\overline{\text{LGD}}\right)}
\end{aligned}
$$

C Proof for the Linear Relationship between AR and AUC.

$$
\begin{aligned}
\text{AUC} = \quad &= \sum_{i=1}^{n} \left(\text{FAR}_i - \text{FAR}_{i-1} \cdot \frac{\text{HR}_i + \text{HR}_{i-1}}{2} \right) \\
&= \sum_{i=1}^{n} \left(\frac{i \cdot \overline{\text{RR}_i}}{n \cdot \overline{\text{RR}}} - \frac{(i-1) \cdot \overline{\text{RR}_{i-1}}}{n \cdot \overline{\text{RR}}} \right) \cdot \frac{\frac{i \cdot \overline{\text{LGD}_i}}{n \cdot \overline{\text{LGD}}} + \frac{(i-1) \cdot \overline{\text{LGD}_{i-1}}}{n \cdot \overline{\text{LGD}}}}{2} \\
&= \frac{\sum_{i=1}^{n} \left(i \cdot \overline{\text{LGD}_i} \right) - 0.5 \cdot n \cdot \overline{\text{LGD}} - 0.5 \cdot n^2 \cdot \overline{\text{LGD}}^2}{n^2 \cdot \overline{\text{RR}} \cdot \overline{\text{LGD}}} \\
&= \frac{\sum_{i=1}^{n} \text{HR}_i - 0.5 - 0.5 \cdot n \cdot \overline{\text{LGD}}}{n \cdot \left(1 - \overline{\text{LGD}} \right)} \\
2 \cdot \text{AUC} - 1 \quad &= \frac{2 \cdot \sum_{i=1}^{n} \text{HR}_i - 1 - n \cdot \overline{\text{LGD}}}{n \cdot \left(1 - \overline{\text{LGD}} \right)} - 1 \\
&= \frac{2 \cdot \sum_{i=1}^{n} \text{HR}_i - 1 - n \cdot \overline{\text{LGD}} - n \cdot \left(1 - \overline{\text{LGD}} \right)}{n \cdot \left(1 - \overline{\text{LGD}} \right)} \\
&= \frac{2 \cdot \sum_{i=1}^{n} \text{HR}_i - 1 - n}{n \cdot \left(1 - \overline{\text{LGD}} \right)} \\
&= \text{AR}
\end{aligned}
$$

D Derivation of HR^e and FAR^e

$$\mathrm{hr}_i^e = \frac{\mathrm{D}_i^e}{\sum\limits_{i=1}^{\max} \mathrm{D}_i^e} \qquad \mathrm{far}_i^e = \frac{\mathrm{ND}_i^e}{\sum\limits_{i=1}^{\max} \mathrm{ND}_i^e}$$

$$\begin{aligned}
\mathrm{HR}_j^e &= \sum_{i=1}^{j} \mathrm{hr}_i^e \\
&= \sum_{l=1}^{j} \frac{\mathrm{D}_l^e}{\sum\limits_{i=1}^{\max} \mathrm{D}_i^e} \\
&= \frac{\sum\limits_{l=1}^{j} \mathrm{D}_l^e}{\sum\limits_{i=1}^{\max} \mathrm{D}_i^e} \cdot \frac{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i}{\sum\limits_{l=1}^{j} \mathrm{EAD}_l} \cdot \frac{\sum\limits_{l=1}^{j} \mathrm{EAD}_l}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i} \\
&= \frac{\sum\limits_{l=1}^{j} \mathrm{EAD}_l}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i} \cdot \frac{\overline{\mathrm{LGD}_j^e}}{\overline{\mathrm{LGD}^e}}
\end{aligned}$$

$$\begin{aligned}
\mathrm{FAR}_j^e &= \sum_{i=1}^{j} \mathrm{far}_i^e \\
&= \sum_{l=1}^{j} \frac{\mathrm{ND}_l^e}{\sum\limits_{i=1}^{\max} \mathrm{ND}_i^e} \\
&= \frac{\sum\limits_{l=1}^{j} \mathrm{ND}_l^e}{\sum\limits_{i=1}^{\max} \mathrm{ND}_i^e} \cdot \frac{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i}{\sum\limits_{l=1}^{j} \mathrm{EAD}_l} \cdot \frac{\sum\limits_{l=1}^{j} \mathrm{EAD}_l}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i} \\
&= \frac{\sum\limits_{l=1}^{j} \mathrm{EAD}_l}{\sum\limits_{i=1}^{\max} \mathrm{EAD}_i} \cdot \frac{\overline{\mathrm{RR}_j^e}}{\overline{\mathrm{RR}^e}}
\end{aligned}$$

E Data of the Exemplary Retail Portfolio

Credit	EAD in €	Loss in €	Credit	EAD in €	Loss in €
1	7,100	0	51	29,900	0
2	7,400	0	52	30,900	0
3	7,500	0	53	31,300	0
4	8,300	0	54	32,200	6,500
5	8,300	0	55	32,700	0
6	8,500	0	56	33,500	11,700
7	9,000	0	57	34,000	0
8	10,100	0	58	34,800	0
9	10,500	0	59	35,400	11,500
10	10,500	0	60	36,100	10,900
11	10,500	0	61	36,700	7,300
12	10,600	4,900	62	37,400	0
13	11,000	2,600	63	38,000	13,700
14	11,400	0	64	38,600	0
15	11,900	4,400	65	39,300	0
16	12,200	2,800	66	39,800	18,600
17	12,400	0	67	40,500	1,700
18	12,500	4,000	68	41,000	0
19	12,900	0	69	41,700	16,500
20	13,400	13,400	70	42,200	10,400
21	13,800	0	71	42,800	10,800
22	13,800	13,800	72	43,200	4,000
23	14,500	0	73	43,900	0
24	14,700	3,600	74	44,300	2,200
25	14,900	0	75	44,900	10,300
26	15,000	0	76	45,300	0
27	15,600	4,300	77	45,900	7,900
28	16,000	0	78	46,200	17,600
29	16,800	0	79	46,800	0
30	16,900	8,300	80	47,100	0
31	17,000	7,700	81	47,600	0
32	18,000	5,700	82	47,900	23,400
33	18,200	0	83	48,400	11,100
34	19,200	0	84	48,600	13,500
35	19,400	0	85	49,100	17,600
36	20,500	3,500	86	49,200	0
37	20,600	0	87	49,700	7,600
38	21,800	4,700	88	49,800	22,500
39	21,800	0	89	50,200	19,100
40	23,000	2,100	90	50,300	11,500
41	23,100	0	91	50,600	16,000
42	24,300	10,300	92	50,700	25,100
43	24,400	24,400	93	51,000	5,100
44	25,600	9,100	94	51,100	20,000
45	25,700	6,400	95	51,200	18,700
46	26,900	10,900	96	51,300	0
47	27,200	0	97	51,400	0
48	28,200	8,800	98	51,400	17,900
49	28,500	7,100	99	51,500	23,200
50	29,600	0	100	51,600	23,300
Sum				3,000,000	600,000