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Exact and heuristic linear-inflation policies for an inventory model with random yield and arbitrary lead times

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We investigate a periodic inventory system for a single item with stochastic demand and random yield. Since the optimal policy for such a system is complicated we study the class of stationary linear-inflation policies where orders are only placed if the inventory position is below a critical stock level, and where the order quantity is controlled by a yield inflation factor. We consider two different models for the uncertain supply: binomial and stochastically proportional yield and we allow positive and constant lead times as well as asymmetric demand and yield distributions. In this paper we propose two novel approaches to derive optimal and near-optimal numerical values for the critical stock level, minimizing the average holding and backorder cost for a given inflation factor. First, we present a Markov chain approach, which is exact in case of negligible lead time. Second, we provide a steady state analysis to derive approximate closed-form expressions for the optimal critical stock level. We conduct an extensive numerical study to test the performance of our approaches. The numerical experiments reveal an excellent performance of both approaches. Since our derived formulas are easily implementable and highly accurate they are very valuable for practical application.

Subject classifications**:** Inventory/Production: Approximations/heuristics, Random Yield, Lead times; Probability: Stochastic model applications

Area of review: Operations and Supply Chains

1 Introduction

It is well-known that in many industries production and inventory systems simultaneously have to cope with uncertainties from two sides, namely from the demand as well as from process side. A major risk concerning production processes stems from yield uncertainties like they frequently occur in the agricultural sector or in chemical, electronics and mechanical manufacturing industries (see Gurnani et al. 2000; Jones et al. 2001; Kazaz 2004). In these environments different reasons such as weather conditions, unreliable processes or imperfect input material can be responsible for the occurrence of random yields. A particularly striking example is given by semiconductor manufacturing in the electronic goods industry where high yield losses with an average of about 80% are met (see Nahmias 2009, p. 392). The most recent field where yield problems have gained specific attention is the remanufacturing industry. Here, the output of disassembly operations often is highly uncertain because of limited knowledge of the condition of used products (see Ilgin and Gupta 2010, Panagiotidou et al. 2012).

In coping with the described yield problems, production and inventory control faces three main challenges. First, it has to be recognized that yield losses often are hard to predict so that their variability is too high to be ignored and randomness of production output must explicitly be taken into account. Second, it is important to be aware that different causes of yield losses exist so that different types of yield randomness have to be dealt with. Finally, many production processes in this context, e.g. in semiconductor manufacturing, are characterized by significant time consumption so that lead times of different length must be considered when control systems are developed and implemented. In this paper, we study different efficient approaches for controlling periodic-review production and inventory systems with random demand and yield, which are easy to implement and address all major challenges mentioned above. We refer to an infinite-horizon problem for a single item under the criterion of average cost minimization. Concerning yield randomness, we consider two different types that are most widely addressed in literature (see Yano and Lee 1995), namely the models of binomial and stochastically proportional yield. Binomial yield applies to cases where the generation of non-defective units within a production batch forms a Bernoulli process while the stochastically proportional yield concept is used if the stochastic process conditions affect the batch as a whole so that the probability distribution of the yield rate does not depend on the batch size.

From prior research it is known (see Gerchak et al. 1988, Henig and Gerchak 1990) that the optimal policy in case of zero lead time and stochastically proportional yield is a rule with a critical stock such that a production order in a period is only released if the inventory does not exceed this stock level. Different from the standard base-stock rule under deterministic yield conditions, however, the order quantity is a non-linear function of the inventory position. Since this rule is computationally cumbersome to evaluate and unattractive for practical application, several approaches have been proposed to develop simple, but still near-optimal rules. Some of these approaches, like the so-called newsvendor and NLH-2 heuristic by Bollapragada and Morton (1999) and the heuristic developed by Li et al. (2008), try to approximate the optimal order quantity rule by a non-linear function that is quite simple to evaluate. More convenient for practical application, however, are those policies where the order release quantity is a linear function of the deviation between critical stock and inventory position. This type of policy for which the term linear-inflation rule has been introduced by Zipkin (2000, p. 393) has the major benefit that it possesses just the structure of the rule that is usually applied in practice when demand and yield variability in MRP-type of production control is handled by a safety stock and a yield inflation factor that accounts for yield losses (see Inderfurth 2009; Nahmias 2009, p. 392; Vollmann et al. 2005, p. 485).

A linear-inflation policy is given by two parameters, which have to be determined before implementation, namely the critical stock S and the yield inflation factor F. Like other research contributions in this field we will focus on this policy type although we are aware that this rule is non-optimal, not only due to the linearization of the order release function but also because it is myopic and uses static policy parameters even in the non-zero lead time case (see Bollapragada and Morton 1999 and Inderfurth and Vogelgesang 2013, respectively). From numerical studies, however, we know that the latter aspects are of minor relevance for suboptimality and that the performance of the linear-inflation policy critically depends on the choice of the numerical values of the policy parameters S and F.

In literature we find four approaches, listed in Table 1, that aim to determine one or both of these policy parameters such that they are optimal or close to optimal. These approaches differ with respect to several problem aspects and application fields which are relevant in the context under consideration. First, these differences relate to which of the two policy parameters is determined and if this parameter evaluation is exact or heuristic. Further, the approaches can be divided into those which only consider zero lead time and others where also non-zero lead times are taken into account. A next criterion is if the approach only refers to a single yield model or if both yield types (binomial, BI, and stochastically proportional, SP) are considered. Finally, we find a difference in the types of probability distributions for demands and yields assumed in the approaches, namely if they are strictly symmetric or if also asymmetric distributions are allowed. Table 1 gives an overview how the four approaches refer to these aspects.

	Exact		Heuristic		Lead time		Yield		Distribution	
	S	F	S	F	zero	non-zero	SP	BI	sym.	asym.
Bollapragada and Morton (1999)			✓		✓		✓		✓	
Zipkin (2000)			✓	✓	✓		✓		✓	
Nagarajan and Huh (2010)	✓			✓	✓		✓		✓	
Inderfurth and Vogelgesang (2013)			✓		✓	✓	✓	✓	✓	✓

Table 1: Approaches for parameter determination of the linear-inflation policy

The first three approaches in Table 1 address a similar limited problem area as they all are restricted to a situation with zero lead time, stochastically proportional yield and symmetric probability distributions. Under these conditions Bollapragada and Morton (1999) propose a heuristic approach (named NLH-1) resulting in a closed-form expression for the critical stock level *S* while they use the reciprocal of the expected yield rate as inflation factor *F*. Although there is a flaw in the calculations for this approach (see Inderfurth and Transchel 2007) a numerical study reveals that this heuristic performs very well in most of the considered cases. The approach suggested by Zipkin (2000) presents a heuristic for determining closed-form solutions for both parameters *S* and *F*. Since these parameter formulas, however, are not derived on the basis of a strict cost minimization approach it turns out that, in general, the performance of the Zipkin heuristic is not satisfactory. This is shown in a computational study carried out by Huh and Nagarajan (2010) whose main own contribution is to develop a method for computing the optimal critical stock *S* for a given yield inflation factor *F* which is computationally tractable. Nevertheless, since for this approach a simulation procedure is needed to generate the stationary inventory distribution the computational burden is still quite considerable. Huh and Nagarajan (2010) also check different simple approaches for determining the inflation factor *F* and suggest a combined approach that works quite well compared to the optimal choice of this parameter. The fourth contribution in this sequence, published in Inderfurth and Vogelgesang (2013), extends the approach of *S* determination in Bollapragda and Morton (1999) and presents closed-form expressions for the critical stock also for binomial yield problems and for environments with non-zero lead times and asymmetric yield rate distributions. For the non-zero lead time case, a numerical study reveals that the performance of the linear-inflation rule with the suggested static parameters is comparable to that of a more sophisticated rule with dynamic critical stock parameters which, in principle, has the potential to perform better under non-zero lead times.

In our paper, we develop two novel approaches to derive optimal and near-optimal expressions for the critical stock *S* under arbitrary inflation factors *F* which apply to all relevant problem instances, i.e. zero and non-zero lead times, binomial and stochastically proportional yield and arbitrary demand and yield distributions which also can incorporate considerable skewness. Both approaches aim to derive the probability distribution of the stationary inventory level under a linear-inflation policy and determine the optimal policy parameter from exploiting this distribution. The first approach uses a Markov chain modeling method and leads to a direct numerical computation of the inventory distribution which is exact for zero and unit lead time and is approximate for lead times greater than one. Thus, this Markov chain approach is able to determine the optimal critical stock level S like the Huh/Nagarajan approach, but with much lower computational effort, because the cumbersome estimation of probabilities by stochastic simulation can be avoided and only a single linear equation system has to be solved. The second approach fits estimations of the moments of the stochastic stationary inventory to standard distribution types so that closed-form expressions for the critical stock parameter S can be found. For estimating the moments in this so-called steady-state approach, an approximate method is employed like it is also used in Zipkin (2000, p. 394) to get results for these moments under equilibrium conditions. This estimation also includes third moments to enable the fitting of asymmetric distributions as for example the gamma distribution, which adequately represents demand for many medium- and fast moving items (Burgin, 1975). In order to appropriately cope with heavy skewness of the inventory distribution that can be caused by asymmetric demand and/or asymmetric yield rate distributions next to a normal distribution also a mirrowed gamma distribution is used to approximate the exact stationary inventory distribution. This innovative way of estimating appropriate distributions is complemented by a sophisticated automatism that decides which type of distribution function (normal or gamma) should be chosen in specific problem instances. Insofar this study also contributes to answering the questions under which circumstances it is feasible to use a normal distribution as approximation in inventory models with asymmetries (see Axsäter 2013). A numerical performance study reveals that in this sophisticated form the steady-state approach leads to close-to-optimal results under almost all problem conditions. The more elaborate Markov chain approach which is optimal for lead times of zero and one, but only approximate in other cases generally shows an even better performance for lead times larger than one. Thus, we are able to present easy-to-apply approaches for highly efficient parameter determination for linear-inflation policies which dominate all other approaches known up to now and can be applied to a wide class of random demand, random yield problems.

The remainder of the paper is organized as follows. In Section 2 we give a detailed description of the model formulation. Section 3 presents the analysis of the approaches in the zero lead time case including a numerical performance study while Section 4 extends this content to the case with lead times larger than zero. Section 5 concludes the paper with a summary and outlook to further research.

2 Model formulation

We consider a single stockpoint for a single item with stochastic customer demand. In order to model the decisions and the demand processes we divide the time axis into discrete time buckets of equal length and assume that demand across the periods is independent and identically distributed. We denote with $D_t \geq 0$ the demand in period *t* and with D the generic demand. We further assume that the cumulative probability distribution of *D* is given. As long as there is enough stock available demand is satisfied directly from stock. Otherwise demand is backlogged.

In order to replenish inventory, orders can be placed at the beginning of each period and are delivered after a deterministic replenishment lead time of λ periods. The order quantity is determined according to a stationary linear-inflation policy, which can formally be described by

$$Q_t = \begin{cases} F\left(S - X_t\right), & X_t < S \\ 0 & , \quad X_t \geq S \end{cases} \tag{1}$$

where Q_t denotes the order quantity, X_t the inventory position at the beginning of a period *t* before the order is placed, and *S* stands for the critical stock level and *F* for the yield inflation factor. If discrete values for the order quantity or the yield are required, then we round the order quantity in policy (1) to the closest integer value and we use the following notation $Q_t = \lfloor F(S - X_t) \rceil$.

Due to uncontrollable conditions the quantity received is not necessarily equal to the quantity ordered. Let us denote with *Q* the order quantity (input) and with *Y (Q)* the uncertain yield (output) for a given order quantity *Q*. The yield is assumed to be independent from the demand and we use the index *t* if the random variables are related to period *t*. In this article we study the two most common random yield models mentioned above. First, we consider a binomial yield model where each unit ordered is defective with the same probability *1-p*. Thus, the yield for a given order quantity *Q* is binomially distributed with the parameters (Q, p) and the probability distribution is given as

$$P(Y(Q) = k) = p^k (1-p)^{Q-k} \binom{Q}{k} \qquad k = 0, 1, 2, \ldots, Q-1, Q$$

Additionally, we study a proportional yield model where the delivered quantity is a random multiple of the ordered quantity $(Y(Q) = Z \cdot Q)$. Let $Z_t \in \left[0,1\right]$ denote the yield rate in period *t*. We assume that the sequence $(Z_t)_{t=1,2,\ldots}$ is iid and denote the mean of the yield rate by μ_Z and the standard deviation by σ_Z. Thus, we find for a given order quantity Q the following moments for the total yield $Y(Q)$ under binomial (stochastic proportional) yield conditions. As expected value we have

$$E[Y(Q)] = p \cdot Q \qquad \left(E[Y(Q)] = \mu_Z \cdot Q\right) \tag{2}$$

and as variance we get

$$VAR[Y(Q)] = p \cdot (1-p) \cdot Q \qquad \left(VAR[Y(Q)] = \sigma_Z^2 \cdot Q^2\right)$$

In addition, our approach takes into account a possible skewness of the yield rate which depends on parameter p for binomial yield and is given by a skewness measure γ_Z for stochastically proportional yield.

After the realization of demand in period t, inventory holding costs of $h \geq 0$ incur for each unit on stock. For each unit backlogged at the end of a period backorder costs $b \geq 0$ are charged. We are interested in minimizing the expected long run average cost of the system that is given as a function of F and S

$$C(F,S) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} C_t(F,S)$$

where $C_t(F,S)$ denotes the average period cost composed of linear holding and backorder costs charged to the net inventory $I_t(F,S)$ at the end of period t:

$$C_t(F,S) = hE\left[I_t(F,S)^+\right] + bE\left[-I_t(F,S)^+\right] \tag{3}$$

If a stationary distribution of the net inventory I_t at the end of a period exists in a continuous model, the average cost in (3) can be written as

$$C = h\int_0^\infty x\varphi_I(x)dx - b\int_{-\infty}^0 x\varphi_I(x)dx \tag{4}$$

where $\varphi_I(.)$ is the density function of I that depends on both policy parameters F and S.

In this study, we focus on the determination of the optimal critical stock level S for any pre-determined yield inflation factor F. Since under policy (1) the choice of S only affects the mean of the distribution of inventory I but not its variability or skewness, the respective influence of S can be considered by its impact on μ_I and by the dependency of $\varphi_I(.)$ on μ_I.

Thus, for optimizing the critical stock S it is sufficient to describe the cost in (4) as function of μ_I which can be formulated as

$$C(\mu_I)=(h+b)\int_0^{\infty} x\varphi_I(x)dx-b\mu_I \tag{5}$$

After normalizing the inventory variable by transforming I to $I_N=(I-\mu_I)/\sigma_I$ and exploiting the optimality condition for minimizing $C(\mu_I)$ we find (see Appendix I)

$$1-\Phi_N\left(-\frac{\mu_I}{\sigma_I}\right)=\frac{b}{h+b} \tag{6}$$

where $\Phi_N(.)$ stands for the cumulative distribution function of the standardized inventory variable. Under a symmetric inventory distribution, condition (6) can be written as

$$\Phi_N\left(\frac{\mu_I}{\sigma_I}\right)=\frac{b}{h+b} \tag{7}$$

Noting that μ_I depends on the critical stock S, conditions (6) and (7) can be exploited to determine the optimal stock level.

In the sequel we perform a Markov chain and a simplified steady-state analysis in order to determine an optimal and a near-optimal critical stock level for the linear-inflation rule. To this end, we have to analyze separately the case without replenishment lead time ($\lambda=0$) and the situation with positive lead times ($\lambda>0$). The Markov chain analysis yields information on the complete inventory distribution and provides exact results for $\lambda=0$ and $\lambda=1$ while it is only approximate for $\lambda>1$. The steady-state analysis delivers closed-form approximations of the moments of the random inventory and generates approximations under all lead time instances. In the remainder of the article the following notation is used:

μ_X: mean of the random variable X

σ_X : standard deviation of the random variable X

ρ_X : coefficient of variation of the random variable X, defined as $\rho_X = \sigma_X / \mu_X$

$\mu_{3,X}$: third moment of random variable X

γ_X : skewness of random variable X, given by $\gamma_X = E\left[\left(X - \mu_X\right)^3\right] / \sigma_X^3$

Our analysis applies to both types of variables, discrete and continuous ones. The discrete Markov chain approach under consideration naturally refers to discrete-type variables, but can approximate continuous ones arbitrarily close. The steady-state approach refers to each variable type according to the selected distribution type for the inventory variable. If a discrete distribution is used, discrete optimality conditions equivalent to (6) and (7) are exploited.

3 Zero lead time (λ = 0)

In this Section we analyze the system for zero lead time (λ = 0). Under this assumption the sequence of events is given as follows. At the beginning of a period the inventory level is observed and an order is placed according to the linear-inflation policy (1). After the instantaneous delivery the demand is realized and at the end of the period costs are charged based on the net inventory level. For $\lambda = 0$ we face the only situation where the order decision is made before the yield realization of the order λ periods ago is known. So this case must be handled separately and cannot be treated as a special case of a general analysis for $\lambda \geq 0$.

3.1 An exact Markov chain analysis

For the analysis of the system we assume a discrete demand distribution and we restrict the order quantities to discrete values as well. We model the difference Δ_t between the inventory position X_t and the critical stock *S* in period *t* as a Markov chain

$$\Delta_t = X_t - S.$$

Note that in the case of zero lead time the inventory position equals the stock on hand. In case of a positive value of Δ an overshoot is observed and no order is placed while for negative values the order quantity is positive. The statespace for the Markov chain is unlimited and given as $SS =$

$(-\infty,+\infty)$. In order to obtain the transition probabilities p_{ij} for the Markov chain, defined as $p_{ij} = P(\Delta_{t+1} = j | \Delta_t = i)$, we use the following recursive relation

$$\Delta_{t+1} = \begin{cases} \Delta_t + Y(\lfloor -F\Delta_t \rceil) - D_t & , \Delta_t < 0 \\ \Delta_t - D_t & , \Delta_t \geq 0 \end{cases}$$

which enables us to derive expressions for $i \geq 0$

$$p_{ij} = P(\Delta_{t+1} = j \mid \Delta_t = i) = P(D_t = i - j), \quad j \in SS$$

independent from the yield distribution. The transition probabilities for $i < 0$ and $j \in SS$ depend on the demand and yield as follows

$$p_{ij} = P(\Delta_{t+1} = j \mid \Delta_t = i) = P(D_t = i + Y(\lfloor -iF \rceil) - j)$$

For a binomial yield we get:

$$p_{ij} = \sum_{k=0}^{\lfloor -Fi \rceil} P(D_t = i + k - j) p^k (1-p)^{\lfloor -Fi \rceil - k} \binom{\lfloor -Fi \rceil}{k}$$

and the discretization in case of stochastically proportional yield is done as follows

$$p_{ij} = \sum_{k=0}^{\lfloor -Fi \rceil} P(D_t = i + k - j) \left(P\left(Z \leq \frac{k+0.5}{\lfloor -Fi \rceil}\right) - P\left(Z \leq \frac{k-0.5}{\lfloor -Fi \rceil}\right) \right)$$

For a given value of F the stationary distribution $v_k = \lim_{t\to\infty} P(\Delta_t = k)$ can be determined numerically solving the system of equations given by

$$v = \mathcal{P}v \quad with \quad \mathcal{P} = (p_{ij})_{(i,j)}, \quad v = (v_k)_k \quad , \quad and \quad \sum_k v_k = 1 \tag{8}$$

Due to the relation $X = S + \Delta$ and since the inventory position at the beginning of a period before ordering is equal to the inventory level at the end of a period in case of zero lead time, the following expression for the average cost per period is obtained:

$$C(F,S) = h \sum_{k=-S}^{+\infty} (S+k) v_k \text{ -} b \sum_{k=-\infty}^{-(S+1)} (S+k) v_k$$

Note that for the optimization of S for a given value of F the stationary distribution of the Markov chain must only be computed once, since it is independent from the numerical value of S. The following lemma shows that the optimal critical stock level S* satisfies a news-vendor like condition.

LEMMA 1: For any given value of F the optimal critical stock level $S^*(F)$ is the smallest value of S satisfying the following condition:

$$\sum_{k=-\infty}^{-1} P(D - Y(\lfloor F(S-k) \rceil) \leq S+k)v_k + \sum_{k=0}^{+\infty} P(D \leq S+k)v_k \geq \frac{b}{b+h}$$

Proof: The proof is similar to the general lead time case and is therefore postponed to the next section.

Note that a similar exact analysis can be done for a lead time equal to one period. The stationary distribution of the inventory position is the same, but the distribution of the inventory level is determined by the relation $I=X-D$.

3.2 An approximate steady-state analysis

In this section we present an approach to derive a simple expression for a near-optimal critical stock level S^{++} which can easily be implemented in a spreadsheet. The approach is based on the presentation of the objective function in (5) keeping in mind that the expected inventory μ_I is a function of the critical stock S.

We will approximate the distribution of the inventory level in two different ways and then we present a decision rule to choose from the two cases. For the optimization of the objective function we further simplify and analyze (5) for a strictly linear control rule. Thus we neglect that the order quantity is zero in case of an overshoot and use

$$Q_t = F\left(S - X_t\right) = F(S - I_{t-1}) \tag{9}$$

For the further analysis we need more information about the expected inventory level.

LEMMA 2: If a strictly linear control rule as in (9) is applied, then the average inventory level is given as

$$\mu_I = S - \frac{\mu_D}{M} \tag{10}$$

where M stands for the scrap loss compensation factor, i.e.

$$M = \begin{cases} Fp & \text{for binomial yield} \\ F\mu_Z & \text{for stochastically proportional yield} \end{cases} \tag{11}$$

If $M > 1$, expected yield losses are overcompensated, otherwise, if $M < 1$, they are undercompensated by the ordering rule. Additionally, the average order quantity is determined by

$$\mu_Q = \frac{F}{M}\mu_D \tag{12}$$

Proof: The system dynamics under the strict linear control rule is given as

$$I_t = I_{t-1} + Y(Q_t) - D_t \tag{13}$$

Which, considering (2) under steady-state conditions immediately leads to (12). The expression for the expected inventory level can then easily be obtained from (9).

In order to exploit the optimality condition in (6), we need knowledge about the distribution of the inventory level. To this end we use two types of common distribution functions as proxy for the unknown real stationary inventory function. Let us first assume that the inventory level is normally distributed with mean μ_I and standard deviation σ_I. Then we can derive the following optimality condition.

LEMMA 3: Under the assumption of a normally distributed inventory level, the optimal critical level S^+_{Norm} for a strict linear control rule (9) minimizing (5) has to satisfy the following condition

$$S^+_{Norm} = \frac{\mu_D}{M} + \Phi^{-1}_{Norm}\left(\frac{b}{b+h}\right)\sigma_I \tag{14}$$

where Φ_{Norm} denotes the cumulative distribution function of a standard normal distribution. This solution can immediately be verified by inserting μ_I from (10) in (7).

Under asymmetric demand and/or yield distributions one must be aware that also the inventory distribution will be asymmetric. Since the normal distribution does not capture this case we also consider another distribution for the inventory level, which can be used to approximate inventory asymmetry. To this end we propose to utilize a gamma distribution which, however, has to be mirrowed in order to approximate an inventory distribution, which is usually expected to be skewed to the left under a critical stock policy. Let us assume now, that the inventory level can be written as

$$I = T(S) - X \tag{15}$$

where $T(S)$ denotes the (due to the unlimited excess stock not really existing) maximum possible value for the inventory level I and X is a random variable distributed according to a gamma distribution with mean $E[X] = E[T(S) - I] = T(S) - \mu_I = T(S) - S + \mu_D / M$ and standard deviation σ_I. Then the following Lemma 4 holds.

LEMMA 4: For the strict linear control rule (9) and under the assumption that the inventory level is given as $I = T(S) - X$ where X is Gamma distributed, from exploiting (6) the following optimality condition is obtained

$$P\left(X \leq T(S)\right) = \frac{b}{b+h} \tag{16}$$

Proof: Appendix II.

Since the expectation of X depends on the difference of $T(S) - S$ we have to determine $T(S)$. Obviously, the probability that the inventory level is larger than the stock level S is equal to the probability that the order quantity Q is negative, resulting in

$$P(Q \leq 0) = P(I \geq S) = P(X \leq T(S) - S)$$

If the demand variability and the yield variability are relatively small, then the probability on the left hand side is close to zero, resulting in $T(S) = S$. We use this equation as an approximation for all situations and, as a consequence, the second approximate critical stock level S^{+}_{Gamm} has to satisfy the following condition

$$P(X \leq S^{+}_{Gamm}) = \frac{b}{b+h} \tag{17}$$

where X is gamma distributed with mean μ_D / M and standard deviation σ_I.

Up to now we have derived two approximate critical levels S^{+}_{Norm} and S^{+}_{Gamm} under the assumption that negative order quantities are possible. Therefore, the real inventory level is underestimated and the obtained approximations for the critical stock level are overestimated. In order to adjust our formulas, we determine the partial expectation for the negative part of the order quantity. This can easily be performed under the assumption that the order quantity is distributed according to a normal distribution with mean μ_Q and standard deviation σ_Q in the following way:

$$E\left[\left(-F(S-X)\right)^{+}\right] = -\int_{-\infty}^{0} y\varphi_Q\left(y\right)dy = \sigma_Q\varphi_N\left(-\frac{\mu_Q}{\sigma_Q}\right) - \mu_Q\Phi_N\left(-\frac{\mu_Q}{\sigma_Q}\right)$$

Since we are interested in a near optimal critical level under a linear control rule as given in (1) we reduce the critical levels by the value of the partial expectation of the negative order quantity.

$$S_i^{++} = S_i^{+} - \left(\sigma_Q\varphi_N\left(-\frac{\mu_Q}{\sigma_Q}\right) - \mu_Q\Phi_N\left(-\frac{\mu_Q}{\sigma_Q}\right)\right) \qquad i = Norm, Gamm \tag{18}$$

For calculating the critical stock levels S_i^{++}, it remains to determine expressions for σ_I and σ_Q. The expressions for the order quantity for binomial yield as well as for stochastic proportional yield can be found in Lemma 5.

LEMMA 5: Given that $M < 2$ holds, the variance for the order quantity under a strict linear control rule $Q = F(S - X)$ and binomial yield is given as:

$$\sigma_Q^2 = \frac{F^2\left(\sigma_D^2 + \left(1-p\right)\mu_D\right)}{1-(1-M)^2} \tag{19}$$

Let us assume that the squared coefficient of variation of the yield factor Z, denoted by ρ_Z^2, is such that the condition $\rho_Z^2 < 2/M - 1$ is fulfilled. Then the variance for the order quantity under a strict linear control rule $Q = F(S - X)$ and stochastically proportional yield is given as

$$\sigma_Q^2 = \frac{F^2\left(\rho_Z^2\mu_D^2 + \sigma_D^2\right)}{1-\left(1-M\right)^2 - M^2\rho_Z^2} \tag{20}$$

Note that these results hold for general lead times and not only for lead time zero. The respective proofs can be found in Appendix III.

In contrast to this, the moments of the inventory level depend on the lead time and are given in Lemma 6.

LEMMA 6: If $M < 2$ then the variance for the inventory level under a strict linear control rule $Q = F(S - X)$, lead time zero and binomial yield is given as:

$$\sigma_I^2 = \frac{\sigma_D^2 + (1-p)\mu_D}{1-(1-M)^2} \tag{21}$$

Under the condition $\rho_Z^2 < 2/M - 1$ the variance for the inventory level under a strict linear control rule $Q = F(S - X)$, lead time zero and stochastically proportional yield is given as:

$$\sigma_I^2 = \frac{\sigma_D^2 + \rho_Z^2 \mu_D^2}{1-(1-M)^2 - M^2 \rho_Z^2} \tag{22}$$

Proof: From (9) we directly get for the inventory variable that $I_{t-1} = S - Q_t/F$. Thus, the steady-state variance of *I* is given by $\sigma_I^2 = \sigma_Q^2/F^2$ so that from (19) and (20) we immediately yield the variance formulas of Lemma 6 in (21) and (22).

It is obvious that for the stability of the inventory process under steady-state conditions the compensation factor *M* (and thereby *F*) must not exceed a certain upper level. Additionally, the above analysis also makes clear that the critical stock level *S* only has an impact on the expected inventory level μ_I, while the variance of the inventory and order distribution is not affected by *S*.

With (18) we have proposed two different approximations for the optimal critical level S^*, and the question arises which of the approaches should be preferred in a specific situation. Since the main difference between the approaches is the distribution assumption of the inventory level, we use a further attribute, the skewness γ_I of this distribution, to support our choice.

LEMMA 7: The skewness of the inventory level γ_I under a strict linear control rule, lead time zero and binomial yield turns out to be

$$\gamma_I = \frac{1}{\sigma_I^3}\left(\frac{\mu_D^3}{M^3} + 3\frac{\sigma_I^2 \mu_D}{M} - \frac{\Omega_{BI}}{1-(1-M)^3}\right) \tag{23}$$

with $$\Omega_{BI} = 3(1-M)\Big[\big((1-p)M + (1-M)\mu_D\big)\big(\sigma_I^2 + \mu_D^2/M^2\big) + \mu_D\big(\sigma_D^2 + \mu_D^2\big)/M\Big] + \\ +(1-p)\mu_D\big(3\mu_D + 2p - 1\big) + \mu_{3,D}$$

For stochastically proportional yield the skewness of the inventory level is given as

$$\gamma_I = \frac{1}{\sigma_I^3}\left(\frac{\mu_D^3}{M^3} + 3\frac{\sigma_I^2 \mu_D}{M} - \frac{\Omega_{SP}}{3M(1-M) - 3M^2\rho_Z^2 + F^3\mu_{3,Z}}\right) \tag{24}$$

with

$$\Omega_{SP} = 3(1-M)\mu_D\Big[\big(\mu_D^2 + \sigma_D^2\big)/M + (1-M)\big(\sigma_I^2 + \mu_D^2/M^2\big)\Big] + 3M^2\rho_Z^2\mu_D\big(\sigma_I^2 + \mu_D^2/M^2\big) + \mu_{3,D}$$

Proof: Appendix IV

Based on the skewness γ_I of the inventory level we choose that critical stock from (18) that is calculated from the inventory distribution with a skewness level which is closest to the γ_I value. Thus the following decision rule is recommended. First compute the approximation S^{++}_{Gamm} according to (17) and (18). Then, a gamma distribution is fit on the corresponding mean (μ_D / M) and standard deviation (σ_I) of the inventory level. In the next step the skewness $\gamma_{I,Gamm}$ of this fitted distribution is computed as $\gamma_{I,Gamm} = -\frac{2\sigma_I}{\mu_D / M}$ and compared with the required skewness γ_I as derived in lemma 7. If the value of γ_I is closer to zero (i.e. the γ value under normal distribution) than to $\gamma_{I,Gamm}$, then we use the approximation S^{++}_{Norm} according to (14) and (18), otherwise S^{++}_{Gamm} is chosen.

3.3 Numerical study

In this section, we present the results of a numerical study where the performance of the approximated critical stock level S^{++}, computed according to the proposed approach presented in Section 3.2, is tested. In our experiments we have used the standard setting for the yield inflation factor, i.e., $F = 1/p$ for binomial yield and $F = 1/\mu_Z$ for stochastically proportional yield which is equivalent to $M = 1$. We fixed the mean demand $\mu_D = 20$ and the holding cost parameter *h*=1. We computed the optimal value *S** with the Markov Chain approach as well as the approximated S^{++} with our proposed approach and compared the exact average costs for both values. For each instance i we have computed the relative cost difference δ_i as

$$\delta_i = \frac{C\left(S^+\right) - C(S^*)}{C(S^*)} \cdot 100\%$$

and we have calculated the average relative difference for N instances as

$$\bar{\delta} = \frac{1}{N} \sum_{i=1}^{N} \delta_i$$

and the maximum relative difference of N instances as

$$\delta_{max} = \max_{i=1,\ldots N} \delta_i.$$

Six different values for the backorder cost parameter are chosen, similar as in Huh and Nagarajan (2010) $b/(b+h) \in \{0.85, 0.9, 0.95, 0.97, 0.99, 0.995\}$. We investigate two distributions for the demand and consider normally as well as gamma distributed demand. The latter is especially considered to allow for large demand variability and non-symmetric demand distributions. Since a discrete demand distribution is needed we compute the probability that demand is equal to $k, k \in \{0,1,2,\ldots\}$ as $P(D \leq k + 0.5) - P(D \leq k - 0.5)$.

For the normal distributed demand three different coefficient of variations are chosen ($\rho_D \in \{0.1, 0.2, 0.3\}$) and for the gamma distributed demand five different values are selected ($\rho_D \in \{0.1, 0.2, 0.3, 0.5, 0.75\}$). Also three different values for p are studied for the binomial distributed yield ($p \in \{0.5, 0.7, 0.9\}$) resulting in 54 instances for the normal distributed demand and 90 instances for the gamma distributed demand. In Table 2 the numerical results for the approximation are summarized.

		Normal demand		Gamma demand	
Parameter	**Value**	**Average relative deviation**	**Maximum relative deviation**	**Average relative deviation**	**Maximum relative deviation**
ρ_D	0.1	0.38	2.89	0.35	2.54
	0.2	0.18	1.12	0.32	2.34
	0.3	0.10	1.01	0.51	2.50
	0.5			0.08	0.30
	0.75			0.02	0.10
p	0.5	0.52	2.89	0.45	2.54
	0.7	0.11	1.09	0.25	2.34
	0.9	0.03	0.59	0.07	0.74
Critical ratio b/(h+b)	0.85	0.21	1.01	0.07	0.80
	0.90	0.21	1.09	0.14	0.97
	0.95	0.45	2.28	0.25	2.03
	0.97	0.32	2.89	0.26	2.54
	0.99	0.02	0.16	0.47	2.50
	0.995	0.12	1.12	0.35	2.34
Total		0.22	2.89	0.26	2.54

Table 2: Results for lead time zero and binomial yield

It can be seen that the approximation works excellent. In about 80 (60) percent of the instances with normal (gamma) distributed demand the optimal critical stock level S^* is found by our approach and for the other instances the cost deviations are very small. A closer look on the numerical results reveals that for the normally distributed demand situation the decision rule always selects S_{Norm}^{++}, while in case of gamma distributed demand S_{Norm}^{++} is only chosen for small demand variability. For large demand variability and high service levels relative cost deviations up to 40% can be observed if only S_{Norm}^{++} is used as an approximation for the critical stock level, which illustrates the necessity of taking the skewness of the distribution of the inventory level into account.

For the part of the numerical study where the stochastically proportional yield is considered we have chosen a beta distribution for the yield, because thereby different scenarios can be modeled easily. We consider three symmetric distributions with different variability $(\mu_Z = 0.5, \rho_Z \in \{0.2, 0.4, 0.5774\})$. While the smallest coefficient of variation leads to a yield distribution similar to a normal distribution, the largest value $\rho_Z = 0.5774$ results in a uniform distribution for the yield. Further, three distributions with a positive skewness $(\mu_Z = 0.75, \rho_Z = 0.2; \mu_Z = 0.85, \rho_Z = 0.2; \mu_Z = 0.85, \rho_Z = 0.1)$ are studied. The densities of the different yield distributions are illustrated in Figure 1.

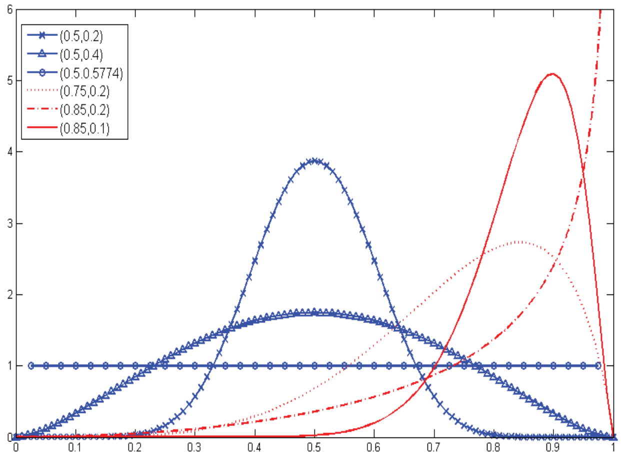

Figure 1: The densities of the different yield models

In total 108 instances are computed for normally distributed demand and 180 instances for gamma distributed demand and stochastically proportional yield. The results of the experiments are summarized in Table 3.

		Normal demand		Gamma demand	
Parameter	**Value**	**Average relative deviation**	**Maximum relative deviation**	**Average relative deviation**	**Maximum relative deviation**
ρ_D	0.1	0.69	7.65	0.64	7.80
	0.2	0.43	3.41	0.47	3.58
	0.3	0.57	6.59	0.62	4.87
	0.5			1.01	8.60
	0.75			2.43	26.85
(μ_Z,ρ_Z)	(0.85;0.1)	0.17	1.33	0.19	1.19
	(0.85;0.2)	1.17	7.65	0.51	7.80
	(0.75;0.2)	0.50	1.71	0.32	1.39
	(0.5;0.2)	0.09	0.69	0.35	4.87
	(0.5;0.4)	0.42	1.95	2.48	26.85
	(0.5;0.5774)	1.01	4.39	2.36	20.31
Critical ratio b/(h+b)	0.85	0.12	0.74	0.07	0.73
	0.90	0.26	1.33	0.12	1.29
	0.95	0.17	1.18	0.23	2.26
	0.97	0.22	1.71	0.51	4.21
	0.99	0.67	2.18	1.76	15.70
	0.995	1.92	7.65	3.52	26.85
Total		0.56	7.65	1.04	26.85

Table 3: Results for stochastic proportional yield and zero lead time

In case of stochastically proportional yield the approximation still works very well, but a little bit worse than in case of binomial yield. In 47 (40) percent of the instances the approximation could find the optimal critical stock level. While on average the relative cost deviations are very small, in a few instances the cost deviation is above or close to ten percent. However, these large deviations occur for extremely high service levels respectively critical ratios of 0.995 or 0.99. It is questionable if in practice the demand and yield distributions can be estimated so precisely that extremely high critical ratios make sense for exact stock determination. We further would like to mention that in case of normal (symmetric) demand and non-symmetric yield the decision rule often selects S_{Gamm}^{++} for the approximated critical stock level and for gamma distributed demand in about 50% of the instances it is chosen as well.

4 Positive lead time λ ≥ 1

Except from Inderfurth and Vogelgesang (2013), random yield models with positive lead times are not studied in detail in the literature and there are several open questions. We consider the situation where the yield is observed when the order arrives. Further, the analysis is based on the following sequence of events. At the beginning of the period the order, placed λ periods before, is delivered and yield is recognized. Second, a new order can be placed based on the inventory position at the beginning of the period and according to the linear inflation rule as given in (1). Then demand occurs and inventory and backorder costs are charged at the end of the period.

The first issue to be discussed is the information to be used for the order decision. Since the yield of an order becomes known at the moment of delivery, we can only include the expected yield in the inventory position resulting in the following definition of the inventory position:

$$X_t = I_{t-1} + Y\left(Q_{t-\lambda}\right) + \sum_{l=1}^{\lambda-1} E\left[Y\left(Q_{t-l}\right)\right] \tag{25}$$

Note that according to the policy in (1) we apply a linear control rule with a static order parameter (critical stock) S. Even under a stationary demand and yield process this has not necessarily to be optimal, since the differently sized open replenishment orders which vary in time incorporate different levels of yield risks. These varying risk positions might be reflected by varying the critical stock from period to period (for a more detailed discussion see Inderfurth and Vogelgesang (2013)).

In the following we first model the system as a Markov chain. Then we show how the Markov model can be used to obtain a near optimal critical stock level and additionally we propose a simple heuristic to compute near-optimal critical stock levels by a steady-state approach.

4.1 A Markov chain analysis

At the moment when the yield becomes known, the inventory position has to be updated, resulting in the following recursive equation:

$$X_{t+1} = X_t + E\left[Y\left(Q_t\right)\right] - D_t - \left(E\left[Y\left(Q_{t+1-\lambda}\right)\right] - Y\left(Q_{t+1-\lambda}\right)\right) \tag{26}$$

We define the correction as the difference between the expected and the actual yield

$$R_{t+1-\lambda} = E\left[Y\left(Q_{t+1-\lambda}\right)\right] - Y(Q_{t+1-\lambda}) \tag{27}$$

Similar as in the case with zero lead time the difference of the inventory position and the critical stock level can be modeled as a Markov chain, but the recursive equation for Δ_t is now given as:

$$\Delta_{t+1} = \begin{cases} \Delta_t + E[Y(\lfloor -F\Delta_t \rceil)] - (D_t + R_{t+1-\lambda}) & , \ \Delta_t < 0 \\ \Delta_t - (D_t + R_{t+1-\lambda}) & , \ \Delta_t \geq 0 \end{cases}$$

Thus, the transition probabilities for $i \geq 0$ and $j \in SS$ can be computed as

$$p_{ij} = P(\Delta_{t+1} = j \middle| \Delta_t = i) = P(D_t + R_{t+1-\lambda} = i - j) \tag{28}$$

and for i<0 and $j \in SS$ as

$$p_{ij} = P(\Delta_{t+1} = j \mid \Delta_t = i) = P(D_t + R_{t+1-\lambda} = i + E[Y(\lfloor -iF \rceil)] - j) \tag{29}$$

The stationary distribution of the inventory position is the solution of the system of equations as presented in (8) where the matrix of the transition probabilities is determined by (28) and (29). In order to obtain the distribution of the inventory level we define a modified demand process

$$\eta\left(\lambda+1\right) = \sum_{i=0}^{\lambda} D_{t+i} + \sum_{i=0}^{\lambda-1} R_{t-i} \tag{30}$$

and use the following relation between the inventory position and the inventory level

$$I_{t+\lambda} = X_t + E\left[Y\left(Q_t\right)\right] - \eta(\lambda+1)$$

LEMMA 8: The optimal critical stock level $S^*(F)$ for any given value of F for the infinite horizon random yield problem with positive lead time and a linear control rule is the smallest value satisfying the following newsboy condition:

$$\sum_{k=-\infty}^{-1} \mathrm{P}(\eta(\lambda+1) \leq S + k + E[\Upsilon(\lfloor -Fk \rfloor)])v_k + \sum_{k=0}^{+\infty} \mathrm{P}(\eta(\lambda+1) \leq S + k)v_k \geq \frac{b}{b+h} \tag{31}$$

where $\eta(\lambda+1)$ is the modified demand as defined in (30).

Proof: Appendix V

In order to apply the approach presented above, the probability distribution of the correction R is needed. Since the distribution of the correction depends on the order quantity, it changes from period to period. Therefore, we rely on an approximation and determine the first two central moments of the correction R in equilibrium and fit a normal distribution on the moments. It is easy to see that for both yield types the first moment of the correction is equal to zero. However, for the variance we have to distinguish between the two yield types.

LEMMA 9: For a binomial yield model the variance of the correction R as defined in (27) is independent on the order quantity and given as

$$\sigma_R^2 = (1-p)\mu_D$$

Under the assumption of a stochastic proportional yield model the variance of the correction R is dependent on the second moment of the order quantity as follows:

$$\sigma_R^2 = \sigma_Z^2\left(\mu_Q^2 + \sigma_Q^2\right)$$

Proof: Appendix VI

Using the approximations (19) and (20) the second moment of the order quantity can be approximated and the approximate stationary distribution of the inventory distribution can be computed. It remains to determine the variance of the sum of the corrections R in order to determine the distribution of the adapted demand process and the inventory level. It can be shown that the covariance of different correction factors is zero and we obtain the following relation

LEMMA10: The variance of the sum of the correction factors is equal to the sum of the variances

$$VAR[\sum_{i=0}^{\lambda-1} R_{t-i}] = \sum_{i=0}^{\lambda\text{ -1}} VAR[R_{t-i}]$$

Proof: Appendix VII

As mentioned above, we always approximate the distribution of the sum of the corrections with a normal distribution and thus, we have to compute the convolution of a normal distribution and the demand distribution to obtain the distribution of the adapted demand process η as given in (30). Due to this approximation the Markov chain approach is not exact. For $\lambda = 1$, however, the correction *R* in (27) is equal to zero so that the exact inventory distribution and optimal critical stock is determined.

4.2 An approximate steady-state analysis

For calculating the approximate critical stock level S^{++} similar as is (18) under non-zero lead times, we can still use the expected order quantity μ_Q from (12) as well as the standard deviation of the order quantity σ_Q from (19) or (20), but need to determine the steady-state values for μ_I and σ_I for positive lead times $\lambda > 0$.

LEMMA 11: If a strictly linear control rule as in (9) is applied, then the average inventory level for both yield types and positive lead time is given as

$$\mu_I = S - \left(\lambda + \frac{1}{M}\right)\mu_D \tag{32}$$

where M denotes the scrap loss compensation factor as defined in (11).

Proof: From the definition of the inventory position X_t in (26) it can be derived for the order quantity that

$$Q_t = F \cdot \left(S - I_{t-1} - \sum_{i=1}^{\lambda-1} E\left[Y(Q_{t-i})\right] - Y(Q_{t-\lambda}) \right) \tag{33}$$

Exploiting $I_{t-1} + Y(Q_t) = I_t + D_t$ from inventory balance equation (13) yields

$$Q_t = F \cdot \left(S - I_t - D_t - \sum_{i=1}^{\lambda-1} E\left[Y(Q_{t-i})\right] \right)$$

Using expected yields from (2) and the μ_Q calculation from (12), the expected inventory formula in (32) can easily be derived.

Concerning the variance of Q, it already has been shown in Lemma 5 that σ_Q^2 does not depend on the lead time λ. With regard to the variance of the inventory level I, however, this independence property does not hold. That is shown in the following Lemma 12.

LEMMA 12: The variance for the inventory level under a strictly linear control rule $Q = F(S - X)$ with M<2, lead time $\lambda > 0$ and binomial yield is given as:

$$\sigma_I^2 = \sigma_D^2 + \left(\lambda - 1 + \frac{1}{M^2}\right) \cdot \frac{M^2\left(\sigma_D^2 + (1-p)\cdot\mu_D\right)}{1-(1-M)^2} \tag{34}$$

For stochastically proportional yield with $\rho_Z^2 < \frac{2}{M} - 1$ the variance of the inventory level is given as:

$$\sigma_I^2 = \sigma_D^2 + \left(\lambda - 1 + \frac{1}{M^2}\right) \cdot \frac{M^2\left(\sigma_D^2 + \rho_Z^2\mu_D^2\right)}{1-(1-M)^2 - M^2\rho_Z^2} \tag{35}$$

Proof: Appendix VIII

Again it is evident that the critical stock S only has an impact on the mean inventory level μ_I Since the variance of the inventory level is also known for positive lead time, S_{Norm}^{+} can be computed using

$$S_{Norm}^{+} = \left(\lambda + \frac{1}{M}\right)\mu_D + \Phi_N^{-1}\left(\frac{b}{b+h}\right)\sigma_I$$

and S_{Gamm}^{+} using (17) where the Gamma distributed random variable X has a mean $\left(\lambda + 1/M\right)\mu_D$ and standard deviation σ_I as given in (34) or (35). For the choice between the two approaches we still need the skewness of the distribution of the inventory level as described in Section 3.2.

LEMMA 13: Under binomial yield, a strictly linear control rule and positive lead time $\lambda > 0$ the skewness of the inventory level is given as

$$\gamma_I = \frac{1}{\sigma_I^3}\left(\begin{array}{l}\left(\lambda+\frac{1}{M^3}\right)\mu_D^3+\frac{3\mu_D}{\lambda-1+\frac{1}{M^2}}\left[\left(\lambda-1+\frac{1}{M^3}\right)\sigma_I^2+\left(\frac{1}{M^2}-\frac{1}{M^3}\right)\sigma_D^2\right]- \\ -\mu_{3,D}-\frac{\lambda-1+1/M^3}{1-(1-M)^3}\cdot\Theta_{BI}\end{array}\right) \tag{36}$$

with

$$\Theta_{BI} = 3M(1-M)\mu_D\left(M(1-p)\mu_D+(1-M)\mu_D^2+\frac{1-M}{\lambda-1+1/M^2}\left(\sigma_I^2-\sigma_D^2\right)+M\left(\mu_D^2+\sigma_D^2\right)\right)- \\ -M^3(1-p)\left(3\mu_D^2+(1-2p)\mu_D\right)+M^3\mu_{3,D}$$

and with σ_I from (34).

Under the same assumptions and stochastically proportional yield we obtain

$$\gamma_I = \frac{1}{\sigma_I^3}\left(\begin{array}{l}\mu_D^3\left(\lambda+\frac{1}{M^3}\right)+\frac{3\mu_D}{\lambda-1+\frac{1}{M^2}}\left[\left(\lambda-1+\frac{1}{M^3}\right)\sigma_I^2+\left(\frac{1}{M^2}-\frac{1}{M^3}\right)\sigma_D^2\right]- \\ -\mu_{3,D}-\frac{\lambda-1+\frac{1}{M^3}}{3M(1-M)-3M^3\rho_Z^2+F^3\mu_{3,Z}}\cdot\Theta_{SP}\end{array}\right) \tag{37}$$

with

$$\Theta_{SP} = 3\mu_D\left[\left(M(1-M)^2+M^2\rho_Z^2\right)\left(\mu_D^2+\frac{\sigma_I^2-\sigma_D^2}{\lambda-1+1/M^2}\right)+M^2(1-M)\left(\mu_D^2+\sigma_D^2\right)\right]+M^3\mu_{3,D}$$

where σ_I is given as in (35).

Proof: Appendix IX

The same decision rule is applied as in case of zero lead time to determine an approximate critical level S^{++}.

4.3 Numerical study

In this part of the numerical study we test the performance of our approaches, the Markov chain approach as well as the steady state solution, for positive lead time. We consider the same instances as in Section 3.3. for lead times $\lambda \in \{2,5,10\}$. For our comparison we first computed the near optimal critical level (S^M) using the Markov chain approach and the approximated critical level S^{++} according to formula (18) with the corresponding moments for the scenario with positive lead times.

Then we simulated the inventory system with both critical levels and we additionally determined the optimal critical level S^* by exhaustive search where the average costs are estimated by simulation. We compare the average costs of our approaches with the optimal simulated costs and report the relative difference for instance i as

$$\delta_{S^M,i} = \frac{C_{sim}\left(S^M\right) - C_{sim}\left(S^*\right)}{C_{sim}\left(S^*\right)} \cdot 100\%$$

for the Markov chain approach and

$$\delta_{S^{++},i} = \frac{C_{sim}\left(S^{++}\right) - C_{sim}(S^*)}{C_{sim}(S^*)} \cdot 100\%$$

for the other approach. The average and the maximum deviation are defined similar as in Section 3.3. For the binomial yield model we have computed 162 (270) instances for normal (gamma) distributed demand and we summarized the results in Table 4.

		Normal demand				**Gamma demand**			
		Markov Chain		**Steady-State**		**Markov Chain**		**Steady-State**	
Para meter	**Value**	**Av. Rel. dev.**	**Max. Rel. dev.**	**Av. Rel. dev.**	**Max. rel. dev.**	**Av. Rel. dev.**	**Max. rel. dev.**	**Av. Rel. dev.**	**Max. rel. dev.**
ρ_D	0.1	0.20	1.83	0.26	1.66	0.16	1.40	0.17	1.40
	0.2	0.00	0.16	0.07	0.76	0.01	0.15	0.11	0.72
	0.3	0.01	0.17	0.06	0.47	0.02	0.28	0.09	0.54
	0.5					0.01	0.09	0.02	0.19
	0.75					0.02	0.08	0.02	0.09
p	0.5	0.00	0.06	0.12	0.86	0.01	0.11	0.07	0.57
	0.7	0.01	0.11	0.19	1.66	0.02	0.60	0.13	1.40
	0.9	0.20	1.83	0.07	0.93	0.11	1.40	0.05	0.72
Lead time	2	0.07	1.83	0.24	1.66	0.01	0.14	0.16	1.40
	5	0.05	1.38	0.11	0.63	0.03	1.40	0.06	0.53
	10	0.10	1.06	0.04	0.51	0.09	1.14	0.03	0.36
Critical ratio b/(h+b)	0.85	0.04	1.06	0.15	1.66	0.03	0.95	0.07	1.39
	0.90	0.09	1.38	0.18	0.86	0.03	1.05	0.07	0.57
	0.95	0.04	0.99	0.14	1.41	0.04	1.14	0.08	0.88
	0.97	0.06	0.97	0.08	0.85	0.04	1.10	0.06	0.54
	0.99	0.09	1.56	0.14	0.76	0.03	0.91	0.11	0.72
	0.995	0.11	1.83	0.08	0.51	0.09	1.40	0.11	1.40
Total		0.07	1.83	0.13	1.66	0.04	1.40	0.08	1.40

Table 4: Results for binomial yield and positive lead time

The performance of both approximations is excellent, but the Markov chain approach is even better. The optimal critical level is found with the approximation based on the Markov chain in 85% of the instances with normal demand and in about 65% in case of gamma distributed demand while for the approach based on the steady-state approximation the numbers are a little bit lower, 57% for normally distributed demand and about 49% for gamma demand distribution. The approximation based on the Markov chain works better with increasing demand variability. Similar as in case of zero lead time, the steady state approximation S_{Norm}^{++} is always chosen in case of normal distributed demand, because the symmetric demand and the little skewness of the binomial distribution seem to result in a nearly normally distributed inventory level. Under gamma distributed demand this can only be observed for small demand variability.

The results for the 864 instances with stochastically proportional yield are presented in Table 5

		Normal demand				**Gamma demand**			
		Markov Chain		**Steady-State**		**Markov Chain**		**Steady-State**	
Para meter	**Value**	**Av. Rel. dev.**	**Max. Rel. dev.**	**Av. Rel. dev.**	**Max. rel. dev.**	**Av. Rel. dev.**	**Max. rel. dev.**	**Av. Rel. dev.**	**Max. rel. dev.**
ρ_D	0.1	0.81	20.31	0.23	5.85	0.64	12.01	0.22	6.20
	0.2	0.32	6.61	0.09	0.86	0.24	4.75	0.07	0.61
	0.3	0.10	1.42	0.09	1.42	0.09	0.80	0.11	1.19
	0.5					0.05	0.31	0.18	4.46
	0.75					0.03	0.22	0.42	9.98
(μ_Z,ρ_Z)	(0.85;0.1)	0.15	2.75	0.05	0.59	0.06	1.66	0.03	0.53
	(0.85;0.2)	1.48	20.31	0.39	5.85	0.75	12.01	0.20	6.20
	(0.75;0.2)	0.60	8.27	0.12	0.86	0.28	2.85	0.06	0.68
	(0.5;0.2)	0.03	0.64	0.07	1.03	0.01	0.20	0.11	1.07
	(0.5;0.4)	0.03	0.49	0.03	0.27	0.03	0.61	0.41	7.19
	(0.5;0.5774)	0.18	0.63	0.17	0.63	0.13	0.72	0.39	9.98
Lead time	2	0.68	20.31	0.24	5.85	0.30	12.01	0.34	9.98
	5	0.35	7.96	0.10	2.11	0.19	7.47	0.18	7.19
	10	0.21	3.36	0.07	0.81	0.14	3.69	0.08	3.05
Critical ratio b/(h+b)	0.85	0.04	0.39	0.13	1.03	0.03	0.39	0.08	1.07
	0.90	0.05	0.35	0.04	0.34	0.04	0.31	0.03	0.29
	0.95	0.13	1.34	0.07	0.66	0.10	1.31	0.07	0.67
	0.97	0.24	2.23	0.07	0.54	0.15	2.11	0.10	1.54
	0.99	0.76	8.16	0.21	3.66	0.37	7.80	0.33	5.41
	0.995	1.26	20.31	0.30	5.85	0.58	12.01	0.60	9.98
Total		0.41	20.31	0.14	5.85	0.21	12.01	0.20	9.98

Table 5: Results for stochastic proportional yield and positive lead time

While both approximations work very well, it can be seen that the steady-state approximation outperforms the approximation based on the Markov chain. In case of normally (gamma) distributed demand the maximum relative deviation is quite large for the Markov chain approach. However, only in 6 (4) instances a larger deviation than 5% is observed and only for very high service levels (99% and 99.5%). Similar, only one (normal demand) and four (gamma demand) instances have a larger deviation than 5% in case of the steady-state approximation. Thus, we can conclude that, for practical relevant parameter instances the relative deviations are more than acceptable and our approximations which are easy to implement show a very good performance.

5 Summary and Outlook

From a practical point of view, it is evident that linear-inflation policies are the most attractive candidates for production and inventory control in systems with stochastic demand and random yield. The two approaches for policy parameter determination presented in this paper are superior to all other approaches developed so far. This is first because they can be used in a wide field of problem instances including relevant circumstances like non-zero lead times and binomial yields where most others cannot be applied. Second, in almost all cases both approaches reveal a very high performance level compared to the optimal linear-inflation rule of the two-parameter type. Third, the computational effort of parameter determination is competitive to other available approaches. The steady-state approach with its closed-form solution can be applied as spread-sheet application, and the Markov chain approach basically only needs the solution of a linear equation system.

Both approaches also have the potential to be used for optimizing the yield inflation factor as second policy parameter since our analysis is valid for arbitrary inflation parameters. It is a matter of future research to investigate if structural solution properties can be found and exploited for easing parameter optimization in this context so that the high effort of simple numerical search procedures can be avoided. A major open problem area refers to the question how the inventory position within a linear-inflation policy should be defined under random yield if lead time is larger than one. First, the consideration of open orders by including expected yields as suggested in our approach is not the only viable way. Second, if multiple open orders from different periods have to be taken into account it will, in general, not be optimal to aggregate them to a single inventory position because each order contributes to the yield risk specifically. Up to now, only the approach in Inderfurth and Vogelgesang (2013) addresses this issue and gives a suggestion how to incorporate the separate yield risks of multiple open orders within a linear-inflation rule. However, it needs much more research in this field to find out how the structure of an optimal linear policy in the case of non-zero lead time might look like and to develop approaches for determining the (most likely more than two) parameters of this policy structure.

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Appendix I

Evaluation of the cost function $C(\mu_I)$ in (5)

Cost function $C(\mu_I) = (h+b)\int_0^\infty x\varphi_I(x)dx - b\mu_I$ can be integrated by terms via

$$C(\mu_I) = (h+b)\{-x(1-\Phi_I(x))_0^\infty + \int_0^\infty (1-\Phi_I(x))dx\} - b\,\mu_I$$

resulting in

$$C(\mu_I) = (h+b)\int_0^\infty (1-\Phi(x))dx - b\mu_I\,.$$

After standardizing the inventory variable we receive

$$C(\mu_I) = (h+b)\sigma_I \int_{-\frac{\mu_I}{\sigma_I}}^\infty (1-\Phi_N(z))dz \text{ -} b\mu_I$$

from which the first-order derivative is given by

$$\frac{dC(\mu_I)}{d\mu_I} = (h+b)(1-\Phi_N(\frac{-\mu_I}{\sigma_I})) - b$$

Eventually, first-order optimization condition $dC(\mu_I)/d\mu_I = 0$ leads to the solution

$$1-\Phi_N(-\frac{\mu_I}{\sigma_I}) = \frac{b}{h+b} \quad \text{in (6).}$$

For a symmetric distribution function it holds that

$\Phi_N\left(-\frac{\mu_I}{\sigma_I}\right) = 1-\Phi_N\left(\frac{\mu_I}{\sigma_I}\right)$ so that we immediately find the solution

$$\Phi_N\left(\frac{\mu_I}{\sigma_I}\right) = \frac{b}{h+b} \quad \text{in (7).}$$

Appendix II

Proof of Lemma 4:

Considering the standardization of the inventory variable, the general optimality condition $1-\Phi_N(-\frac{\mu_I}{\sigma_I})=\frac{b}{h+b}$ in (6) can be rewritten as

$$1-\mathrm{P}\left(I_N\sigma_I+\mu_I\le 0\right)=\frac{b}{h+b}.$$

Since, by definition, $I_N\sigma_I+\mu_I=I$ we get for the original inventory variable

$$1-\mathrm{P}\left(I\le 0\right)=\mathrm{P}\left(I>0\right)=\mathrm{P}\left(-I\le 0\right)=\frac{b}{h+b}$$

From the definition $I=T(S)-X$ in (15) we directly find that

$\Rightarrow\ \mathrm{P}\left(X-T(S)\le 0\right)=\mathrm{P}\left(X\le T(S)\right)=\frac{b}{b+h}$ as given in (16).

Appendix III

Proof of Lemma 5:

A: σ_Q derivation for binomial yield

A.1: Lead time $\lambda=0$

The derivation is based on the time development of order quantities

$$Q_t=Q_{t-1}-FY(Q_{t-1})+FD_{t-1} \tag{A1}$$

which results from inserting the inventory balance equation in (13) in form of $I_{t-1}=I_{t-2}+Y\left(Q_{t-1}\right)-D_{t-1}$ into the control rule $Q_t=F(S-I_{t-1})$ from (9).

Exploiting (A1) and noting that Q_{t-1} and D_{t-1} are independent, we find

$V[Q_t]=V\left[Q_{t-1}-FY(Q_{t-1})\right]+F^2\sigma_D^2$

Furthermore, it holds

$E\left[Q_{t-1}-FY(Q_{t-1})\right]=\left(1-Fp\right)E[Q_{t-1}]$.

Additionally, for binomially distributed yield we have $V[Y(Q_{t-1})]=p(1-p)Q_{t-1}$ so that we get

$$
\begin{aligned}
V\left[Q_{t-1}-FY(Q_{t-1})\right]&=E\left[\left(Q_{t-1}-FY(Q_{t-1})\right)^2\right]-E\left[Q_{t-1}-FY(Q_{t-1})\right]^2=\\
&=E\left[Q_{t-1}^2-2FQ_{t-1}Y(Q_{t-1})+F^2Y(Q_{t-1})^2\right]-\left(1-Fp\right)^2E[Q_{t-1}]^2=\\
&=E[Q_{t-1}^2]-2FpE[Q_{t-1}^2]+F^2E\left[V[Y(Q_{t-1})]\right]+F^2p^2E[Q_{t-1}^2]-\left(1-Fp\right)^2E[Q_{t-1}]^2=\\
&=\left(1-Fp\right)^2\left(E[Q_{t-1}^2]-E[Q_{t-1}]^2\right)+F^2E\left[p(1-p)Q_{t-1}\right]=\\
&=\left(1-Fp\right)^2V[Q_{t-1}]+F^2p(1-p)\mu_D/p
\end{aligned}
$$

Thus, noting that $M=Fp$ we finally find the following expression under steady-state conditions, i.e. $V[Q]=V[Q_t]\ \forall t$,

$$V[Q]=\left(1-M\right)^2V[Q]+F^2(1-p)\mu_D+F^2\sigma_D^2.$$

From that we receive the variance formula for the order quantity as shown in (19):

$$\Rightarrow\quad V[Q]=\frac{F^2\left(\sigma_D^2+(1-p)\mu_D\right)}{1-\left(1-M\right)^2}\quad.$$

It is obvious that this variance is only feasible and finite if the denominator is larger than zero. Thus a necessary condition for the stability of the steady-state development of order quantities is given by $M<2$.

A.2: Lead time $\lambda\geq 1$

This derivation is based on the time development of order quantities

$$Q_t=\left(1-Fp\right)Q_{t-1}+F\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)+FD_{t-1}\tag{A2}$$

which results from applying the order quantity formula in (33) to the binomial yield case with $E[Q_t]=p$.

Based upon the Q_t formula in (A2) for $\lambda\geq 1$, we can derive the steady-state variance of Q in the following way:

$$
\begin{aligned}
V[Q_t]&=V[\left(1-Fp\right)Q_{t-1}+F\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)+FD_{t-1}]\\
&=\left(1-Fp\right)^2V[Q_{t-1}]+F^2V[pQ_{t-\lambda}-Y(Q_{t-\lambda})]+Cov[\left(1-Fp\right)Q_{t-1},F\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)]+F^2\sigma_D^2
\end{aligned}
$$

Evaluating single terms results in

$$V[pQ_{t-\lambda}-Y(Q_{t-\lambda})]=E[\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)^2]-E[pQ_{t-\lambda}-Y(Q_{t-\lambda})]^2=(1-p)\mu_D$$

where

$E[pQ_{t-\lambda}-Y(Q_{t-\lambda})]=0$ and

$$\begin{aligned}E[\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)^2]&=p^2E[Q_{t-\lambda}^2]-2pE[Q_{t-\lambda}Y(Q_{t-\lambda})]+E[Y(Q_{t-\lambda})^2]=\\&=p^2E[Q_{t-\lambda}^2]-2p^2E[Q_{t-\lambda}^2]+p^2E[Q_{t-\lambda}^2]+E\left[V[Y(Q_{t-\lambda})]\right]=\\&=E[p(1-p)Q_{t-\lambda}]=p(1-p)\mu_D/p=(1-p)\mu_D\end{aligned}$$

Additionally, we find

$$\begin{aligned}Cov[(1-Fp)Q_{t-1},F\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)]&=E[F\left(1-Fp\right)Q_{t-1}\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)]\\&\quad-E[(1-Fp)Q_{t-1}]E[F\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)]\end{aligned}.$$

Due to $E[\left(p\cdot Q_{t-\lambda}-Y(Q_{t-\lambda})\right)]=0$, the covariance term equals zero,

i.e. $Cov[(1-Fp)Q_{t-1},F\left(pQ_{t-\lambda}-Y(Q_{t-\lambda})\right)]=0$, so that for the steady state variance the following expression holds where *Fp* is replaced by *M*:

$$V[Q]-\left(1-M\right)^2\cdot V[Q]=F^2\cdot(1-p)\cdot\mu_D+F^2\cdot\sigma_D^2.$$

Thus, we finally get the variance formula described in (19).

$$\Rightarrow\quad V[Q]=\frac{F^2\left(\sigma_D^2+(1-p)\mu_D\right)}{1-\left(1-M\right)^2}$$

and see that this variance is independent of the lead time λ and identical to the formula for $\lambda=0$. The same holds for the steady-state stability condition $M<2$.

B: σ_Q derivation for stochastically proportional yield

B.1: Lead time $\lambda=0$

The derivation is based on the time development of order quantities that, corresponding to the binomial yield case in (A1), under stochastically proportional yield is expressed by

$$Q_t=Q_{t-1}-FZ_{t-1}Q_{t-1}+FD_{t-1}=\left(1-FZ_{t-1}\right)Q_{t-1}+FD_{t-1}\qquad\text{(A3)}$$

In the following we apply the variance rule for the product of independent random variables *X* and *Y*:

$$V[X\cdot Y]=\left(V[X]+E[X]^2\right)\cdot V[Y]+V[X]\cdot E[Y]^2$$

From applying the above variance rule to the Q_t formula in (A3) with $X=\left(1-F\cdot Z_{t-1}\right)$ and $Y=Q_{t-1}$ and by noting that all random variables in (A3) are independent, we get $V\left[Q\right]=F^2\sigma_Z^2\left(V[Q]+E[Q]^2\right)+\left(1-F\mu_Z\right)^2V[Q]+F^2\sigma_D^2$.

Finally, noting that $E[Q]=\mu_D/\mu_Z$ and replacing $F\cdot\mu_Z$ by M and $F^2\cdot\sigma_Z^2$ by $M^2\rho_Z^2$ yields

$$V\left[Q\right]-\rho_Z^2V[Q]-\left(1-M\right)^2V[Q]=F^2\rho_Z^2\mu_D^2+F^2\sigma_D^2.$$

Thus we receive the following variance formula shown in (20):

$$\Rightarrow\quad V\left[Q\right]=\frac{F^2\left(\sigma_D^2+\rho_Z^2\mu_D^2\right)}{1-\left(1-M\right)^2-M^2\rho_Z^2}\quad.$$

Like in the case of binomial yield, it is obvious that this variance only is feasible and finite if the denominator is larger than zero. Thus, here a necessary condition for the stability of the steady-state development of order quantities is given by $M<2/\ 1+\rho_Z^2$.

B.2: Lead time $\lambda\geq 1$

This derivation is based on the time development of order quantities

$$Q_t=\left(1-F\mu_Z\right)Q_{t-1}+F\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}+FD_{t-1}\tag{A4}$$

which results from applying the order quantity formula in (33) to the stochastically proportional yield case with $E[Q_t]=\mu_Z$.

Based upon the Q_t formula in (A4), we can derive the steady-state variance of Q in the following way:

$$\begin{aligned}V[Q_t]&=V[\left(1-F\mu_Z\right)Q_{t-1}+F\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}+FD_{t-1}]\\&=\left(1-F\mu_Z\right)^2V[Q_{t-1}]+F^2V[\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}]+Cov[\left(1-F\mu_Z\right)Q_{t-1},F\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}]+F^2\sigma_D^2\end{aligned}$$

Evaluating single terms results in

$V[\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}]=\sigma_Z^2\left(V[Q_{t-\lambda}]+E[Q_{t-\lambda}]^2\right)=\sigma_Z^2V[Q_{t-\lambda}]+\rho_Z^2\mu_D^2$ and

$$\begin{aligned}Cov[\left(1-F\mu_Z\right)Q_{t-1},F\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}]&=E[F\left(1-F\cdot\mu_Z\right)Q_{t-1}Q_{t-\lambda}\left(\mu_Z-Z_{t-\lambda}\right)]\\&\quad-E[\left(1-F\mu_Z\right)Q_{t-1}]E[F\left(\mu_Z-Z_{t-\lambda}\right)Q_{t-\lambda}]\end{aligned}.$$

Due to $E[\mu_Z-Z_{t-\lambda}]=0$ we find that the covariance term equals zero, i.e. $Cov[(1-F\mu_Z)Q_{t-1}, F(\mu_Z - Z_{t-\lambda})Q_{t-\lambda}]=0$, so that for the steady-state variance the following expression holds

$$V[Q]-(1-F\mu_Z)^2 V[Q]-F^2\sigma_Z^2 V[Q]=F^2\rho_Z^2\mu_D^2+F^2\sigma_D^2.$$

where we additionally exploit that $F\mu_Z=M$ and $F^2\sigma_Z^2=M^2\rho_Z^2$.

From that we finally receive the variance formula in (20)

$$\Rightarrow \quad V[Q]=\frac{F^2\left(\sigma_D^2+\rho_Z^2\mu_D^2\right)}{1-(1-M)^2-M^2\rho_Z^2}$$

and see that this variance is the same as in the case $\lambda=0$ and, thus, is completely independent of the lead time λ.

From that it additionally follows that under stochastically proportional yield in any lead time case the necessary condition for the stability of the steady-state development of order quantities is given by $M<2/\ 1+\rho_Z^2$.

Appendix IV

Proof of Lemma 7:

A: γ_I derivation for binomial yield und lead time $\lambda=0$

The derivation departs from skewness definition

$$\gamma_I=\frac{E\left[(I-\mu_I)^3\right]}{\sigma_I^3} \tag{A5}$$

where we make use of the inventory parameters for μ_I and σ_I^2 from (10) and (21)

$$\mu_I=S-\frac{\mu_D}{M} \quad \text{and} \quad \sigma_I^2=\frac{\sigma_D^2+(1-p)\mu_D}{1-(1-M)^2}$$

and of the order quantity parameters for μ_Q and σ_Q^2 according to (12) and (19)

$$\mu_Q=\frac{\mu_D}{p} \quad \text{and} \quad \sigma_Q^2=F^2\sigma_I^2.$$

For the third central moment of I we have

$$E\left[(I-\mu_I)^3\right]=E\left[I^3\right]-3\mu_I\sigma_I^2-\mu_I^3. \tag{A6}$$

Furthermore, due to $I=S-Q/F$ we find

$$E\left[I^3\right]=E\left[(S-Q/F)^3\right]=S^3-3S^2\frac{E[Q]}{F}+3S\frac{E\left[Q^2\right]}{F^2}-\frac{E\left[Q^3\right]}{F^3}.$$

Noting that $Fp=M$, with $\mu_Q/F=\mu_D/M$ and $E\left[Q^2\right]=\sigma_Q^2+\mu_Q^2=F^2\sigma_I^2+\mu_Q^2$ we get

$$E\left[I^3\right]=S^3-3S^2\frac{\mu_D}{M}+3S\left(\sigma_I^2+\frac{\mu_D^2}{M^2}\right)-\frac{E\left[Q^3\right]}{F^3}.$$

From (A6) it can be derived that

$$\begin{aligned}E\left[(I-\mu_I)^3\right]&=S^3-3S^2\frac{\mu_D}{M}+3S\left(\sigma_I^2+\frac{\mu_D^2}{M^2}\right)-\frac{E\left[Q^3\right]}{F^3}-3\left(S-\frac{\mu_D}{M}\right)\sigma_I^2-\left(S-\frac{\mu_D}{M}\right)^3=\\&=\frac{\mu_D^3}{M^3}+3\frac{\sigma_I^2\mu_D}{M}-\frac{E\left[Q^3\right]}{F^3}\end{aligned} \tag{A7}$$

Next we evaluate $E\left[Q^3\right]$, considering that $Q_t=Q_{t-1}-FY(Q_{t-1})+FD_{t-1}$ holds

$$\begin{aligned}E\left[Q_t^3\right]=&E\left[(Q_{t-1}-FY(Q_{t-1}))^3\right]+3E\left[(Q_{t-1}-FY(Q_{t-1}))^2FD_{t-1}\right]+\\&+3E\left[(Q_{t-1}-FY(Q_{t-1}))F^2D_{t-1}^2\right]+E\left[F^3D_{t-1}^3\right]\end{aligned} \tag{A8}$$

The different terms in (A8) can be expressed as follows (note that Q_{t-1} and D_{t-1} are independent) exploiting the following moments of $Y(Q)$ under a binomial distribution:

$E\left[Y(Q)\right]=pQ$,

$E\left[Y(Q)^2\right]=p(1-p)Q+p^2Q^2$ and

$E\left[Y(Q)^3\right]=p(1-p)(1-2p)Q+3p^2(1-p)Q^2+p^3Q^3$.

Thus, we find the following expressions

$$\begin{aligned}&E\left[(Q_{t-1}-FY(Q_{t-1}))^3\right]=E\left[Q^3-3Q^2FY(Q)+3QF^2Y(Q)^2-F^3Y(Q)^3\right]=\\&=E\left[Q^3-3FpQ^3+3F^2p(1-p)Q^2+3F^2p^2Q^3-F^3p(1-p)(1-2p)Q-3F^3p^2(1-p)Q^2-F^3p^3Q^3\right]=\\&=(1-M)^3E\left[Q^3\right]+3M(1-M)F(1-p)E\left[Q^2\right]-F^3(1-p)(1-2p)\mu_D\end{aligned}$$

$$E\left[\left(Q_{t-1}-FY\left(Q_{t-1}\right)\right)^2 FD_{t-1}\right]=F\mu_D E\left[Q^2-2FQY(Q)+F^2Y(Q)^2\right]=$$
$$=F\mu_D E\left[Q^2-2FpQ^2+F^2p(1-p)Q+F^2p^2Q^2\right]=$$
$$=(1-M)^2 F\mu_D E\left[Q^2\right]+F^3(1-p)\mu_D^2$$

and

$$E\left[\left(Q_{t-1}-FY\left(Q_{t-1}\right)\right)F^2D_{t-1}^2\right]=F^2E\left[D^2\right]E\left[(1-Fp)Q\right]=(1-M)F^2\left(\sigma_D^2+\mu_D^2\right)\mu_D/p$$

Altogether, inserting these expressions in (A8) yields

$$\begin{aligned}E\left[Q^3\right]&=(1-M)^3E\left[Q^3\right]+3(1-M)F\left((1-p)M+(1-M)\mu_D\right)E\left[Q^2\right]+3(1-M)F^2\left(\sigma_D^2+\mu_D^2\right)\mu_D/p\\&\quad+3F^3(1-p)\mu_D^2-F^3(1-p)(1-2p)\mu_D+F^3\mu_{3,D}=\\&=(1-M)^3E\left[Q^3\right]+3(1-M)F^3\left\{\left((1-p)M+(1-M)\mu_D\right)\left(\sigma_I^2+\mu_D^2/M^2\right)+\mu_D\left(\sigma_D^2+\mu_D^2\right)/M\right\}+\\&\quad+F^3\left\{3(1-p)\mu_D^2-(1-p)(1-2p)\mu_D+\mu_{3,D}\right\}\end{aligned}$$

Isolating for $E\left[Q^3\right]$ results in

$$E\left[Q^3\right]=\frac{F^3}{1-(1-M)^3}\begin{Bmatrix}3(1-M)\left[\left((1-p)M+(1-M)\mu_D\right)\left(\sigma_I^2+\mu_D^2/M^2\right)+\mu_D\left(\sigma_D^2+\mu_D^2\right)/M\right]+\\+3(1-p)\mu_D^2-(1-p)(1-2p)\mu_D+\mu_{3,D}\end{Bmatrix}$$

Finally, from inserting in (A7) we get

$$\begin{aligned}&E\left[\left(I-\mu_I\right)^3\right]=\frac{\mu_D^3}{M^3}+3\frac{\sigma_I^2\mu_D}{M}-\\&-\frac{1}{1-(1-M)^3}\begin{Bmatrix}3(1-M)\left[\left((1-p)M+(1-M)\mu_D\right)\left(\sigma_I^2+\mu_D^2/M^2\right)+\mu_D\left(\sigma_D^2+\mu_D^2\right)/M\right]+\\+(1-p)\mu_D\left(3\mu_D+2p-1\right)+\mu_{3,D}\end{Bmatrix}\end{aligned}$$

By inserting this expression in the skewness measure (A5) we directly find the γ_I formula shown in (23).

B: γ_I derivation for stochastically proportional yield und lead time $\lambda=0$

Next to the skewness definition in (A5) this derivation relies on the μ_I, μ_Q, σ_I^2, and σ_Q^2 formulas given in (10), (12), (22), and (20).

The derivation of $E\left[\left(I-\mu_I\right)^3\right]$ is the same as in (A7) for binomial yield, i.e.

$$E\left[\left(I-\mu_I\right)^3\right]=S^3-3S^2\frac{\mu_D}{M}+3S\left(\sigma_I^2+\frac{\mu_D^2}{M^2}\right)-\frac{E\left[Q^3\right]}{F^3}-3\left(S-\frac{\mu_D}{M}\right)\sigma_I^2-\left(S-\frac{\mu_D}{M}\right)^3= \\ =\frac{\mu_D^3}{M^3}+3\frac{\sigma_I^2\mu_D}{M}-\frac{E\left[Q^3\right]}{F^3} \tag{A9}$$

For evaluating $E\left[Q^3\right]$ we consider that $Q_t=Q_{t-1}-FZ_{t-1}Q_{t-1}+FD_{t-1}$ holds and get

$$E\left[Q_t^3\right]=E\left[\left(Q_{t-1}-FZ_{t-1}Q_{t-1}\right)^3\right]+3E\left[\left(Q_{t-1}-FZ_{t-1}Q_{t-1}\right)^2FD_{t-1}\right]+ \\ +3E\left[\left(Q_{t-1}-FZ_{t-1}Q_{t-1}\right)F^2D_{t-1}^2\right]+E\left[F^3D_{t-1}^3\right] \tag{A10}$$

The different terms in (A10) can be expressed as follows (note that Q_{t-1}, Z_{t-1} and D_{t-1} are independent)

$$E\left[\left(Q_{t-1}-FZ_{t-1}Q_{t-1}\right)^3\right]=E\left[\left(1-FZ\right)^3\right]E\left[Q^3\right]=\left(1-3FE[Z]+3F^2E\left[Z^2\right]-F^3E\left[Z^3\right]\right)E\left[Q^3\right]= \\ =\left(1-3M+3M^2+3F^2\sigma_Z^2-F^3\mu_{3,Z}\right)E\left[Q^3\right]$$

$$E\left[\left(Q_{t-1}-FZ_{t-1}Q_{t-1}\right)^2FD_{t-1}\right]=F\mu_DE\left[\left(1-FZ\right)^2Q^2\right]=F\mu_D\left(1-2M+M^2+F^2\sigma_Z^2\right)E\left[Q^2\right]= \\ =F\mu_D\left(\left(1-M\right)^2+F^2\sigma_Z^2\right)\left(F^2\sigma_I^2+\mu_D^2/\mu_Z^2\right)$$

and

$$E\left[\left(Q_{t-1}-FZ_{t-1}Q_{t-1}\right)F^2D_{t-1}^2\right]=\left(1-F\mu_Z\right)\mu_QF^2\left(\mu_D^2+\sigma_D^2\right)=\frac{F^2}{\mu_Z}(1-M)\left(\mu_D^3+\mu_D\sigma_D^2\right).$$

Altogether, inserting these expressions in (A10) yields

$$E\left[Q^3\right]=\left(1-3M(1-M)+3F^2\sigma_Z^2-F^3\mu_{3,Z}\right)E\left[Q^3\right]+3\left(\left(1-M\right)^2+F^2\sigma_Z^2\right)\left(F^2\sigma_I^2+\mu_D^2/\mu_Z^2\right)F\mu_D+ \\ +3\frac{F^2}{\mu_Z}(1-M)\left(\mu_D^3+\mu_D\sigma_D^2\right)+F^3\mu_{3,D}$$

Isolating for $E\left[Q^3\right]$ results in

$$E\left[Q^3\right]=\frac{1}{3M(1-M)-3F^2\sigma_Z^2+F^3\mu_{3,Z}}\begin{cases}3F\mu_D(1-M)\left[F\left(\mu_D^2+\sigma_D^2\right)/\mu_Z+(1-M)\left(F^2\sigma_I^2+\mu_D^2/\mu_Z^2\right)\right]+\\+3F^3\sigma_Z^2\mu_D\left(F^2\sigma_I^2+\mu_D^2/\mu_Z^2\right)+F^3\mu_{3,D}\end{cases}$$

$$=\frac{F^3}{3M(1-M)-3F^2\sigma_Z^2+F^3\mu_{3,Z}}\begin{cases}3(1-M)\mu_D\left[\frac{\mu_D^2+\sigma_D^2}{M}+(1-M)\left(\sigma_I^2+\frac{\mu_D^2}{M^2}\right)\right]+\\+3F^2\sigma_Z^2\mu_D\left(\sigma_I^2+\frac{\mu_D^2}{M^2}\right)+\mu_{3,D}\end{cases}$$

Finally, from inserting in (A9) we get

$$E\left[(I-\mu_I)^3\right]=\frac{\mu_D^3}{M^3}+3\frac{\sigma_I^2\mu_D}{M}-$$
$$-\frac{3(1-M)\mu_D\left[\left(\mu_D^2+\sigma_D^2\right)/M+(1-M)\left(\sigma_I^2+\mu_D^2/M^2\right)\right]+3M^2\rho_Z^2\mu_D\left(\sigma_I^2+\mu_D^2/M^2\right)+\mu_{3,D}}{3M(1-M)-3M^2\rho_Z^2+F^3\mu_{3,Z}}$$

By inserting this expression in the skewness measure (A5) we directly get the γ_I formula shown in (24).

Appendix V

Proof of Lemma 8:

Let us define with Y the inventory position after ordering. Then the average costs can be reformulated as

$$\begin{aligned}C(F,S)&=hE[I]+bE[B]\\&=\sum_y\left(hE[(y-\eta(\lambda+1))^+]+bE[(\eta(\lambda+1)-y)^+]\right)P(Y=y)\\&=\sum_{k=0}^{\infty}\left((h+b)E[(S+k-\eta(\lambda+1))^+]-(S+k)b+bE[\eta(\lambda+1)]\right)v_k\\&\quad+\sum_{k=-\infty}^{-1}\left((h+b)E[(S+k-pFk-\eta(\lambda+1))^+]-(S+k-pFk)b+bE[\eta(\lambda+1)]\right)v_k\end{aligned}\tag{A11}$$

For a fixed value of F the optimal S has to satisfy the following conditions

$$C(F,S+1)-C(F,S)>0\quad\text{and }C(F,S-1)-C(F,S)>0$$

We first consider the difference related to $S+1$ and S of the first term in (A11)

$$\sum_{k=0}^{\infty}\Big((h+b)E[(S+1+k-\eta(\lambda+1))^{+}]-(S+1+k)b+bE[\eta(\lambda+1)]\Big)v_k$$

$$-\sum_{k=0}^{\infty}\Big((h+b)E[(S+k-\eta(\lambda+1))^{+}]-(S+k)b+bE[\eta(\lambda+1)]\Big)v_k$$

$$= \quad (h+b)\sum_{k=0}^{\infty}\Big(E[(S+1+k-\eta(\lambda+1))^{+}]-E[(S+k-\eta(\lambda+1))^{+}]-b\Big)v_k$$

$$= \quad (h+b)\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{S+k}(S+1+k-i)P(\eta(\lambda+1)=i)-\sum_{i=0}^{S+k}(S+k-i)P(\eta(\lambda+1)=i)-b\Big)v_k$$

$$= \quad (h+b)\sum_{k=0}^{\infty}\Big(P(\eta(\lambda+1)\le S+k)-b\Big)v_k$$

The difference of the second term in (A11) is given as:

$$\sum_{k=-\infty}^{-1}\Big((h+b)E[(S+1+k-pFk-\eta(\lambda+1))^{+}]-(S+1+k-pFk)b+bE[\eta(\lambda+1)]\Big)v_k$$

$$-\sum_{k=-\infty}^{-1}\Big((h+b)E[(S+k-pFk-\eta(\lambda+1))^{+}]-(S+k-pFk)b+bE[\eta(\lambda+1)]\Big)v_k$$

$$= (h+b)\sum_{k=-\infty}^{-1}\Big(E[(S+1+k-pFk-\eta(\lambda+1))^{+}]-E[(S+k-pFk-\eta(\lambda+1))^{+}]-b\Big)v_k$$

$$= (h+b)\sum_{k=-\infty}^{-1}\Big(\sum_{i=0}^{S+k-pFk}(S+1+k-pFk-i)P(\eta(\lambda+1)=i)-\sum_{i=0}^{S+k-pFk}(S+k-pFk-i)P(\eta=i)-b\Big)v_k$$

$$= (h+b)\sum_{k=-\infty}^{-1}\Big(P(\eta(\lambda+1)\le S+k-pFk)-b\Big)v_k$$

Thus, we obtain the following condition

$$(h+b)\sum_{k=0}^{\infty}\Big(P(\eta\le S+k)-b\Big)v_k+(h+b)\sum_{k=-\infty}^{-1}\Big(P(\eta(\lambda+1)\le S+k-pFk)-b\Big)v_k\ge 0$$

$$(h+b)\Big(\sum_{k=0}^{\infty}P(\eta(\lambda+1)\le S+k)+\sum_{k=-\infty}^{-1}P(\eta(\lambda+1)\le S+k-pFk)\Big)v_k\ge b$$

This is equivalent with

$$\sum_{k=-\infty}^{S-1}P\Big(\eta(\lambda+1)\le k+p\cdot F(S-k)\Big)v_k+\sum_{k=S}^{+\infty}P\Big(\eta(\lambda+1)\le k\Big)v_k\ge\frac{b}{b+h}$$

what is equivalent to condition (31) in Lemma 8. The smallest value of S satisfying the inequality is the optimal S for a given value of F .

Appendix VI

Proof of Lemma 9:

For binomial yield the variance of the correction can be computed as

$$
\begin{aligned}
\sigma_R^2 &= VAR[R_{t+1-\lambda}] = VAR\big[pQ_{t+1-\lambda} - Y(Q_{t+1-\lambda})\big] \\
&= E\big[\big(pQ_{t+1-\lambda} - Y(Q_{t+1-\lambda})\big)^2\big] \\
&= \sum_{q=0}^{+\infty} E\big[\big(pQ_{t+1-\lambda} - Y(Q_{t+1-\lambda})\big)^2 \mid Q_{t+1-\lambda} = q\big] P(Q_{t+1-\lambda} = q) \\
&= \sum_{q=0}^{+\infty} E\big[p^2q^2 - 2pqY(q) + Y(q)^2 \mid Q_{t+1-\lambda} = q\big] P(Q_{t+1-\lambda} = q) \\
&= p^2E[Q_{t+1-\lambda}^2] - 2p^2E[Q_{t+1-\lambda}^2] + p(1-p)E[Q_{t+1-\lambda}] + p^2E[Q_{t+1-\lambda}^2] \\
&= p(1-p)E[Q_{t+1-\lambda}] = (1-p)\mu_D
\end{aligned}
$$

For stochastically proportional yield we obtain

$$
\begin{aligned}
VAR[R] &= VAR[\mu_Z \cdot Q - Z \cdot Q] \\
&= E[(\mu_Z \cdot Q - Z \cdot Q)^2] \\
&= E[\mu_Z^2 Q^2 - 2\mu_Z Z \cdot Q^2 + Z^2 Q^2] \\
&= \mu_Z^2 E[Q^2] - \int_0^\infty E[2\mu_Z Z \cdot Q^2 + Z^2 Q^2 \mid Q = q] f_Q(q)dq \\
&= \mu_Z^2 E[Q^2] - 2\mu_Z \int_0^\infty E[Z \cdot q^2] f_Q(q)dq + \int_0^\infty E[Z^2 q^2] f_Q(q)dq \\
&= \mu_Z^2 E[Q^2] - 2\mu_Z \int_0^\infty \mu_Z q^2 f_Q(q)dq + \int_0^\infty q^2 (\sigma_Z^2 + \mu_Z^2) f_Q(q)dq \\
&= \mu_Z^2 E[Q^2] - 2\mu_Z^2 E[Q^2] + E[Q^2](\sigma_Z^2 + \mu_Z^2) \\
&= \sigma_Z^2 E[Q^2] = \sigma_Z^2 \left(\mu_Q^2 + \sigma_Q^2\right)
\end{aligned}
$$

Thus, the respective formulas of Lemma 9 are verified.

Appendix VII

Proof of Lemma 10:

In general, for the variance of a sum of random variables R_{t-i} it holds that

$$
VAR\Big[\sum_{i=0}^{\lambda-1} R_{t-i}\Big] = VAR\Big[\sum_{i=0}^{\lambda-1} R_{t-i}\Big] + 2\sum_{i=0}^{\lambda-2}\sum_{j=i+1}^{\lambda-1} COV[R_{t-i}R_{t-j}] \ .
$$

For the correction variables R_{t-i} as defined in (27), however, the covariances can be shown to be equal to zero, because all values Q_{t-i} and Q_{t-k} are independent as long as $|i-k| < \lambda$,

$$
\begin{aligned}
COV[R_{t-i}, R_{t-j}] &= E[R_{t-i} \cdot R_{t-j}] - E[R_{t-i}] \cdot E[R_{t-j}] \\
&= E\Big[\big(E[Y(Q_{t-i})] - Y(Q_{t-i})\big) \cdot \big(E[Y(Q_{t-j})] - Y(Q_t - j)\big)\Big] \\
&= E\Big[\big(pQ_{t-i} - Y(Q_{t-i})\big) \cdot \big(pQ_{t-j} - Y(Q_t - j)\big)\Big] \\
&= E\Big[p^2 Q_{t-i} Q_{t-j} - pQ_{t-i} Y(Q_{t-j}) - pQ_{t-j} Y(Q_{t-i}) + Y(Q_{t-i}) Y(Q_{t-j})\Big] \\
&= p^2 E[Q_{t-i} Q_{t-j}] - p^2 E[Q_{t-i} Q_{t-j}] - p^2 E[Q_{t-j} Q_{t-i}] + p^2 E[Q_{t-i} Q_{t-j}] \\
&= 0
\end{aligned}
$$

Thus Lemma 10 is proven.

Appendix VIII

Proof of Lemma 12:

A: σ_I derivation for binomial yield under lead time $\lambda \geq 1$

In order to find an expression for the variance of *I*, we exploit the order equation in (33) which for binomial yield is

$$
Q_t = F\left(S - I_t - D_t - p \cdot \sum_{i=1}^{\lambda-1} Q_{t-i} \right)
$$

and thus can be transformed into

$$
I_t = S - D_t - p \cdot \sum_{i=1}^{\lambda-1} Q_{t-i} - Q_t / F \, .
$$

Due to the independence of D_t and the above *Q* values, as variance we find

$$
V[I_t] = \sigma_D^2 + p^2 \cdot V\left[\sum_{i=1}^{\lambda-1} Q_{t-i} \right] + \frac{1}{F^2} \cdot V[Q_t] \, .
$$

Since all values Q_{t-i} and Q_{t-k} are independent as long as $|i-k| < \lambda$, we get as variance term

$V\left[\sum_{i=1}^{\lambda-1} Q_{t-i} \right] = \sum_{i=1}^{\lambda-1} V[Q_{t-i}]$ and find for the steady state variance $V[I]$

$$V[I]=\sigma_D^2+(\lambda-1)p^2V[Q]+\frac{1}{F^2}V[Q].$$

(A12)

Finally, after inserting $V[Q]=\sigma_Q^2$ from (19), the steady-state variance of I results in

$$V[I]=\sigma_I^2=\sigma_D^2+\left((\lambda-1)\cdot M^2+1\right)\cdot\frac{\left(\sigma_D^2+(1-p)\mu_D\right)}{1-(1-M)^2}$$

which easily can be seen to be identical to the variance formula (34) in Lemma 12.

B: σ_I derivation for stochastically proportional yield under lead time $\lambda\geq 1$

For the start, the derivation is completely identical to that for the binomial yield case. It only has to be considered that instead of p the expected yield rate equals μ_Z so that analogously to (A12) for the variance of I we get

$$V[I]=\sigma_D^2+(\lambda-1)\mu_Z^2V[Q]+\frac{1}{F^2}V[Q].$$

Thus, after inserting $V[Q]=\sigma_Q^2$ from (20), the steady-state variance of I results in

$$V[I]=\sigma_I^2=\sigma_D^2+\left((\lambda-1)M^2+1\right)\cdot\frac{\left(\sigma_D^2+\rho_Z^2\mu_D^2\right)}{1-(1-M)^2-M^2\cdot\rho_Z^2}$$

which easily can be seen to be identical to the variance formula (35) in Lemma 12.

Appendix IX

Proof of Lemma 13:

A: γ_I derivation for binomial yield under lead time $\lambda\geq 1$

The derivation departs from skewness definition given by

$$\gamma_I=\frac{E\left[(I-\mu_I)^3\right]}{\sigma_I^3}$$

(A13)

where we make use of the inventory expressions for μ_I and σ_I^2 from (32) and (34)

$$\mu_I = S - \left(\lambda + \frac{1}{M}\right)\mu_D \quad \text{and} \quad \sigma_I^2 = \sigma_D^2 + \left(\lambda - 1 + \frac{1}{M^2}\right) \cdot \frac{M^2\left(\sigma_D^2 + (1-p)\cdot\mu_D\right)}{1-(1-M)^2}$$

and of the order quantity expressions for μ_Q and σ_Q^2 according to (12) and (19)

$$\mu_Q = \frac{\mu_D}{p} \quad \text{and} \quad \sigma_Q^2 = F^2\sigma_I^2 .$$

For the third central moment of I we have

$$E\left[(I-\mu_I)^3\right] = E\left[I^3\right] - 3\mu_I\sigma_I^2 - \mu_I^3 \tag{A14}$$

Furthermore, due to $I_t = S - D_t - p\sum_{i=1}^{\lambda-1} Q_{t-i} - Q_t / F$ we find

$$I^3 = (S-D)^3 - 3(S-D)^2 A + 3(S-D)A^2 - A^3 \tag{A15}$$

where we use as abbreviation A for: $A = p\sum_{i=1}^{\lambda-1} Q_{t-i} + Q_t / F = p\left(\sum_{i=1}^{\lambda-1} Q_{t-i} + Q_t / M\right)$.

In order to calculate $E\left[I^3\right]$ we need to take the expectations of A, A^2 and A^3.

$$E[A] = (\lambda-1)p\mu_Q + p\mu_Q\frac{1}{M} = \left(\lambda - 1 + \frac{1}{M}\right)\mu_D$$

For A^2 we get

$$A^2 = p^2\left(\left(\sum_{i=1}^{\lambda-1} Q_{t-i}\right)^2 + \frac{1}{M^2}Q_t^2 + \frac{2}{M}Q_t\sum_{i=1}^{\lambda-1} Q_{t-i}\right).$$

Note that we can formulate

$$\left(\sum_{i=1}^{\lambda-1} Q_{t-i}\right)^2 = (\lambda-1)Q_j^2 + (\lambda-1)(\lambda-2)Q_jQ_k \quad \text{with} \quad j \neq k$$

where Q_j and Q_k can be seen as *iid* random variables.

Thus A^2 can be written as

$$A^2 = p^2\left((\lambda-1)Q_j^2 + (\lambda-1)(\lambda-2)Q_jQ_k + \frac{1}{M^2}Q_t^2 + \frac{2}{M}Q_t(\lambda-1)Q_j\right) \quad \text{with} \quad j \neq k \neq t \ .$$

Due to independence of all respective Q_t variables the expectation of A^2 is given by

$$E\left[A^2\right]=p^2\left((\lambda-1)E\left[Q^2\right]+(\lambda-1)(\lambda-2)\mu_Q^2+\frac{1}{M^2}E\left[Q^2\right]+\frac{2}{M}(\lambda-1)\mu_Q^2\right)=$$
$$=p^2\left(\left((\lambda-1)\left(\lambda-1+\frac{2}{M}\right)+\frac{1}{M^2}\right)\mu_Q^2+\left(\lambda-1+\frac{1}{M^2}\right)\sigma_Q^2\right)$$

For A^3 we find

$$A^3=p^3\left(\left(\sum_{i=1}^{\lambda-1}Q_{t-i}\right)^3+3\frac{1}{M}Q_t\left(\sum_{i=1}^{\lambda-1}Q_{t-i}\right)^2+3\frac{1}{M^2}Q_t^2\sum_{i=1}^{\lambda-1}Q_{t-i}+\frac{1}{M^3}Q_t^3\right)$$

where we can further evaluate

$$E\left[A^3\right]=p^3\begin{pmatrix}(\lambda-1)E\left[Q^3\right]+3(\lambda-1)(\lambda-2)\mu_Q E\left[Q^2\right]+(\lambda-1)(\lambda-2)(\lambda-3)\mu_Q^3+\frac{3}{M}\mu_Q(\lambda-1)E\left[Q^2\right]+\\+\frac{3}{M}(\lambda-1)(\lambda-2)\mu_Q^3+\frac{3}{M^2}(\lambda-1)\mu_Q E\left[Q^2\right]+\frac{1}{M^3}E\left[Q^3\right]\end{pmatrix}$$

Noting that $E\left[Q^2\right]=\mu_Q^2+\sigma_Q^2$ and $p\mu_Q=\mu_D$, after some algebra the above expectation can be reformulated as follows

$$E\left[A^3\right]=p^3\left(\lambda-1+\frac{1}{M^3}\right)E\left[Q^3\right]+\mu_D^3\left(\lambda(\lambda-1)(\lambda-2)+\frac{3}{M}(\lambda-1)^2+\frac{3}{M^2}(\lambda-1)\right)+$$
$$+\mu_D p^2\sigma_Q^2\left(3(\lambda-1)(\lambda-2)+\left(\frac{3}{M}+\frac{3}{M^2}\right)(\lambda-1)\right) \tag{A16}$$

For determining $E\left[I^3\right]$ we now evaluate the different terms of I^3 from (A15).

We start with

$$E\left[(S-D)^3\right]=S^3-3S^2\mu_D+3S\left(\mu_D^2+\sigma_D^2\right)-\mu_{3,D}.$$

As second term we evaluate

$$E\left[-3(S-D)^2A\right]=-3E\left[(S-D)^2\right]E[A]=-3\left(S^2-2S\mu_D+\mu_D^2+\sigma_D^2\right)\left(\lambda-1+\frac{1}{M}\right)\mu_D.$$

For the third term we find

$$E\left[3(S-D)A^2\right]=3E[S-D]E\left[A^2\right]=$$
$$=3(S-\mu_D)\left(\left((\lambda-1)\left(\lambda-1+\frac{2}{M}\right)+\frac{1}{M^2}\right)\mu_D^2+\left(\lambda-1+\frac{1}{M^2}\right)p^2\sigma_Q^2\right)$$

The expectation of the last term in (A15) finally is given by (A16).

After considering all parts of $E\left[I^3\right]$ we turn to the second and third term of $E\left[\left(I-\mu_I\right)^3\right]$ in (A14).

For the second term we simply get

$$-3\mu_I\sigma_I^2=-3\left(S-\left(\lambda+\frac{1}{M}\right)\mu_D\right)\sigma_I^2 .$$

As third term we find

$$\begin{aligned}-\mu_I^3&=-\left(S-\left(\lambda+\frac{1}{M}\right)\mu_D\right)^3=\\&=-S^3+3S^2\left(\lambda+\frac{1}{M}\right)\mu_D-3S\left(\lambda^2+\frac{2}{M}\lambda+\frac{1}{M^2}\right)\mu_D^2+\left(\lambda^3+\frac{3}{M}\lambda^2+\frac{3}{M^2}\lambda+\frac{1}{M^3}\right)\mu_D^3\end{aligned}$$

Summing up all parts of $E\left[I^3\right]$, it first turns out that all expressions which depend on S will cancel down. Thus the following expression remains

$$\begin{aligned}E\left[\left(I-\mu_I\right)^3\right]=&-\mu_{3,D}-3\left(\mu_D^2+\sigma_D^2\right)\left(\lambda-1+\frac{1}{M}\right)\mu_D-3\mu_D\left(\left((\lambda-1)\left(\lambda-1+\frac{2}{M}\right)+\frac{1}{M^2}\right)\mu_D^2+\left(\lambda-1+\frac{1}{M^2}\right)p^2\sigma_Q^2\right)-\\&-p^3\left(\lambda-1+\frac{1}{M^3}\right)E\left[Q^3\right]-\mu_D^3\left(\lambda(\lambda-1)(\lambda-2)+\frac{3}{M}(\lambda-1)^2+\frac{3}{M^2}(\lambda-1)\right)-\\&-\mu_D p^2\sigma_Q^2\left(3(\lambda-1)(\lambda-2)+\left(\frac{3}{M}+\frac{3}{M^2}\right)(\lambda-1)\right)+3\left(\lambda+\frac{1}{M}\right)\mu_D\sigma_I^2+\left(\lambda^3+\frac{3}{M}\lambda^2+\frac{3}{M^2}\lambda+\frac{1}{M^3}\right)\mu_D^3\end{aligned}$$

Exploiting that from the σ_Q^2 and σ_I^2 formulas we find that

$$p^2\sigma_Q^2=\frac{\sigma_I^2-\sigma_D^2}{\lambda-1+1/M^2},$$

after some algebra and rearrangement of terms we come to the following expression for the numerator of the skewness measure in (A13)

$$\begin{aligned}E\left[\left(I-\mu_I\right)^3\right]=&-\mu_{3,D}+\mu_D^3\left(\lambda+\frac{1}{M^3}\right)+3\mu_D\sigma_I^2\frac{\lambda-1+\dfrac{1}{M^3}}{\lambda-1+\dfrac{1}{M^2}}+3\mu_D\sigma_D^2\frac{\dfrac{1}{M^2}-\dfrac{1}{M^3}}{\lambda-1+\dfrac{1}{M^2}}-\\&-p^3\left(\lambda-1+\frac{1}{M^3}\right)E\left[Q^3\right]\end{aligned}\tag{A17}$$

Now we still have to evaluate the $E\left[Q^3\right]$ term in (A17). To this end we revert to the order quantity formula $Q_t=(1-M)Q_{t-1}+MQ_{t-\lambda}-FY\left(Q_{t-\lambda}\right)+FD_{t-1}$.

Using as abbreviation $B=(1-M)Q_{t-1}+MQ_{t-\lambda}-FY(Q_{t-\lambda})$, we can formulate

$$Q_t^3=B^3+3B^2F\cdot D_{t-1}+3BF^2D_{t-1}^2+F^3D_{t-1}^3. \qquad \text{(A18)}$$

We will evaluate the expectation of Q_t^3 term by term. Thereby, we start with considering

$$B^3=\left((1-M)Q_{t-1}+MQ_{t-\lambda}\right)^3-3\left((1-M)Q_{t-1}+MQ_{t-\lambda}\right)^2FY(Q_{t-\lambda})+3\left((1-M)Q_{t-1}+MQ_{t-\lambda}\right)F^2Y(Q_{t-\lambda})^2-F^3Y(Q_{t-\lambda})^3$$

Taking the expectation and exploiting that $Fp=M$ we get

$$\begin{aligned}E\left[B^3\right]=&(1-M)^3E\left[Q^3\right]+3(1-M)^2M\mu_QE\left[Q^2\right]+3(1-M)M^2\mu_QE\left[Q^2\right]+M^3E\left[Q^3\right]-\\&-3(1-M)^2M\mu_QE\left[Q^2\right]-6(1-M)M^2\mu_QE\left[Q^2\right]-3M^3E\left[Q^3\right]+3(1-M)M^2\mu_QE\left[Q^2\right]-\\&+3(1-M)MF(1-p)\mu_Q^2+3M^3E\left[Q^3\right]+3M^2F(1-p)E\left[Q^2\right]-M^3E\left[Q^3\right]-3M^2F(1-p)E\left[Q^2\right]-\\&-MF^2(1-p)(1-2p)\mu_Q\end{aligned}$$

After some algebra we find

$$E\left[B^3\right]=(1-M)^3E\left[Q^3\right]+3(1-M)MF(1-p)\mu_Q^2-MF^2(1-p)(1-2p)\mu_Q$$

B^2 is given by

$$B^2=(1-M)^2Q_{t-1}^2+M^2Q_{t-\lambda}^2+F^2Y(Q_{t-\lambda})^2+2(1-M)MQ_{t-1}Q_{t-\lambda}-2(1-M)FQ_{t-1}Y(Q_{t-\lambda})-2MFQ_{t-\lambda}Y(Q_{t-\lambda})$$

The expectation of B^2 then is given by

$$\begin{aligned}E\left[B^2\right]=&(1-M)^2E\left[Q^2\right]+M^2E\left[Q^2\right]+MF(1-p)\mu_Q+M^2E\left[Q^2\right]+2(1-M)M\mu_Q^2-2(1-M)M\mu_Q^2-2M^2E\left[Q^2\right]=\\&=(1-M)^2E\left[Q^2\right]+F^2p(1-p)\mu_Q=(1-M)^2\left(\mu_Q^2+\sigma_Q^2\right)+F^2(1-p)\mu_D\end{aligned}$$

so that the expected value of the second term in (A18) equals

$$E\left[3B^2F\cdot D\right]=3F\mu_D\left((1-M)^2\left(\mu_Q^2+\sigma_Q^2\right)+F^2(1-p)\mu_D\right)$$

The expected value of B is just given by

$$E[B]=(1-M)\mu_Q+M\mu_Q-M\mu_Q=(1-M)\mu_Q$$

so that the third term in (73) is equal to

$$E\left[3BF^2D^2\right]=3E[B]F^2E\left[D^2\right]=3(1-M)F^2\mu_Q\left(\mu_D^2+\sigma_D^2\right)$$

The last term's expectation is simply

$$E\left[F^3D^3\right]=F^3\mu_{3,D}$$

Tying all $E\left[Q^3\right]$ terms from (A18) together yields

$$E\left[Q^3\right]=(1-M)^3 E\left[Q^3\right]+3(1-M)MF(1-p)\mu_Q^2-MF^2(1-p)(1-2p)\mu_Q+$$
$$+3F(1-M)^2\mu_D\left(\mu_Q^2+\sigma_Q^2\right)+3F^3(1-p)\mu_D^2+3F^2(1-M)\mu_Q\left(\mu_D^2+\sigma_D^2\right)+F^3\mu_{3,D}$$

Solving for $E\left[Q^3\right]$ and noting that $\mu_Q=\mu_D/p$ and $\sigma_Q^2=\left(\sigma_I^2-\sigma_D^2\right)/p^2\left(\lambda-1+1/M^2\right)$ results in

$$E\left[Q^3\right]=\frac{1/p^3}{1-(1-M)^3}\cdot\left\{\begin{array}{l}3M(1-M)\mu_D\left(M(1-p)\mu_D+(1-M)\mu_D^2+\dfrac{1-M}{\lambda-1+1/M^2}\left(\sigma_I^2-\sigma_D^2\right)+M\left(\mu_D^2+\sigma_D^2\right)\right)+\\+3M^3(1-p)\mu_D^2-M^3(1-p)(1-2p)\mu_D+M^3\mu_{3,D}\end{array}\right\}$$

Inserting this $E\left[Q^3\right]$ expression into the $E\left[\left(I-\mu_I\right)^3\right]$ formula in (A17) finally yields

$$E\left[\left(I-\mu_I\right)^3\right]=-\mu_{3,D}+\left(\lambda+\frac{1}{M^3}\right)\mu_D^3+\frac{3\mu_D}{\lambda-1+\dfrac{1}{M^2}}\left[\left(\lambda-1+\frac{1}{M^3}\right)\sigma_I^2+\left(\frac{1}{M^2}-\frac{1}{M^3}\right)\sigma_D^2\right]-\frac{\lambda-1+1/M^3}{1-(1-M)^3}\cdot\Theta_{BI}$$

and with

$$\Theta_{BI}=3M(1-M)\mu_D\left(M(1-p)\mu_D+(1-M)\mu_D^2+\frac{1-M}{\lambda-1+1/M^2}\left(\sigma_I^2-\sigma_D^2\right)+M\left(\mu_D^2+\sigma_D^2\right)\right)-$$
$$-M^3(1-p)\left(3\mu_D^2+(1-2p)\mu_D\right)+M^3\mu_{3,D}$$

Inserting this result in the γ_I formula (A13), we immediately arrive at the skewness expression (36) of Lemma 13.

B: γ_I derivation for stochastically proportional yield under lead time $\lambda\geq 1$

Next to the skewness definition in (A13) this derivation relies on the μ_I, μ_Q, σ_I^2, and σ_Q^2 formulas given in (32), (12), (35), and (20).

The derivation of $E\left[\left(I-\mu_I\right)^3\right]$ is the same as in (A17) for binomial yield so that after replacing the expected yield rate p by we have

$$E\left[\left(I-\mu_I\right)^3\right]=-\mu_{3,D}+\mu_D^3\left(\lambda+\frac{1}{M^3}\right)+3\mu_D\sigma_I^2\frac{\lambda-1+\dfrac{1}{M^3}}{\lambda-1+\dfrac{1}{M^2}}+3\mu_D\sigma_D^2\frac{\dfrac{1}{M^2}-\dfrac{1}{M^3}}{\lambda-1+\dfrac{1}{M^2}}-$$
$$-\mu_Z^3\left(\lambda-1+\frac{1}{M^3}\right)E\left[Q^3\right]$$

(A19)

For evaluating $E\left[Q^3\right]$ we have to consider the order quantity formula $Q_t=(1-M)Q_{t-1}+\left(M-FZ_{t-\lambda}\right)Q_{t-\lambda}+FD_{t-1}$.

Using for abbreviation $B=(1-M)Q_{t-1}+\left(M-FZ_{t-\lambda}\right)Q_{t-\lambda}$, we can formulate

$$Q_t^3=B^3+3B^2F\cdot D_{t-1}+3BF^2D_{t-1}^2+F^3D_{t-1}^3. \tag{A20}$$

We will evaluate the expectation of Q_t^3 term by term. Thereby, we start with

$$\begin{aligned}E\left[B^3\right]=&(1-M)^3E\left[Q^3\right]+3(1-M)^2\left(M-F\mu_Z\right)\mu_Q E\left[Q^2\right]+\\&+3(1-M)\left(M^2-2MF\mu_Z+F^2\left(\mu_Z^2+\sigma_Z^2\right)\right)\mu_Q E\left[Q^2\right]+\\&+\left(M^3-3M^2F\mu_Z+3MF^2\left(\mu_Z^2+\sigma_Z^2\right)-F^3\mu_{3,Z}\right)E\left[Q^3\right]\end{aligned}$$

Exploiting that $F\mu_Z=M$ and doing some algebra eventually yields

$$\begin{aligned}E\left[B^3\right]=&\left(1-3M(1-M)+3MF^2\sigma_Z^2-F^3\mu_{3,Z}\right)E\left[Q^3\right]+\\&+3(1-M)F^2\sigma_Z^2\left(\mu_Q^3+\mu_Q\sigma_Q^2\right)\end{aligned}$$

For the expectation of B^2 we find

$$\begin{aligned}E\left[B^2\right]&=(1-M)^2E\left[Q^2\right]+2(1-M)\left(M-F\mu_Z\right)\mu_Q^2+\left(M^2-2MF\mu_Z+F^2\left(\mu_Z^2+\sigma_Z^2\right)\right)E\left[Q^2\right]\\&=\left((1-M)^2+F^2\sigma_Z^2\right)\left(\mu_Q^2+\sigma_Q^2\right)\end{aligned}$$

so that the expected value of the second term in (A20) equals

$$E\left[3B^2F\cdot D\right]=3F\mu_D\left((1-M)^2+F^2\sigma_Z^2\right)\left(\mu_Q^2+\sigma_Q^2\right).$$

The expected value of B is just given by

$$E[B]=(1-M)\mu_Q+\left(M-F\mu_Z\right)\mu_Q=(1-M)\mu_Q$$

so that the third term in (A20) is equal to

$$E\left[3BF^2D^2\right]=3E[B]F^2E\left[D^2\right]=3(1-M)\mu_Q F^2\left(\mu_D^2+\sigma_D^2\right).$$

The last term's expectation is simply

$$E\left[F^3D^3\right]=F^3\mu_{3,D}.$$

Tying all $E\left[Q^3\right]$ terms according to (A20) together yields

$$E\left[Q^3\right]=\left(1-3M\left(1-M\right)+3MF^2\sigma_Z^2-F^3\mu_{3,Z}\right)E\left[Q^3\right]+3\left(1-M\right)F^2\sigma_Z^2\left(\mu_Q^3+\mu_Q\sigma_Q^2\right)+$$
$$+3F\mu_D\left(\left(1-M\right)^2+F^2\sigma_Z^2\right)\left(\mu_Q^2+\sigma_Q^2\right)+3\left(1-M\right)\mu_Q F^2\left(\mu_D^2+\sigma_D^2\right)+F^3\mu_{3,D}$$

Solving for $E\left[Q^3\right]$ and noting that $\mu_Q=\mu_D/\mu_Z$ and $\mu_Q^2+\sigma_Q^2=\left(\mu_D^2+\mu_Z^2\sigma_Q^2\right)/\mu_Z^2$ results in

$$E\left[Q^3\right]=\frac{1/\mu_Z^3}{3M\left(1-M\right)-3MF^2\sigma_Z^2+F^3\mu_{3,Z}}\cdot\left\{\begin{array}{l}3\left(1-M\right)F^2\sigma_Z^2\left(\mu_D^3+\mu_D\mu_Z^2\sigma_Q^2\right)+\\+3\mu_D\mu_Z F\left(\left(1-M\right)^2+F^2\sigma_Z^2\right)\left(\mu_D^2+\mu_Z^2\sigma_Q^2\right)+\\+3\left(1-M\right)\mu_D\mu_Z^2F^2\left(\mu_D^2+\sigma_D^2\right)+\mu_Z^3F^3\mu_{3,D}\end{array}\right\}$$

This expression can be exploited further by using $\mu_Z F=M$, $F^2\sigma_Z^2=M^2\rho_Z^2$ and $\mu_Z^2\sigma_Q^2=\left(\sigma_I^2-\sigma_D^2\right)/\left(\lambda-1+1/M^2\right)$ so that we get

$$E\left[Q^3\right]=\frac{1/\mu_Z^3}{3M\left(1-M\right)-3M^3\rho_Z^2+F^3\mu_{3,Z}}\cdot\left\{3\mu_D\left[\begin{array}{l}\left(M\left(1-M\right)^2+M^2\rho_Z^2\right)\left(\mu_D^2+\dfrac{\sigma_I^2-\sigma_D^2}{\lambda-1+1/M^2}\right)+\\+M^2\left(1-M\right)\left(\mu_D^2+\sigma_D^2\right)\end{array}\right]+M^3\mu_{3,D}\right\}$$

Inserting this in the $E\left[\left(I-\mu_I\right)^3\right]$ formula in (A19) finally yields

$$E\left[\left(I-\mu_I\right)^3\right]=-\mu_{3,D}+\mu_D^3\left(\lambda+\frac{1}{M^3}\right)+\frac{3\mu_D}{\lambda-1+\dfrac{1}{M^2}}\left[\left(\lambda-1+\frac{1}{M^3}\right)\sigma_I^2+\left(\frac{1}{M^2}-\frac{1}{M^3}\right)\sigma_D^2\right]-$$
$$-\frac{\lambda-1+\dfrac{1}{M^3}}{3M\left(1-M\right)-3M^3\rho_Z^2+F^3\mu_{3,Z}}\cdot\Theta_{SP}$$

and with $\Theta_{SP}=3\mu_D\left[\left(M\left(1-M\right)^2+M^2\rho_Z^2\right)\left(\mu_D^2+\dfrac{\sigma_I^2-\sigma_D^2}{\lambda-1+1/M^2}\right)+M^2\left(1-M\right)\left(\mu_D^2+\sigma_D^2\right)\right]+M^3\mu_{3,D}$

Inserting this result in the γ_I formula (A13), we immediately arrive at the skewness expression of Lemma 13.

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