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Equilibrium Selection and Simple Signaling Games

Michael Mitzkewitz

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Verantwortlich für diese Ausgabe:
Michael Mitzkewitz
Otto-von-Guericke-Universität Magdeburg
Fakultät für Wirtschaftswissenschaft
Postfach 4120
39016 Magdeburg
Germany

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Abstract

This paper calculates the Harsanyi-Selten solutions for a class of simple signaling games. This means that for each generic game belonging to this class one of its equilibrium points is selected according to the principles developed by John C. Harsanyi and Reinhard Selten (Harsanyi & Selten, A General Theory of Equilibrium Selection in Games, 1988). For almost fifty years signaling games have been of great interest for both normative game theorists and scientists interested in the analysis of social, cultural and biological phenomena. The paper provides an introduction into the Harsanyi-Selten theory, solves all generic games and subsumes the results. Thus comparisons to Nash refinement concepts can easily be done and the solution of more complex games is facilitated.

JEL Classification Number: C 72

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Contact:
Michael Mitzkewitz
Bonner Str. 18
53424 Remagen, Germany
Phone: +49 2228 37 80936
Email: m.mitzkewitz@gmx.de

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1. Introduction

A *signaling game* is a game where one of the players (called the “sender”) can be of different types. The actual type is chosen by random and is informed about his identity. The type can choose some action (called the “signal”) observable for an other player (called the “receiver”). The receiver does not know the actual type but the probability distribution of the possible types (or, as a Bayesian, he forms prior beliefs about the probabilty distribution). The receiver can use the observed signal to update his beliefs about the actual type. Hence the actual type can choose the signal strategically to influence the receiver’s updated beliefs about his identity. It is easy to imagine situations where a type has a strong incentive that his true identity becomes public and other situations where the type is interested to feign an honourable character.

Signaling games have been of great attractiveness in the last decades for both economists and game theorists, and the interest seems to increase unbrokenly. Starting from the pioneering works of Akerlof (1970) and Spence (1973, 1974) economists have realized that many situations of substantial economic significance are characterized by incomplete information where privately informed agents can strategically choose actions to affect the beliefs of uninformed agents about the true state of the world. A series of papers of Harsanyi (1967-1968) provided the framework to analyze situations of incomplete information with the appropriate game-theoretical tools. Harsanyi demonstrated that a game of incomplete information can be sensibly transformed into a game of imperfect information. This was a breakthrough because before no satisfying solution concept existed for games of incomplete information. Selten (1965, 1975) refined the concept of Nash equlibrium point (Nash (1950, 1951)) by eliminating incredible threats and he proposed the concept of subgame perfect equilibrium point and (especially for games of imperfect information) the concept of perfect equilibrium point. So almost at the same time the insight into the necessity to analyze models of incomplete information and the possibility to do this in an appropriate way appeared.

In the subsequent years a large number of articles and books has been published that apply signaling games in different economic arenas. Michael Spence, one of the pioneers, dedicated his 2001 Nobel prize lecture to “Signaling in Retrospect and the Informational Structure of Markets” (Spence (2002)). From the vast literature let me list only a small sample of economic or related fields to which signaling games of the described structure or similar structures have been applied and some of the corresponding arcticles:

- Labor market (Spence (1973, 1974, 1976), Nöldeke & van Damme (1990), Austen-Smith & Fryer (2005), Delfgaauw & Dur (2007))
- Market entry (Milgrom & Roberts (1982a, 1982b), De Bijl (1997))
- Competition in product quality (Gal-Or (1989), Bagwell & Riordan (1991))
- Advertising (Milgrom & Roberts (1986), Bagwell (2001), Anand & Shachar (2009))
- Insurances (Wilson (1977), Puelz & Snow (1996), Aarbu (2017))
- Finance ((Ross (1977), Allen & Morris (2001), Levine & Hughes (2005))
- Economics of Law (Reinganum & Wilde (1986), Schweizer (1989), Friedman & Wittman (2007), Dari-Mattiacci & Saraceno (2017))
- Money Laundering (Takáts (2011))
- Bargaining (Rubinstein (1985a, 1985b), Admati & Perry (1987), Feinberg & Skrzypacz (2005))
- Political Science (Banks (1991), Potters, van Winden & Mitzkewitz (1991), Prat (2002), Gavious & Mizrahi (2003)).

Of course the articles mentioned above are usually not solely based on "pure" signaling games as described before but on games with more sophisticated signaling structures or on games where simple signaling games are embedded.

Let me mention that (besides the fact that game theory as a whole has an unexpected predictive power in evolutionary biology) signaling games provide also a useful framework to study animal behavior. Impressive examples are presented e. g. by Grafen (1990), Godfray (1991) and Getty (2006). The philosophical theories of the evolution of conventions (Lewis (1969)) and of the emergence of language (Zollman (2005), Huttegger (2007), Skyrms (2010)) benefit also from the analysis of signaling games.

Signaling games are, however, also under special observation of *pure* game theorists not mainly driven by interests in economic or whatever applications. The point is that simple numerical examples for some signaling games reveal the weakness of certain equilibrium concepts, especially of the *sequential equilibrium* (Kreps & Wilson (1982)). This means that a nontrivial signaling game can have (or usually has) sequential equilibrium points labelled "unreasonable", "nonsensible" or "counterintuitive" by some straightforward criteria. This gave rise to doubts on the claim stated above that the appropriate tool to analyze games of imperfect (and, à la Harsanyi, incomplete) information is not really given by perfect equilibrium or its non-uniovalur twin sequential equilibrium.

After realizing this in the 1980ies a series of papers was published which demonstrated the weakness of existing equilibrium concepts and tried to overcome this weakness by "refining" these concepts. I will call this the "refinement programme". Refinements are usually made in notions of the sequential equilibrium concept and are concerned with restrictions on the beliefs a player can sensibly form at information sets *off* the equilibrium path. The aim of the refinement program is to reduce the multiplicity of sequential equilibria by putting more and more requirements to the players' "rational" choices.

Contributions to the refinement program are for example Banks & Sobel (1987), Cho (1987), Cho & Kreps (1987), Cho & Sobel (1990) and Okunu-Fujiwara, Postlewaite, & Suzumura (1990). The most important contribution to the refinement program was the introduction of *stable equilibria* by Kohlberg & Mertens (1986). Stable equilibria are based on *forward induction.* This means that a player's *past behavior indicates his future behavior* (which is something different from that his *past behavior indicates his identity).* Many of the papers mentioned above are concerned among other things with the question of how the set of stable equilibria can be characterized for signaling games. Surveys on the different refinement concepts and their implications for signaling games are presented by van Damme (1987) and Kreps & Sobel (1994).

Completely different to the refinement program John C. Harsanyi and Reinhard Selten claimed that in any case the *rational solution* for a game must be a *unique* equilibrium point and that this solution cannot be derived by putting more and more restrictions on the equilibrium concept. Instead, given a particuliar equilibrium concept, one and only one equilibrium points out of the set of all equilibrium points of this kind. Hence, the problem of normative game theory is not to create sophisticated refinement procedures but to develop reasonable selection criteria. This should be done from the point of view of an "expert" outside the game who is asked by the players (or by some of the players and, maybe, independent of each other) for a *rational strategic recommendation*. A professional game theorist must be an expert for "*how to play a game*", and, of course, he has to recommend each of his clients *an equilibrium strategy* and, if he tries to live on his new job, he has to recommend *strategies belonging to the same equilibrium point.* Therefore, *a game theorist should have a theory which equilibrium point is the solution of a given game*. Of course, such a theory has to reflect carefully all the strategic relationships and opportunities the game includes. Harsanyi & Selten (1988) present a theory that selects a unique equilibrium point for each finite game as its solution. To quote from Robert J. Aumann's foreword of the Harsanyi-Selten book: "The major implication, like that of the first heavier-than-air flying machine, is that it can be done."

In this study we will calculate the Harsanyi-Selten solution for a class of simple signaling games. This class is characterized as follows: There are just two types of the *sender* possible, each of these two types has just two different choices, and only after one of these two choices, called the "*inside*" choice, the *receiver* comes into play, not knowing, which type has sent the signal. After being alarmed the receiver has two different responses which both terminate the game. If the active type chooses his "*outside*" choice the game ends immediately. The game tree for this class of signaling games is later shown in section 3. Probably this is the simplest class of games which can capture the essence of signaling.

In the following we calculate the Harsanyi-Selten solution for all *generic* games of the class described above. What "generic" means in our context is explained at the end of section 4. The author, however, also find the solution for the *nongeneric games* but to write down all the calculations will exceed the limits of this study. The results are available on request.

As the reader will see, even for the generic games it takes much effort (not only for the author) to go through all the case distinctions which appear to be necessary. The reader may ask whether the aim of this study is not too modest to justify such a fatiguing exertion. I give four answers to this question.

- First, despite its frugal game-theoretical structure the class of signaling games we will consider can be applied to different elementary situations of economic relevance. Having computed the solutions for the whole class, the solutions for games of special interest are easily available in our overview of results.
- Secondly, more complex and interesting economic and other models may have games belonging to our class as subgames. The Harsanyi-Selten theory has the property that the solution of a game prescribes for all agents in a subgame the same local strategies as if the subgame is solved as a game by itself. This *subgame-consistency property* makes it valuable to have complete overviews of the solutions of simple games in order to facilitate to solve more complex games where the simple games arise as subgames.
- Thirdly, it would be interesting to compare the results of the Harsanyi-Selten theory with the results of certain refinement concepts in the latters' domain, the signaling games. Unfortunately, an overwiew how the sets of, e.g., stable equilibria for the whole class of signaling games considered here is not available. It is obvious that for a large part of the parameter space the refinement concepts fail to contract successfully the set of equilibrium points contrary to the ingenious numerical examples

presented in the literature. For a special model such a comparison is made in Potters, van Winden & Mitzkewitz (1991).

- Finally, a lot of the concepts introduced by Harsanyi and Selten are involved in solving our class of signaling games. The interested reader can observe the concepts "at work". So this study can also been taken as a learning-by-doing introduction to the Harsanyi-Selten theory.

This paper is organized as follows. After this introdution section 2 presents a brief digest of the Harsanyi-Selten theory. Section 3 defines the class of games we will consider and presents the solution for special members of this class, called the "decomposable and reducible games". In section 4 we normalize the "indecomposable and irreducible games" and in section 5 we compute for generic cases, i. e. "for almost all" games, their solutions. Section 6 presents an overview of the results and section 7 summarizes.

2. Relevant Elements of the General Theory of Equilibrium Selection

The theory of equilibrium selection developed by John C. Harsanyi and Reinhard Selten (Harsanyi & Selten (1988)) singles out a unique equilibrium point for each finite noncooperative game as its solution. In this section we will sketch the Harsanyi-Selten theory only briefly. Some important ingredients of this theory which will not be involved in the course of our analysis, like "*strategic distance*", are not mentioned here. Other components are explained only to such a degree of complexity which is sufficient to understand the procedures in the following sections. We omit detailed discussions and justifications of the concepts and refer the interested reader to the book of Harsanyi and Selten. Given these limitations, this section could be considered as a small user's guide for the Harsanyi-Selten theory.

In the class of games we will analyze each player has just one information set, so there is no distinction between a player and his single agent. Because of this *normal-form structure* we can omit the explanation of the *"standard form"* of a game which distinguishes thoroughly between players and their agents.

2.1. Some Notations and Definitions

NORMAL FORM. A n-player game in *normal form* $G = (\Phi_1, \dots, \Phi_n; H)$ consists of n nonempty finite sets $\Phi_1, \dots, \Phi_n$ and a *payoff function* H. The set of *pure strategies* of player i ($i = 1, \dots, n$) is represented by Φ_i. A *pure strategy combination* is denoted by φ:

$$\boldsymbol{\varphi = (\varphi_1, \dots, \varphi_n)\ with\ \varphi_i \in \Phi_i} \qquad \textbf{(1)}$$

The payoff function $\boldsymbol{H}$ assigns a payoff vector $\boldsymbol{H(\varphi)}$ to each $\boldsymbol{\varphi}$:

$$\boldsymbol{H(\varphi) = (H_1(\varphi), \dots, H_n(\varphi))} \qquad \textbf{(2)}$$

MIXED STRATEGIES. A *mixed strategy* of player i is a probability distribution over Φ_i and is denoted by q_i. The notation $q_i(\varphi_i)$ represents the probability that player i will choose his pure strategy φ_i. Given a *mixed strategy combination* $q = (q_1, \dots, q_n)$, a particular pure strategy combination $\varphi = (\varphi_1, \dots, \varphi_n)$ occurs with the following probability:

$$\boldsymbol{q(\varphi) = q_1(\varphi_1) \cdot \ldots \cdot q_n(\varphi_n)} \tag{3}$$

Thus the payoff function H can be extended to mixed strategy combinations in the following way:

$$\boldsymbol{H(q) = \sum_{\varphi \in \Phi} q(\varphi) H(\varphi)} \tag{4}$$

Here Φ represents the set of all pure strategy combinations.

In the class of games we will consider each player has just two pure strategies. For this reason we can represent a mixed strategy of player i by a single number q_i, which means the probability to choose the player's *first* pure strategy (it will always be clear what is meant by "*first*"). Hence $1 - q_c$ is the probability to choose his second pure strategy. Pure strategy choices can be represented by $q_i = 1$ and $q_i = 0$. Therefore we can describe any strategy combination (pure or mixed) by a n-tuple of the following kind:

$$\boldsymbol{q = (q_1, \ldots, q_n) \quad with \quad 0 \leq q_i \leq 1 \quad for \quad i = 1, \ldots, n} \tag{5}$$

***i*-INCOMPLETE MIXED STRATEGY COMBINATIONS.** An *i-incomplete mixed strategy combination*q_{-i} is a $(n-1)$-tuple of mixed strategies:

$$\boldsymbol{q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)} \tag{6}$$

Using this notation a mixed strategy combination q can also be written as follows:

$$\boldsymbol{q = (q_i q_{-i})} \tag{7}$$

This means that q contains player i's mixed strategy q_i and the other players' mixed strategies in q_{-i} as its components.

BEST REPLIES. A mixed strategy r_i is called a *best reply* to the i-incomplete strategy combination q_{-i} if:

$$\boldsymbol{H_i(r_i q_{-i}) = \max_{q_i} H_i(q_i q_{-i})} \tag{8}$$

We say that r_i is a *strong best reply* to q_{-i} if all other strategies yield a lower payoff than r_i. Of course a strong best reply must be a pure strategy.

EQUILIBRIUM POINTS. A mixed strategy combination $q = (q_1, \dots, q_n)$ is called an *equilibrium point* of game G if for each player i ($i = 1, \dots, n$) his mixed strategy q_i is a best reply to q_{-i}. If all q_i are strong best replies to q_{-i}, than q is called a *strong equilibrium point*. Notice that we use the term "strong equilibrium point" different from Aumann (1959).

UNIFORM PERTURBATIONS. The Harsanyi-Selten theory is not applied directly to the game G under consideration but to *uniform perturbations* of this game, denoted by G_ε. Each pure strategy of a player must be chosen with a minimal probability ε, where ε is supposed to be close to zero but positive. ε can be interpreted as the probabiltity to choose the "wrong" pure strategy by error due to "trembling hands". The term "uniform" refers to the fact that the perturbation parameter ε is the same for all players and for all pure strategies. This differs from Selten's general definition of perfectness (Selten (1975)).

In the class of games we will consider each player has just two pure strategies. So we can describe each mixed strategy combination which is admissible in the uniformly perturbed game as follows:

$$\boldsymbol{q_\varepsilon = \left(q_{1_\varepsilon}, \dots, q_{n_\varepsilon}\right) \; with \quad \varepsilon \leq q_{i_\varepsilon} \leq 1 - \varepsilon \quad for\; i = 1, \dots, n} \tag{9}$$

Of course $\varepsilon < 0.5$ is supposed. If player i chooses $q_{i_\varepsilon} = 1 - \varepsilon$ or $q_{i_\varepsilon} = \varepsilon$, we say that he plays an *ε-extreme strategy.* He "tries" to play one of his pure strategies and the other pure strategy can only appear by mistake. We will indicate by φ_ε the *ε-extreme strategy combination* which corresponds to the pure strategy combination φ.

UNIFORMLY PERFECT EQUILIBRIUM POINTS. The *limit equilibrium points* of the uniformly games G_ε for $\varepsilon \to 0$ are called the *uniformly perfect equilibrium points* of the unperturbed game G.

The Harsanyi-Selten theory requires that the solution of a game must be one of its uniformly perfect equilibrium points. But Harsanyi and Selten do not select directly among these equilibrium points (if there are more than one). They first *solve* (i.e., they single out a unique equilibrium of) the perturbations of the game and then, by letting $\varepsilon \to 0$, they obtain the *limit solution* of the game.

Hence it must be kept in mind that in the following descriptions of *how to solve a game* we deal with (uniformly) perturbed games.

2.2. Decomposition and Reduction

The first step in solving a game is to check whether this game is *decomposable.* To understand what this means, we need some further definitions.

CELLS. A proper subset of players *forms a cell* if for each of these players the strategic situation only depends on the other members of the cell and is completely independent of the strategic choices of the players outside the subset. In other words, this subset is closed with respect to the best-reply correspondence. A cell is called *elementary* if it contains no proper subset of players which forms a cell by itself.

DECOMPOSABLE GAMES. A game is called *decomposable* if it has at least one cell. Otherwise it is called *indecomposable.* Obviously an elementary cell is indecomposable.

FIXING A PLAYER. We say that *a player is fixed at a particular strategy* if after this fixing a game is considered which results from the substitution of this player's strategy set by this particular strategy and from modification of the payoff function in the appropriate way. We emphasize that with such a strategy fixing always a *new game* results from a more complex one.

INFERIOR CHOICES. A pure strategy φ_i of player i is called *inferior* if he has a pure strategy ψ_i which is always a best reply whenever φ_i is a best reply, but also in some cases where φ_i is not a best reply. Since in our class of games each player just has two pure strategies, the term "inferior" is here equivalent to "weakly dominated". This is obviously not true for more than two pure strategies. Notice that the original Harsanyi-Selten definition of inferiority is concerned with choices of an agent and not with pure strategies of a player. We do not need such a distinction here because in our games each player has only one information set (and, therefore, no agents). But in order to match Harsanyi's and Selten's terminology, we will speak of *inferior choices* instead of "weakly dominated strategies".

ELIMINATION OF INFERIOR CHOICES. If a player has an inferior choice, this choice is eliminated from his strategy set. But notice that this elimination takes place within the perturbed game. In the class of games we will consider the elimination of an inferior choice means nothing else but fixing the respective player at his ε-extreme strategy concentrated on his superior pure strategy. The inferior choice is still chosen "erroneously" with probability ε.

SEMIDUPLICATE CLASSES. If some pure strategies of a player yield always the same payoff *to him* independent of the strategies chosen by the other players, we say that these pure strategies are *semiduplicates* or that they *form a semiduplicate class.*

CENTROID STRATEGY. The mixed strategy of a player which assigns the same probability to each of his pure strategies is called his *centroid strategy.* Hence, $q_i = {}^1/_2$ is player i's centroid strategy if he has two pure strategies. This is not the exact definition proposed by Harsanyi and Selten, but sufficient for our purposes and more convenient.

ELIMINATION OF SEMIDUPLICATE CLASSES. If the pure strategies of a player form a semiduplicate class, this class is eliminated by fixing this player at his centroid strategy.

IRREDUCIBLE GAMES. A game is called *irreducible* if it is indecomposable and has neither inferior choices nor semiduplicate classes. Otherwise the game is called *reducible*.

DECOMPOSITION AND REDUCTION. The *procedure of decomposition and reduction* tries to facilitate the task of solving games to the simpler task of solving irreducible games. How to solve an irreducible game is explained in the following subsections. The precise procedure of decomposition and reduction is best explained by the flowchart on page 127 in Harsanyi & Selten (1988) or by the flowchart in Güth & Kalkofen (1989) on page 39. For our purposes a much more superficial description is sufficient. It will turn out that games of our class are only decomposable if they contain inferior choices and/or semiduplicate classes. Within our framework we can describe the procedure of decomposition and reduction by the following steps:

- **STEP 1:** If the game is irreducible, carry on with STEP 4. Otherwise carry on with STEP 2.
- **STEP 2:** If the game contains inferior choices, eliminate them and carry on with STEP 1. Otherwise carry on with STEP 3.
- **STEP 3:** Eliminate the semiduplicate classes and carry on with STEP 1.
- **STEP 4:** Compute the solution of the irreducible game (see the following subsections).

Here the term "game" always means the original perturbed game *after* previous elimination steps. So each game will be reduced to an irreducible game after finitely many steps.

2.3. The Linear Tracing Procedure

LINEAR TRACING PROCEDURE. An important component of the Harsanyi-Selten theory is the so-called *linear tracing procedure,* introduced by Harsanyi (1975). The linear tracing procedure is an attempt to extend principles of Bayesian rationality from one-person decision problems to n-person noncooperative games. It is assumed that players form *prior beliefs* about the other players' strategic intentions, maximize their expected payoffs on the base of these beliefs, modify continuously the prior beliefs by "observing" more and more of the other players' maximizing behavior, and change in case of need their own actions on the base of these modified beliefs. Formally, player i's payoff function H_i of a given game G is transformed to:

$$\boldsymbol{H_i^t(q_i q_{-i}) = tH_i(q_i q_{-i}) + (1-t)H_i(q_i p_{-i})} \qquad \textbf{(10)}$$

Here t is the so-called *tracing parameter* with $0 \leq t \leq 1$. The tracing parameter can be loosely interpreted as "time", so $t = 0$ marks the beginning and $t = 1$ marks the end of the process generated by the linear tracing procedure. The prior beliefs (or simply the *priors*) of player i about the other players' strategic intentions are expressed by the i-incomplete mixed strategy combination p_{-i} , whereas q_i and q_{-i} are player i's and the other players' *actual mixed strategies at time t.* Each player is assumed to choose q_i at time t in order to maximize H_i^t. To put it differently, player i plays at time t a best reply to the following i-incomplete mixed strategy (see also Harsanyi & Selten (1988), p. 142n):

$$\boldsymbol{tq_{-i} + (1-t)p_{-i}} \qquad \textbf{(11)}$$

Hence, at time $t = 0$ player i plays a best reply to his priors independent of the other players' actual strategies, which are in fact their best replies to *their* priors. When t increases player i lays less stress on his priors and lays more stress on the "observed" actual strategies of the other players. At time $t = 1$ the influence of the priors completely vanished and, since all players choose best replies to the other players' *actual* strategies, an equilibrium point of the original game G is reached.

PATH AND RESULT OF THE LINEAR TRACING PROCEDURE. We will say that the set of pairs (q, t) for all $0 \leq t \leq 1$ describes the *path of the linear tracing procedure.* In some cases, however, the path of the linear tracing procedure is not well-defined. We will discuss this problem at the end of this subsection. For the moment let us assume that no difficulties of this kind arise. Then it is clear that the path of the linear tracing procedure ends in an equilibrium point of the considered game. This equilibrium point is called the *result of the linear tracing procedure.*

If the strategy combination given by the best replies to the priors (i. e. q at $t = 0$) forms an equilibrium point of the considered game, it is obvious that this strategy combination will be played along the whole path of the linear tracing procedure up to the end (because in this case for each player i his strategy is a best reply to *both* q_{-i} and p_{-i} and, therefore, also a best reply to each convex combination of these two i-incomplete strategy combinations).

DESTABILIZATION POINTS. If the vector of best replies to the priors does not form an equilibrium point of the considered game, it is clear that at least one player must alter his strategy along the path of the linear tracing procedure at some time t. This value for t is called this player's *destabilization point*.

STRATEGY SHIFT. At a player's destabilization point a player *shifts his strategy.* Maybe after the *strategy sh*ift an equilibrium point is reached and then this strategy combination is followed in the further path of the linear tracing procedure. But it is also possible that a series of strategy shifts is necessary to reach an equilibrium point at the end. Notice that with each strategy shift of a player i at time t payoff shifts for all players are usually connected (q_{-j} changes in the modified payoff functions given by **(10)** for all players j with $j \neq i$). So it is important to calculate who is the *first player to shift his strategy.* It is the player with the smallest value for t at his destabilization point. After his strategy shift the new destabilization points are calculated (if there are some) and the *next player to shift his strategy* is determined, etc. Let us mention that even if the path of the linear tracing procedure is well defined, it can have so-called *backward-moving variable segments* (see section 4.19 in Harsanyi & Selten (1988)). In our analysis this phenomenon will not arise and so we omit any discussion of this issue.

The tracing procedure is involved in three ways in Harsanyi's and Selten's solution concept. The three jobs of the linear tracing procedure are:

- *Risk-dominance comparison between two equilibrium points*
- *Forming a substitute of a candidate set*
- *Computation of the solution of a basic game.*

In the next subsection we will explain at which steps of the solution procedure these three jobs come into play. Here we describe the implications for the construction of the priors.

RISK-DOMINANCE COMPARISON. Consider the situation that all players are convinced that the solution of a game is one out of two equilibrium points, say q^1 and q^2, with the property that $q_i^1 \neq q_i^2$ holds for each player *i*. It will turn out that in our analysis only comparisons between *strong* equilibrium points are necessary, so the following explanations are restricted to such a situation. Each player *i* is assumed to be initially doubtful about the "correct" equilibrium point, but he believes that all the other players *know* the correct one and consequently they will *play jointly either* q_{-i}^1 *or* q_{-i}^2. According to Bayesian rationality player *i* must form a *subjective* probability, say z_i, for the event that the other players choose q_{-i}^1 and a *subjective* probability $1 - z_i$ for the event q_{-i}^2. Therefore player *i* is assumed to play initially ($t = 0$) a best reply to the following *i*-incomplete *joint mixture*:

$$\boldsymbol{z_i q_{-i}^1 + (1 - z_i) q_{-i}^2} \qquad \textbf{(12)}$$

Since q^1 and q^2 are strong equilibrium points, there must exist for each player *i* a particular value $\hat{z}_i$ with $0 < \hat{z}_i < 1$, such that q_i^1 is for all $z_i \in (\hat{z}_i, 1]$ a strong best reply to the joint mixture given in (12), but for all $z_i \in [0, \hat{z}_i)$ q_i^2 is a strong best reply.

But how does player *i* form his subjective probability z_i about the "correct" equilibrium point? Or, to put it more precisely, what should the *other* players think about the way player *i* forms z_i? As Bayesians the other players have to construct a distribution function of z_i over the interval [0,1]. Because the initial state must be considered as a situation of complete naivety, there is no reason whatsoever to put more weight on a specific value of z_i than on another one. Hence, Harsanyi and Selten assume that z_i is uniformly distributed over the interval [0,1].

This has the consequence that player *i* is assumed to choose initially (at $t = 0$) q_i^1 with probability $1 - \hat{z}_i$ and q_i^2 with probability $\hat{z}_i$, where as explained above $\hat{z}_i$ is that particular value of z_i of player *i* that makes him indifferent between q_i^1 and q_i^2. So the *prior beliefs about player i* are that he plays the following mixed strategy:

$$\boldsymbol{p_i = (1 - \hat{z}_i) q_i^1 + \hat{z}_i q_i^2} \qquad \textbf{(13)}$$

These priors are also called the *bicentric priors* because just two equilibrium points are compared.

Given these priors for all players, the path of the linear tracing procedure can be computed. If the result of the linear tracing procedure is q^1, than we will say that q^1 *risk-dominates* q^2. If q^2 is the result, q^2 risk-dominates q^1.

SUBSTITUTION OF A CANDIDATE SET. Sometimes in the calculation of the Harsanyi-Selten solution for a given game the linear tracing procedure is used to *substitute* a set of equilibrium points by a single equilibrium point. In such a case the priors about player *i* are formed by the equally weighted average of his mixed strategies used in the equilibrium points of the set that should be substituted. In our case it will turn out that only sets of two pure equilibrium points must be substituted, so the priors are simply given by the players' *centroid strategies* (see subsection 2.2). With the term "*substitution of a candidate set*" used in the next subsection we mean the following: If we replace a set of equilibrium points (the *candidate set*) by that equilibrium point which is the result of the linear tracing procedure using the players' centroid strategies as their priors, then we say that this set of equilibrium points is *substituted*.

SOLUTION OF A BASIC GAME. In the next subsection we introduce the concept of a *basic game*. Here we want to state that the *solution of a basic game* is the result of the linear tracing procedure using the players' centroid strategies in that basic game as their priors.

EXISTENCE OF A WELL-DEFINED PATH OF THE LINEAR TRACING PROCEDURE. Hitherto, we have excluded any discussion about the uniqueness of the path (and, therefore, the result) of the linear tracing procedure. Unfortunately, such a *well-defined path* exists only for "almost all" games. For example, in a game of complete symmetry (or complete asymmetry) between two players, they will have the same destabilization points and the path of the linear tracing procedure does not have a unique continuation after this point (think of a symmetric "battle of sexes" game).

Harsanyi and Selten attempted to single out a unique equilibrium point for *all finite* games and not only for the generic subset. So they could not be satisfied that the linear tracing procedure as one of their most important tools in solving games lead to dubious results in nongeneric cases. Therefore they introduced the *logarithmic tracing procedure.* The logarithmic tracing procedure generates a well-defined path for *all finite* games and the result of the logarithmic tracing procedure is the same as the result of the linear tracing procedure if the latter's path is well-defined. Hence the logarithmic tracing procedure can be considered as a generalization of the linear tracing procedure.

The payoff function along the logarithmic tracing procedure differs from that of the linear tracing procedure (see **(10)**) by an additional logarithmic term which "punishes" to some extent each deviation from the player's centroid strategy. This term ensures that for each $t < 1$ each player has a unique best reply in completely mixed strategies to any strategy combination of the other players.

The logarithmic tracing procedure only comes into play in nongeneric games. In this work we will only determine the solution of the generic elements in our class of signaling games, therefore the linear tracing procedure is sufficient. Since 1988, when Harsanyi's and Selten's book was published, some properties of the tracing procedure and its computability are investigated in more detail (Schanuel, Simon & Zame (1991), van den Elzen & Talman (1995), van den Elzen (1996), Herings & van den Elzen (2002)). However, for our purposes these advances are of no relevance.

2.4. Solution of Irreducible Games

After the preparations given in subsection 2.2 (the process of decomposition and reduction) and in subsection 2.3 (the linear tracing procedure), we want to explain in this subsection how Harsanyi and Selten solve an *irreducible* game (for the definition see subsection 2.2). However, we need some further definitions.

FORMATIONS. Consider a game F which results from a game G by eliminating some pure strategies (and changing the payoff functions in the appropriate way). If for each i-incomplete mixed strategy combination permissible in F player i's best replies in G are all contained in F, and if this holds for each player, we call F a *formation*.

PRIMITIVE FORMATIONS. A formation is called *primitive* if it contains no proper subformations. For example, a strong equilibrium point *generates* a primitive formation. However, strong equilibrium points do not always exist. Harsanyi and Selten introduced the concept of a primitive formation to have a concept with similar stability properties as a strong equilibrium point.

BASIC GAMES. A game is called *basic* if it is irreducible and if it contains no formations. Hence, each irreducible game must be basic or it must contain some primitive formations.

INITIAL CANDIDATES. The *initial candidates* for the solution of an irreducible game are defined as follows: If the game is basic, then the solution of this basic game is the only initial candidate. If the game is not basic, then the solutions of the primitive formations of this game are the initial candidates. The set of initial candidates is also called the *first candidate set*.

It will turn out that in the class of games we consider a game can have two primitive formations at most, and that in this case these two primitive formations must be generated each by a strong pure equilibrium point. So the first candidate set contains either one (pure or mixed) or two (pure) equilibrium points.

If there is only one candidate, this equilibrium point is the solution of the game. If we have two initial candidates, we first look whether one of them strictly payoff-dominates the other one. If this is not the case, a risk-dominance comparison via the linear tracing procedure between these two equilibrium points is necessary (see subsection 2.3). The solution of the game is then the equilibrium point that dominates (strictly payoff-dominates or risk-dominates) the other one, where priority is given to payoff-dominance. However, it is possible that neither (strict) payoff-dominance nor risk-dominance exist between two equilibrium points. No risk-dominance relationship between two equilibrium points is given if the path of the (logarithmic) tracing procedure with the bicentric priors does not end in one of these equilibrium points. This can only happen in degenerate cases. Then a substitution step becomes necessary.

SUBSTITUTION OF A CANDIDATE SET (see also subsection 2.3). If the first candidate set consists of two equilibrium points without dominance relationship, we substitute this set by the equilibrium point which is the result of the tracing procedure using the players' *centroid strategies* as their priors. This equilibrium point is the solution of the game. Notice that this resulting equilibrium is generally not among the two initial candidates. For example, in a symmetric battle-of-sexes game the first candidate set consists of the two pure equilibrium points, but its substitute (and, therefore, the solution of the game) is the mixed equilibrium point.

SUMMARY OF PROCEDURES. In subsection 2.2 we explained how games are transformed to become *irreducible games*. In the present subsection we defined how an irreducible game is *solved*. First, we check whether the game is *basic*. If the game is basic we compute its solution, which is the *result of the linear tracing procedure using the players' centroid strategies as their priors*. If the game is not basic, we compute the *solutions of its primitive formations*. If the game has two primitive formations (generated by two pure equilibrium points), we make a *payoff-dominance comparison* and, if necessary, a *risk-dominance comparison* between the two generating pure equilibrium points. If no dominance relationship exists we compute the *result of the linear tracing procedure using the players' centroid strategies as their priors*. In any case we come out with a unique equilibrium point which is called *the solution of the game.*

Pay special attention to the fact that all procedures mentioned above are done within the *perturbed game*. The *solution of the unperturbed game* is obtained as the limit of the solutions of its perturbations letting ε go to zero.

2.5. Solution of 2x2 Games with Two Strong Equilibrium Points

In many game-theoretical models 2x2-games arise as subgames or cells (see subsection 2.2). Therefore their *solutions* are of special interest. Here we are concerned only with the equilibrium selection problem resulting of a 2x2-game with two strong equilibrium points. Let such a game be given as follows (figure 1):

Player 2

		U_2	V_2
Player 1	U_1	a_{11} / b_{11}	a_{12} / b_{12}
	V_1	a_{21} / b_{21}	a_{22} / b_{22}

Figure 1: A 2x2-game with the two strong equilibrium points $(\boldsymbol{U_1}, \boldsymbol{U_2})$ and $(\boldsymbol{V_1}, \boldsymbol{V_2})$ because we assume that $\boldsymbol{a_{11}} > \boldsymbol{a_{21}}$, $\boldsymbol{a_{22}} > \boldsymbol{a_{12}}$, $\boldsymbol{b_{11}} > \boldsymbol{b_{12}}$ and $\boldsymbol{b_{22}} > \boldsymbol{b_{21}}$ hold. For each strategy combination, player 1's payoff is shown in the upper left corner and player 2's payoff is shown in the lower right corner of the respective square.

The game described in figure 1 can be transformed in a *best-reply structure preserving game,* as shown in figure 2:

		Player 2	
		U_2	V_2
Player 1	U_1	u_1 ; u_2	0 ; 0
	V_1	0 ; 0	v_1 ; v_2

Figure 2: A 2x2-game with the two strong equilibrium points $(\boldsymbol{U_1}, \boldsymbol{U_2})$ and $(\boldsymbol{V_1}, \boldsymbol{V_2})$ which results from the following best-reply structure preserving transformations: $\boldsymbol{u_1} = \boldsymbol{a_{11}} - \boldsymbol{a_{21}} > 0$, $\boldsymbol{v_1} = \boldsymbol{a_{22}} - \boldsymbol{a_{12}} > 0$, $\boldsymbol{u_2} = \boldsymbol{b_{11}} - \boldsymbol{b_{12}} > 0$ and $\boldsymbol{v_2} = \boldsymbol{b_{22}} - \boldsymbol{b_{21}} > 0$.

The term "*best-reply preserving transformations*" simply means that after some payoff manipulations of a game *G* a game *G'* is received with the property that for each player his best replies against all opponents' strategy combinations are the same in both games.

Harsanyi and Selten provide an axiomatic foundation for the risk-dominance comparison between two pure equilibrium points of a 2x2-game like in figure 2. In their book they proof that three plausible requirements on the solution of the selection problem between the two equilibrium points (U_1, U_2) and (V_1, V_2) are only fulfilled by the following criterion:

- (U_1, U_2) is the solution if $u_1 u_2 > v_1 v_2$ holds.
- (V_1, V_2) is the solution if $u_1 u_2 < v_1 v_2$ holds.

The mixed equilibrium of the game is its solution if $u_1 u_2 = v_1 v_2$ holds.

Furthermore, Harsanyi and Selten show that these results are equivalent to those obtained by making a risk-dominance comparison between the two equilibrium points via the linear tracing procedure (see subsection 2.3). This means that the axiomatically founded solution concept for 2x2 games with two strong equilibrium points is embedded into the general solution theory for all games roughly described in this section. The comparison of the *payoff products* $u_1 u_2$ and $v_1 v_2$ is similar to the analysis of the *Nash product* (Nash (1953)). In consequence, this embedding is called the *Nash property* of the Harsanyi-Selten theory.

The solution of a 2x2-game can therefore be obtained without explicitly making use of the tracing procedure. If the game is given as in figure 1, then you have to check whether one of the two equilibrium points strictly payoff-dominates the other one. In this case, the payoff-dominating equilibrium point is the solution of the game. Otherwise, you have to transform the game into a game as in figure 2 preserving the best-reply structure. Then you have to compute which of the equilibrium points yields the higher "Nash product". This one is the solution of the game. If both equilibrium points yield the same Nash product, the mixed equilibrium of the game is its solution.

In the following lemma we show that there exists a simple measure equivalent to the Nash product criterion in 2x2-games, which is in some applications easier to compute (see Potters, van Winden & Mitzkewitz (1991)).

LEMMA. Given a 2x2-game as in figure 2. Then the two pure strategies (one for each player) chosen in the pure equilibrium with the higher product of payoffs (Nash product) are chosen in the mixed equilibrium point of the game with probabilities adding up to <u>*less*</u> than 1.

PROOF. In the mixed equilibrium (q_1, q_2) of the game in figure 2 the strategies U_1 and U_2 are chosen with the following probabilities:

$$\boldsymbol{q_1 = \frac{v_2}{u_2 + v_2}} \tag{14}$$

$$\boldsymbol{q_2 = \frac{v_1}{u_1 + v_1}} \tag{15}$$

It follows:

$$\boldsymbol{q_1 + q_2 = 1 + \frac{v_1 v_2 - u_1 u_2}{(u_1 + v_1)(u_2 + v_2)}} \tag{16}$$

Since all u_i and v_i are greater than zero, it follows that $q_1 + q_2 > 1$ if $v_1 v_2 > u_1 u_2$ and that $q_1 + q_2 < 1$ if $v_1 v_2 < u_1 u_2$. ■

We make use of this result in our analysis.

3. The Class of Games Considered and the Solution of Its Decomposable and Reducible Members

Consider the following class of signaling games. The *sender* is one of two types which occur with known positive probabilities α and $1-\alpha$. Each type has to choose between two alternatives: the move "*inside*" and the move "*outside*". If the activated type chooses "*outside*" the game is finished, but if he chooses "*inside*" a *receiver* observes this *message* without being informed about the sender's type. Afterwards, the receiver has to choose between two responses, called "*left*" and "*right*" to terminate the game. At each of the six possible endpoints of the game the players receive their respective payoffs. Following Harsanyi (1967-1968) we consider the two types as different players, hence the payoff vectors have three components. Figure 3 shows the extensive form of the game without specifying the payoff vectors. The two types of the sender are called player 1 and player 2, and the receiver is called player 3. Nature choosing sender's type by chance is called player 0.

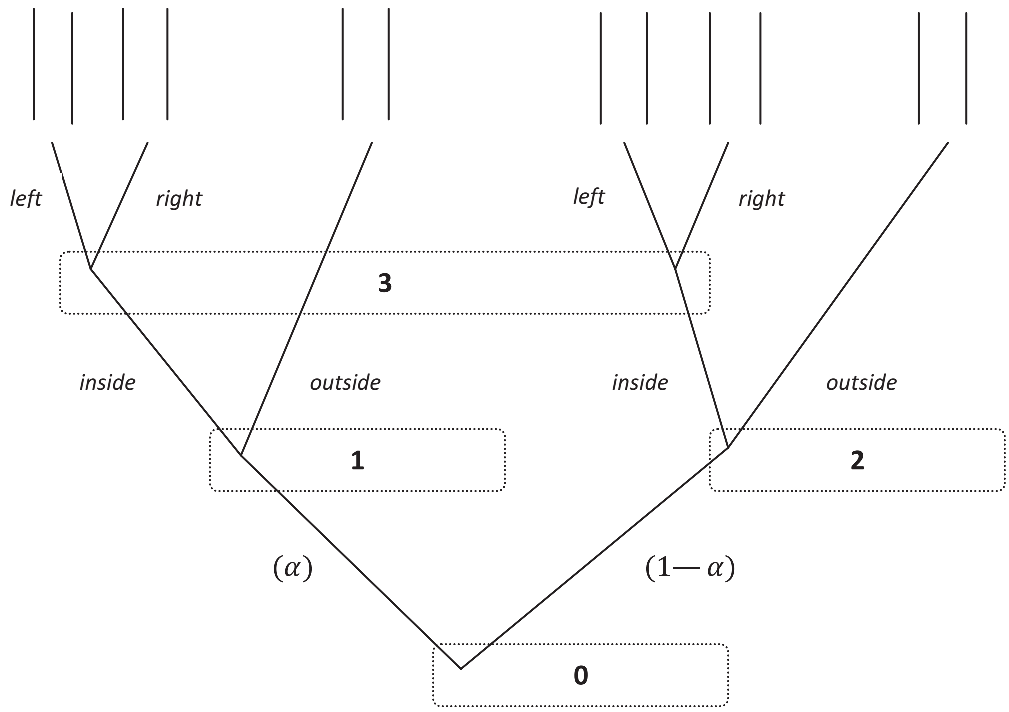


Figure 3: The extensive form of the considered class of games. Information sets are indicated by the dotted lines. Payoff vectors are unspecified.

NORMALIZATION. In this section, we make some steps to normalize this class of signaling games as follows: A type always receives nothing if he is not active. Furthermore, we subtract the payoff vector after an *"outside"* choice of a type from all three payoff vectors which can be achieved if this type has become active. This transformation preserves the best-reply structure for all players. By this procedure the new payoff vectors of the normalized game are obtained. The payoffs are named as in figure 4. We will call this steps *semi-normalization.* In section 4, dealing with the indecomposable and irreducible games, we will proceed with the normalization.

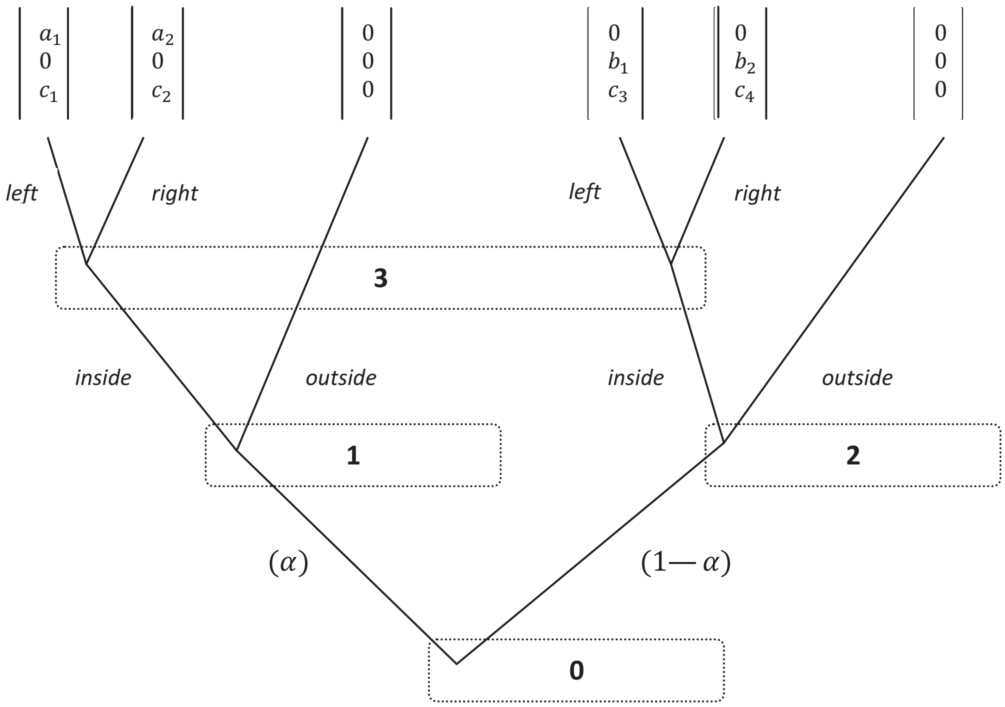


Figure 4: The extensive form of the semi-normalized games.

DECOMPOSITION AND REDUCTION. Now we explain the meaning of *"decomposable"* and *"reducible"* (see subsection 2.2) in the normalized signaling games. Fortunately, for the simple game structure considered here the two concepts are closely connected.

ELEMENTARY CELLS. First we look on possible kinds of *elementary cells* (see subsection 2.2). Obviously, the two types *together* cannot form an elementary cell because their best replies are always independent from each other. Furthermore, the receiver *together with one type* cannot form an elementary cell by the following reason. If they form a cell, the receiver must be independent from the strategy of the other type and, therefore, the receiver should calculate only for the situation after an *"inside"* choice of the cell type. But this means that the receiver's best reply is independent of the probability of this move. Consequently, in this case the receiver forms an elementary cell by himself. It follows, that,*if signaling games of our class are decomposable, an elementary cell is formed by a single player.*

CONDITIONS THAT A TYPE FORMS A CELL. By definition, the best-reply structure of a type forming a cell must be independent of the receiver's strategy. This situation can occur in three ways:

1. The cell type receives in one case more than null and in the other case at least null after an *"inside"* choice in dependence on the receiver's response. This means that the cell type's *"outside"* choice is inferior.
2. The cell type receives in one case less than null and in the other case he receives null at most after an *"inside"* choice in dependence on the receiver's response. This means that the cell type's *"inside"* choice is inferior.
3. The cell type receives always null. This means that his two pure strategies are semiduplicates (see subsection 2.2).

CONDITIONS THAT THE RECEIVER FORMS A CELL. This situation is given in two cases :

1. One of the receiver's choices is (weakly) dominated. Then this pure strategy is, of course, inferior.
2. The payoffs of the receiver only depend on the active type but not on his own choice. Then his two pure strategies are semiduplicate classes.

REDUCTION. In our simple games, the process of solving first the one-person cells is equivalent to the process of reduction. Every player who forms a cell *is fixed* at his superior choice (if he has an inferior choice) or at his centroid strategy (if his pure strategies are semiduplicates). If all three players form cells for themselves the solution of the game is obtained immediately by such strategy fixing. Otherwise, the reduced game has to be analyzed further. Solutions for all decomposable signaling games of our class are developed in the following subsections. The results of the somewhat tedious case-by-case analysis are summarized in an overview presented in section 6 after the results of the indecomposable and irreducible games have also been calculated in section 5.

3.1. At least the receiver forms a cell

The situations are quite similar if only the receiver forms a cell or if the receiver and one type form cells or if all three players form cells. This similarity arises from the fact that *at the latest after the elimination* of the receiver's inferior choice or of his semiduplicate class both types will form cells by themselves. In these cases, the solution of the reduced game is obtained by fixing the types at their superior choices (if they have inferior choices) or at their centroids (if their pure strategies are semiduplicates).

In the remaining subsections those situations are considered in which at least one type forms a cell but the receiver does not.

3.2. Both types form cells

In this case it is necessary to look at the ε-perturbed game. First, both types are fixed at their superior choices or at their centroid strategies. But notice that in the perturbed game inferior choices still occur with probability ε. Table 1 presents the conditional probabilities that the node after player 1's *"inside"* choice (the left node in player 3's information set in figure 3) is reached, given that the receiver has observed an *"inside"* choice.

Probability for player 3's left node after fixing the types		Player 2		
		Inferior choice *"inside"*	Inferior choice *"outside"*	Semiduplicate Class
Player 1	Inferior choice *"inside"*	α	$\frac{\alpha}{\alpha\varepsilon + (1-\alpha)(1-\varepsilon)}$	$\frac{2\alpha\varepsilon}{1-(1-2\varepsilon)\alpha}$
	Inferior choice *"outside"*	$\frac{\alpha(1-\varepsilon)}{\alpha(1-\varepsilon) + (1-\alpha)\varepsilon}$	α	$\frac{2\alpha(1-\varepsilon)}{1+(1-2\varepsilon)\alpha}$
	Semi-duplicate class	$\frac{\alpha}{\alpha + 2(1-\alpha)\varepsilon}$	$\frac{\alpha}{\alpha + 2(1-\alpha)(1-\varepsilon)}$	α

Table 1: Conditional probabilities that the node after player 1's *"inside"* choice is reached, given that the receiver observed an *"inside"* choice.

Given these conditional probabilities, the receiver is able to compute which of his two responses yields a higher expected payoff. This response is his ε-extreme strategy in the perturbed game. If both responses yield the same expected payoff, the receiver has to choose his centroid strategy. By letting $\varepsilon \to 0$, the limit solution of the game is obtained.

3.3. Only one type forms a cell – he has the inferior choice "outside"

In the remaining parts of section 3 the cell forming type is always called player 1. In this subsection we consider the reduced game after elimination of an inferior *"outside"* choice of player 1. However, this choice occurs with positive probability due to the perturbation. The two responses of the receiver are called r_1 and r_2, and player 2's *"inside"* choice is called m_1 and his *"outside"* choice is called m_2. The payoffs are named as in figure 4.

Since player 2 and player 3 do not initially form cells in the case considered in this subsection, the following conditions for the payoffs must hold:

$$\boldsymbol{b_1 \neq 0, \qquad b_2 \neq 0, \qquad sgn\, b_1 \neq sgn\, b_2} \tag{17}$$

$$\boldsymbol{c_1 \neq c_2, \qquad c_3 \neq c_4, \qquad sgn(c_1 - c_2) \neq sgn(c_3 - c_4)} \tag{18}$$

Without loss of generality we can assume that the receiver's responses are named in such a way that $c_3 - c_4 > 0$ holds. In the reduced perturbed game (after elimination of player 1's inferior *"outside"*) player 2 does obviously not form a cell. But player 3 gets an inferior choice r_1 if the following inequality holds:

$$\boldsymbol{\alpha(c_2 - c_1) \geq (1 - \alpha)(c_3 - c_4)} \tag{19}$$

This inferiority results from the fact that the left node in player 3's information set is reached in the reduced game with probability $\alpha(1 - \varepsilon)$, but the right node is reached with probability $(1 - \alpha)(1 - \varepsilon)$ at most. If **(19)** holds, player 3 is fixed at r_2 and finally player 2 has to choose m_1 if $b_1 < 0$ and m_2 if $b_1 > 0$.

If **(19)** does not hold the reduced game is not further decomposable and reducible. Therefore, the equilibrium points of this game are examined. The probability that player 2 chooses m_1 is called q_2 and the probability that player 3 chooses r_1 is called q_3. Best reply of player 2 is m_1 if either **(20)** or **(21)** holds:

$$q_3 \geq \frac{-b_2}{b_1 - b_2} \equiv b \ \ and \ \ b_1 > 0 \tag{20}$$

$$q_3 \leq \frac{-b_2}{b_1 - b_2} \equiv b \ \ and \ \ b_1 < 0 \tag{21}$$

Since **(17)** holds, we have always $0 < b < 1$.

r_1 is a best reply of player 3 if:

$$q_2 \geq (1-\varepsilon)\frac{\alpha(c_2 - c_1)}{(1-\alpha)(c_3 - c_4)} \equiv (1-\varepsilon)c \tag{22}$$

Since **(18)** holds and **(19)** does not hold in the situation considered, it follows that $0 < c < 1$.

First, consider the case that $b_1 < 0$. The *best-reply correspondences* are shown in figures 5 and 6 for arbitrary values of ε, b and c. For sufficiently small values of ε we must have $\varepsilon < b < 1 - \varepsilon$ and $\varepsilon < (1-\varepsilon)c < 1 - \varepsilon$.

Figure 5: Best-reply correspondence of player 2 in a subclass of subsection 3.2.

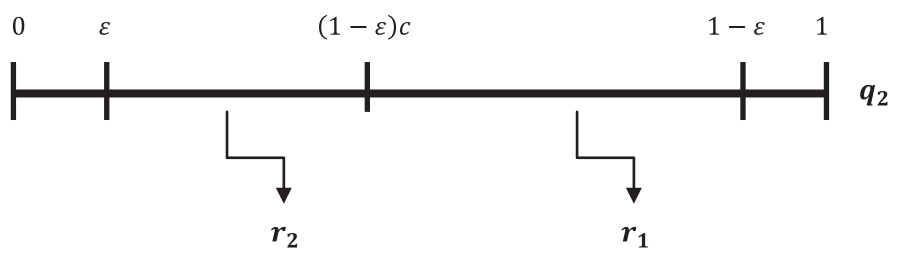


Figure 6: Best-reply correspondence of player 3 in a subclass of subsection 3.2.

Obviously, the (mixed) strategy combination $(q_2, q_3) = ((1-\varepsilon)c, b)$ is the only equilibrium point of the reduced game. Defining q_1 as player 1's probability to choose m_1 (his *"inside"* choice), the limit solution of the whole game is therefore $(q_1, q_2, q_3) = (1, c, b)$.

The situation is quite different for $b_1 < 0$. Player 3's best-reply correspondence is the same as in figure 6, and player 2's best-reply correspondence is obtained by interchanging m_1 and m_2 in figure 5. The reduced game has three strategy combinations (q_2, q_3) as equilibrium points: $(1-\varepsilon, 1-\varepsilon), (\varepsilon, \varepsilon)$ and $((1-\varepsilon)c, b)$. The third one is not in the *first candidate set* for the solution of the reduced game because it is not the solution of a *primitive formation* (see subsection 2.4). Therefore, the first candidate set contains only the two ε-extreme equilibrium points $(1-\varepsilon, 1-\varepsilon)$ and $(\varepsilon, \varepsilon)$. We first analyze under which conditions there is a payoff-dominance relationship between these two equilibrium points.

At the first equilibrium point, player 2's (expected) payoffs are approximately $(1-\alpha)b_1$ for sufficiently small ε, hence they are strictly positive (since $b_1 > 0$). At the second equilibrium point, his expected payoffs are approximately null. Simple computations show that the (expected) payoffs of player 3 are at least as much at the first equilibrium point than at the second one, if the following inequality holds:

$$\boldsymbol{c_3 \geq \frac{\alpha}{1-\alpha}(1-\varepsilon)(c_2 - c_1)} \qquad \textbf{(23)}$$

This inequality is independent of c_4 because the knot at which player 3 receives this payoff is reached in both equilibrium points with the same probability (i.e. $(1-\alpha)(1-\varepsilon)\varepsilon$). In the case we consider, **(19)** does not hold. Therefore we have:

$$\boldsymbol{c_3 - c_4 > \frac{\alpha}{1-\alpha} \cdot (c_2 - c_1)} \qquad \textbf{(24)}$$

Thus, **(23)** always holds if c_4 is nonnegative or if it is negative but its absolute value is small enough. Hence, the equilibrium point $(1-\varepsilon, 1-\varepsilon)$ payoff dominates the equilibrium point $(\varepsilon, \varepsilon)$ for sufficiently small ε if:

$$\boldsymbol{c_3 > \frac{\alpha}{1-\alpha} \cdot (c_2 - c_1)} \qquad \textbf{(25)}$$

In this case, the limit solution of the game is $(q_1, q_2, q_3) = (1,1,1)$.

If **(25)** does not hold (this implies that c_4 is negative and its absolute value is large enough) there is no payoff-dominance relationship between the two equilibrium points in the first candidate set. Therefore a *risk-dominance comparison* between the two candidates becomes necessary.

Since the reduced game is a 2x2-game, the lemma of subsection 2.5 can be applied. It implies that the sums of the probabilities chosen in the mixed equilibrium point for those strategies used in the *first* pure equilibrium point determines the result of the risk-dominance comparison. In our case it follows:

- If $b+(1-\varepsilon)c<1$, the equilibrium point $(q_2,q_3)=(1-\varepsilon,1-\varepsilon)$ risk-dominates the equilibrium point $(\varepsilon,\varepsilon)$. This condition is satisfied for each $\varepsilon>0$ if $b+c\leq 1$ holds. Hence, in this case we obtain $(q_1,q_2,q_3)=(1,1,1)$ as the limit solution of the game, too.
- On the other hand, if $b+c>1$ holds, the inequality $b+(1-\varepsilon)c>1$ is implied for sufficiently small ε. In this case the equilibrium point $(\varepsilon,\varepsilon)$ risk-dominates the equilibrium point $(1-\varepsilon,1-\varepsilon)$ and we obtain $(q_1,q_2,q_3)=(1,0,0)$ as the limit solution of the game.

3.4. Only one type forms a cell – he has the inferior choice "inside"

Now we consider the reduced perturbed game after fixing player 1 at his *"outside"* choice. Clearly, **(17)**, **(18)** and $c_3-c_4>0$ still hold. Player 3 obtains (after the fixing) an inferior choice r_2 if the following inequality holds:

$$\boldsymbol{\alpha(c_1-c_2)\leq(1-\alpha)(c_3-c_4)} \qquad \textbf{(26)}$$

The definition of c given by **(22)** implies that **(26)** is equivalent to $c\leq 1$. If **(26)** holds, player 3 is fixed at his r_1 choice, and, finally, player 2 must choose m_1 if $b_1>0$ or m_2 if $b_1<0$.

If **(26)** does not hold, we must look at the equilibrium points of the reduced perturbed game. The best-reply structure is still given by **(20)** and **(21)**. Different to **(22)**, r_1 is a best reply of player 3 if the following inequality holds:

$$\boldsymbol{q_2\geq\varepsilon\frac{\alpha(c_2-c_1)}{(1-\alpha)(c_3-c_4)}\equiv\varepsilon c} \qquad \textbf{(27)}$$

Since **(26)** does not hold, we have $\varepsilon<\varepsilon c$ for each $\varepsilon>0$ and we have $\varepsilon c<1-\varepsilon$ for sufficiently small ε. For $b_1<0$ the situation is illustrated in figures 7 and 8.

Figure 7: Best-reply correspondence of player 2 in a subclass of subsection 3.3.

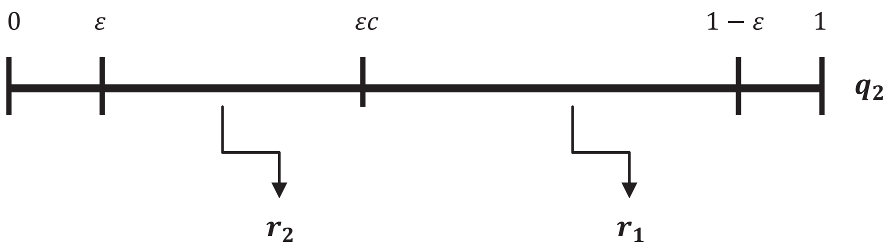


Figure 8: Best-reply correspondence of player 3 in a subclass of subsection 3.3.

The mixed-strategy combination $(q_2, q_3) = (\varepsilon c, b)$ is the only equilibrium point of the reduced perturbed game. Hence, for $\varepsilon \to 0$ we obtain $(q_1, q_2, q_3) = (0,0,b)$ as the limit solution of the whole game.

In the case $b_1 > 0$ the best-reply correspondence of player 2 is obtained by interchanging m_1 and m_2 in figure 7. Now the reduced game has the three equilibrium points $(1-\varepsilon, 1-\varepsilon)$, $(\varepsilon, \varepsilon)$ and $(\varepsilon c, b)$. The first two equilibrium points form the *first candidate set*. Like in subsection 3.3 player 2's (expected) payoffs at the $(1-\varepsilon, 1-\varepsilon)$-equilibrium point are approximately $(1-\alpha)b_1$ (hence, strictly positive) and approximately null at the $(\varepsilon, \varepsilon)$-equilibrium point. The (expected) payoffs of player 3 at the first equilibrium point are not smaller than at the second one if the following holds:

$$\boldsymbol{c_3 \geq \frac{\alpha}{1-\alpha}\varepsilon(c_2 - c_1)} \tag{28}$$

Since $c_2 - c_1 > 0$, inequality **(28)** is fulfilled for sufficiently small ε if $c_3 > 0$ holds. In this case the $(1-\varepsilon, 1-\varepsilon)$-equilibrium point payoff-dominates the $(\varepsilon, \varepsilon)$-equilibrium point. The limit solution of the game is $(q_1, q_2, q_3) = (0,1,1)$.

If $b_1 > 0$ and $c_3 \leq 0$, a risk-dominance comparison between the two ε-extreme equilibrium points is necessary. If ε goes to zero, then in the mixed equilibrium point of the reduced game the sum of the probabilities of the pure strategies used in the equilibrium point $(\varepsilon, \varepsilon)$ approaches $2 - b > 1$, whereas the sum of the probabilities of the pure strategies used in the equilibrium point $(1 - \varepsilon, 1 - \varepsilon)$ approaches $b < 1$. According to the lemma of subsection 2.5, we obtain the result that the $(1 - \varepsilon, 1 - \varepsilon)$-equilibrium point risk-dominates the $(\varepsilon, \varepsilon)$-equilibrium point. Again, the limit solution is $(q_1, q_2, q_3) = (0,1,1)$. Hence, for $b_1 > 0$ the limit solution is independent of the sign of c_3.

3.5. Only one type forms a cell – his pure strategies are semiduplicates

In this subsection we consider the reduced game after fixing player 1 at his centroid strategy (because his two pure strategies are semiduplicates). As before, **(13)**, **(14)** and $c_3 - c_4 > 0$ hold. After fixing player 1, player 3 gets an inferior choice r_1 if the following inequality holds:

$$\boldsymbol{\alpha(c_2 - c_1) \geq 2(1 - \alpha)(c_3 - c_4)} \qquad \textbf{(29)}$$

This inequality is equivalent to $c \geq 2$ (see the implicit definition of c given in **(22)**). If **(29)** holds player 3 is fixed at his choice r_2. Then, player 2 must choose m_1 if $b_1 > 0$ and m_2 if $b_1 < 0$.

If $c < 2$, player 3 has the best reply r_1 if we have:

$$\boldsymbol{q_2 \geq \frac{\alpha(c_2 - c_1)}{2(1 - \alpha)(c_3 - c_4)} \equiv \frac{c}{2}} \qquad \textbf{(30)}$$

The following analysis is quite similar to that of subsection 3.3, replacing $(1 - \varepsilon)c$ by $c/2$ (compare **(22)** and **(30)**). Thus, we present the results only briefly. For $c < 2$ and $b_1 < 0$ the mixed strategy combination $(q_2, q_3) = (c/2\,, b)$ is the only equilibrium point of the reduced game. Therefore, the limit solution of the game is $(q_1, q_2, q_3) = (1/2\,, c/2\,, b)$.

For $c < 2$ and $b_1 < 0$ there are three equilibrium points $(1 - \varepsilon, 1 - \varepsilon)$, $(\varepsilon, \varepsilon)$ and $(c/2\,, b)$. The $(1 - \varepsilon, 1 - \varepsilon)$-equilibrium point payoff-dominates the $(\varepsilon, \varepsilon)$-equilibrium point, if (compare with **(23)**):

$$\boldsymbol{c_3 \geq \frac{\alpha}{2(1 - \alpha)}(c_2 - c_1)} \qquad \textbf{(31)}$$

$c < 2$ implies:

$$\boldsymbol{c_3 - c_4 > \frac{\alpha}{2(1-\alpha)}(c_2 - c_1)} \qquad \textbf{(32)}$$

Thus **(31**) always holds if $c_4 > 0$ or it holds if c_4 is negative but with a small absolute value. In these cases, $(q_1, q_2, q_3) = (1/2\,, 1{,}1)$ is the limit solution of the game.

If **(31)** does not hold, a risk-dominance comparison between the two ε-extreme equilibrium points becomes necessary. Similar to subsection 3.3 it follows:

- If $b + c/2 < 1$ holds, the equilibrium point $(q_2, q_3) = (1-\varepsilon, 1-\varepsilon)$ risk-dominates the equilibrium point $(\varepsilon, \varepsilon)$. We obtain $(q_1, q_2, q_3) = (1/2\,, 1{,}1)$ as the limit solution of the game.
- If $b + c/2 > 1$ holds, the risk-dominance comparison is the very opposite and we obtain $(q_1, q_2, q_3) = (1/2\,, 0{,}0)$ as the limit solution of the game.
- If $b + c/2 = 1$ holds, the mixed equilibrium point $(q_2, q_3) = (c/2\,, b)$ is obtained as the solution of the reduced game. The limit solution of the whole game is $(q_1, q_2, q_3) = (1/2\,, c/2\,, b)$.

4. Normalization of the Indecomposable and Irreducible Games

In section 3 we characterized the conditions under which one of the types or the receiver form a cell and analyzed these cases. If none of the three players forms a cell by himself the signaling game is *indecomposable and irreducible.* This situation allows the following steps of normalization:

1. Call the "*inside"* choice of the two types m_1 and the *"outside"* choice m_2.
2. Call the receiver *player 3*.
3. A type receives nothing if he is inactive.
4. Subtract the payoff vector after a m_2-choice of each type from those three payoff vectors which can be reached if this type is the active one. By this subtraction the new payoff vectors of the normalized game are obtained.
5. For each type compute the difference of the *receiver's payoffs* achieved after his two responses, given that this type has become active and has chosen m_1. Multiply these differences with the respective probability of occurrence of the two types. Call that type *player 1* who induces the greater absolute value of these "weighted differences". If both types induce the same weighted difference, call by random some type *player 1*. Call the remaining type *player 2.*
6. Call player 1's probability of becoming active α.
7. Call that *response* of player 3 r_1 that yields the *smaller* payoff to him if player 1 becomes active and chooses m_1. Call the other response r_2. Indifference is not possible because the games considered in this section are indecomposable.

The extensive form of the normalized game is shown in figure 9. The following properties of the payoff structure result from the process of normalization described above and from the fact that in this section only indecomposable and irreducible games are considered.

$$\boldsymbol{a_1 \neq 0, \;\; a_2 \neq 0, \;\; sgn\, a_1 \neq sgn\, a_2} \tag{33}$$

$$\boldsymbol{b_1 \neq 0, \;\; b_2 \neq 0, \;\; sgn\, b_1 \neq sgn\, b_2} \tag{34}$$

$$\boldsymbol{c_1 < c_2} \tag{35}$$

$$\boldsymbol{c_3 > c_4} \tag{36}$$

$$\boldsymbol{\alpha(c_2 - c_1) \geq (1 - \alpha)(c_3 - c_4)} \qquad \textbf{(37)}$$

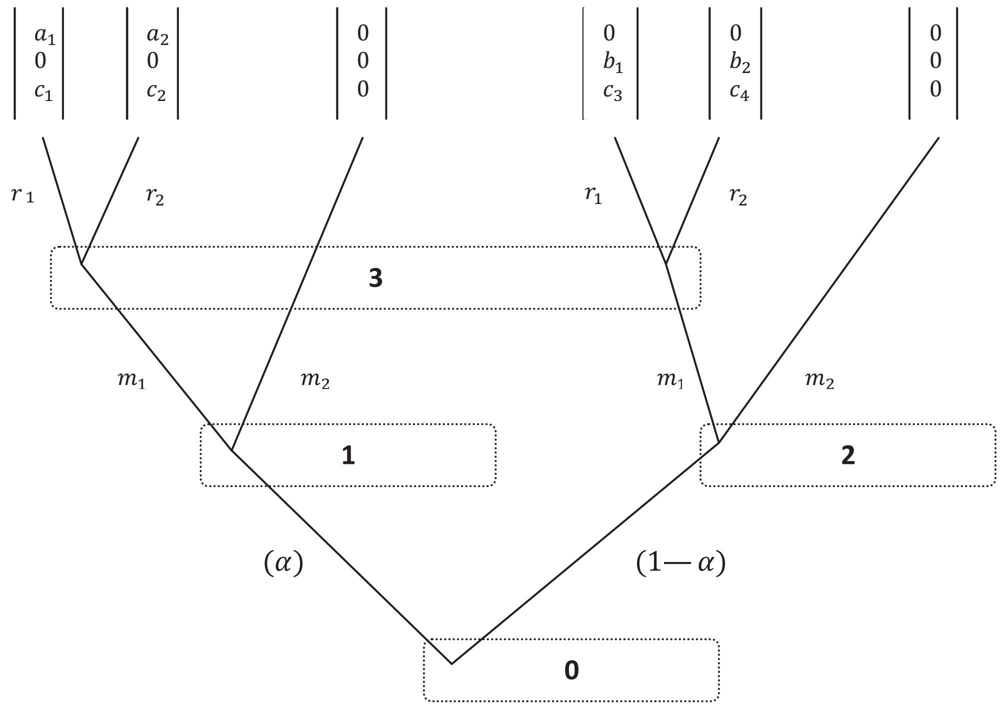


Figure 9: The extensive form of the normalized games.

In the following the probabilities that player 1 and player 2 choose m_1 are called q_1 and q_2, respectively. q_3 is the probability that player 3 chooses r_1. All three players have just two pure strategies, hence, player i's mixed strategy is completely described by q_i. The conditions that a player becomes indifferent between his two choices are calculated now.

INDIFFERENCE POINT OF PLAYER 1. The pure strategies m_1 and m_2 yield the same (expected) payoffs for player 1 if the following holds:

$$\boldsymbol{q_3 a_1 + (1 - q_3) a_2 = 0} \qquad \textbf{(38)}$$

This is equivalent to:

$$\boldsymbol{q_3 = \frac{-a_2}{a_1 - a_2} \equiv a} \qquad \textbf{(39)}$$

From **(33)** we can see that $0 < a < 1$ holds. If $a_1 < 0$, then m_1 is a best reply of player 1 if $q_3 \leq a$. If $a_1 > 0$, then m_1 is a best reply of player 1 if $q_3 \geq a$. The next results are obtained in a similar way.

INDIFFERENCE POINT OF PLAYER 2.

$$\boldsymbol{q_3 = \frac{-b_2}{b_1 - b_2} \equiv b} \qquad \textbf{(40)}$$

(34) ensures $0 < b < 1$. A best reply of player 2 is m_1 if $b_1 < 0$ and $q_3 \leq b$ hold simultaneously or if $b_1 > 0$ and $q_3 \geq b$ hold simultaneously.

INDIFFERENCE LINE OF PLAYER 3. The pure strategies r_1 and r_2 yield the same (expected) payoffs for player 3 if the following holds:

$$\boldsymbol{\alpha q_1 c_1 + (1-\alpha) q_2 c_3 = \alpha q_1 c_2 + (1-\alpha) q_2 c_4} \qquad \textbf{(41)}$$

This is equivalent to:

$$\boldsymbol{q_2 = \frac{\alpha(c_2 - c_1)}{(1-\alpha)(c_3 - c_4)} q_1 \equiv c q_1} \qquad \textbf{(42)}$$

Since **(35)**, **(36)** and **(37)** hold, it follows that $1 \leq c < \infty$. If $q_2 \geq c q_1$, then r_1 is a best reply of player 3, if $q_2 \leq c q_1$, then r_2 is a best reply of player 3.

We have to mention that the indifference points for players 1 and 2 and the indifference line for player 3 given **(39)**, **(40)** and **(42)** matter not only for the unperturbed game, but represent also the exact values of the perturbed game. We show this only for player 1. In the perturbed game his two ε-extreme strategies yield the same expected payoffs if:

$$\boldsymbol{(1-\varepsilon)(q_3 a_1 + (1-q_3) a_2) = \varepsilon (q_3 a_1 + (1-q_3) a_2)} \qquad \textbf{(43)}$$

Or, equivalently:

$$\boldsymbol{(1-2\varepsilon)(q_3 a_1 + (1-q_3) a_2) = 0} \qquad \textbf{(44)}$$

Since $\varepsilon < \frac{1}{2}$, equation **(44)** is equivalent to $q_3 = a$ (see **(38)** and **(39)**).

In this work we are only concerned with the generic cases of signaling games. The indecomposable and irreducible games are nongeneric if $a = b$ and/or if $c = 1$. Their solutions have also been calculated by the author (using if necessary the *logarithmic tracing procedure* and numerical methods), but their presentation will go beyond the scope of this work.

5. Solution of the Generic Indecomposable and Irreducible Games

In this section we solve the indecomposable and irreducible games with the additional properties $a \neq b$ and $c > 1$. Let $A_i(r_1)$ for $i = 1, 2$ be player i's best reply to a r_1 choice of player 3. We have to distinguish eight cases which are analyzed in the following subsections:

Subsection 5.1.: Case $a < b,\ A_1(r_1) = A_2(r_1) = m_1$

Subsection 5.2.: Case $a < b,\ A_1(r_1) = m_1,\ A_2(r_1) = m_2$

Subsection 5.3.: Case $a < b,\ A_1(r_1) = m_2,\ A_2(r_1) = m_1$

Subsection 5.4.: Case $a < b,\ A_1(r_1) = A_2(r_1) = m_2$

Subsection 5.5.: Case $a > b,\ A_1(r_1) = A_2(r_1) = m_1$

Subsection 5.6.: Case $a > b,\ A_1(r_1) = m_1,\ A_2(r_1) = m_2$

Subsection 5.7.: Case $a > b,\ A_1(r_1) = m_2,\ A_2(r_1) = m_1$

Subsection 5.8.: Case $a > b,\ A_1(r_1) = A_2(r_1) = m_2$.

Throughout this section we always assume that in the uniformly perturbed game the trembling hand parameter ε is sufficiently small, i.e.:

$$\boldsymbol{\varepsilon < \min\left(a, 1 - a, b, 1 - b, \frac{1}{c+1}\right)} \tag{45}$$

5.1. Case $a < b,\ A_1(r_1) = A_2(r_1) = m_1$

The best-reply correspondences of players 1 and 2 in the case considered are given in figure 10. In the following cases we omit the corresponding figures. They can be obtained easily by interchanging m_1 and m_2 for player i if $A_i(r_1) = m_2$ holds instead of $A_i(r_1) = m_1$, and by interchanging the positions of the markings of a and b if $a > b$ holds instead of $a < b$.

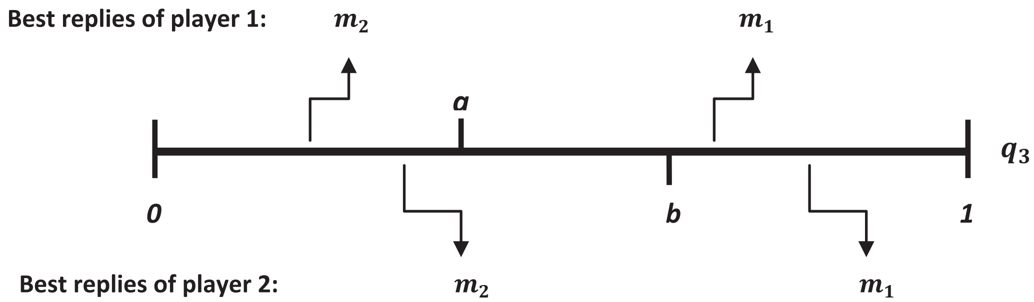
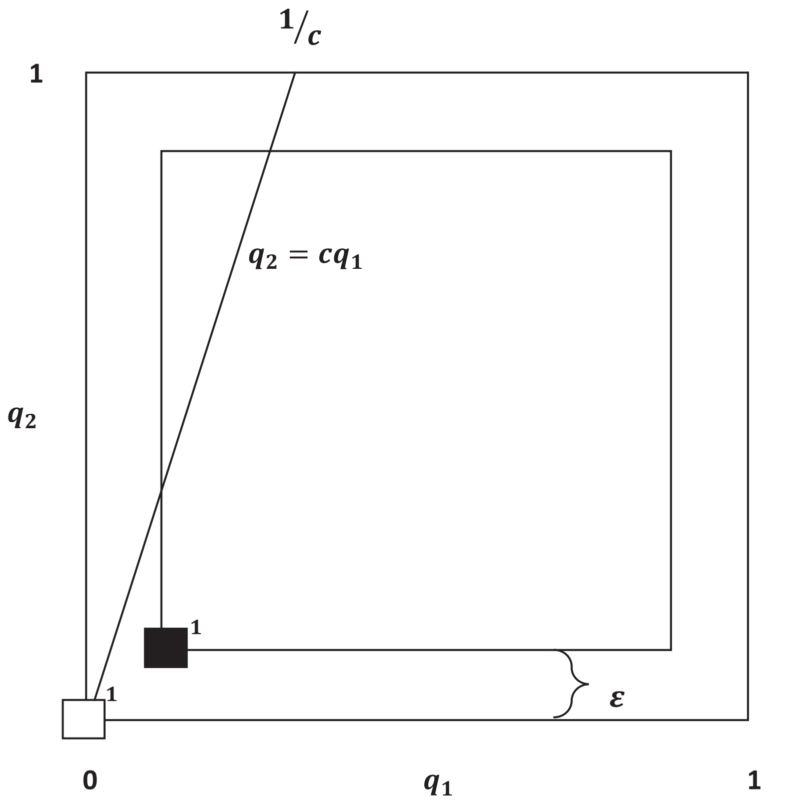


Figure 10: Best-reply correspondences of players 1 and 2 in dependence of player 3's strategy in case 5.1.

The best-reply correspondence of player 3 is shown in figure 11. Such figures are presented for all eight cases considered in this section. The horizontal axis of figure 10 refers to player 1's mixed strategy q_1 and the vertical axis to player 2's mixed strategy q_2. The inner square corresponds to the perturbed game whereas the outer square to the unperturbed game. The straight line $q_2 = cq_1$ shows the set of points at which player 3 is indifferent between his two choices (see **(42)**). Points above this line have the property that r_1 is player 3's unique best reply. The same is true for r_2 if we consider points below the indifference line. This follows from the discussion from **(42)** in section 4 and is true both for the unperturbed and the perturbed game.

In figure 11 and the corresponding figures of the remaining subsections we also mark equilibrium points or connected sets of equilibrium points by the symbol $\square^{\mathrm{i}}$ if we deal with the unperturbed game and by the symbol $\blacksquare^{\mathrm{i}}$ if we deal with the uniformly ε-perturbed game. The exact mathematical description of an equilibrium point q^i or of a set of equilibrium points Q^i is given in the text. q^i or Q^i correspond to $\square^{\mathrm{i}}$ in the following figures. Likewise, q^i_ε or Q^i_ε correspond to $\blacksquare^{\mathrm{i}}$. The index i is the number of different equilibrium points or sets of equilibrium points starting with $i = 1$ in case 5.1. It should be clear that equilibrium points in the lower left or the upper right corner of figures correspond to *pooling equilbria* because both types choose the same signal, whereas equilibrium points in the upper left or lower right corner are so-called *separating equilibria.*



<u>Figure 11:</u> Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.1.

As one can easily check with the help of figures 10 and 11, in case 5.1 the unperturbed game has the set Q^1, indicated by $\square^1$ in figure 11, as equilibrium points and no others:

$$\boldsymbol{Q^1 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, q_3 \leq a\}} \qquad \textbf{(46)}$$

But each perturbed game has for sufficiently small ε (see condition **(45)**) a unique equilibrium point, as indicated by $\blacksquare^1$ in figure 11:

$$\boldsymbol{q^1_\varepsilon = (\varepsilon, \varepsilon, \varepsilon)} \qquad \textbf{(47)}$$

Therefore, in case 5.1 the *limit solution* of the game is $q^1 = (0,0,0)$. Of course, $q^1 \in Q^1$.

5.2. Case $a < b$, $A_1(r_1) = m_1$, $A_2(r_1) = m_2$

Figure 12 illustrates the case considered in this subsection. The unperturbed game has a unique equilibrium point $q^2 = (1/c\,, 1, a)$ which is therefore the solution of the game. The unique equilibrium point of the perturbed game is:

$$\boldsymbol{q_\varepsilon^2 = ((1-\varepsilon)/c\,, 1-\varepsilon, a)} \qquad \textbf{(48)}$$

Clearly, $\lim_{\varepsilon\to 0} q_\varepsilon^2 = q^2$.

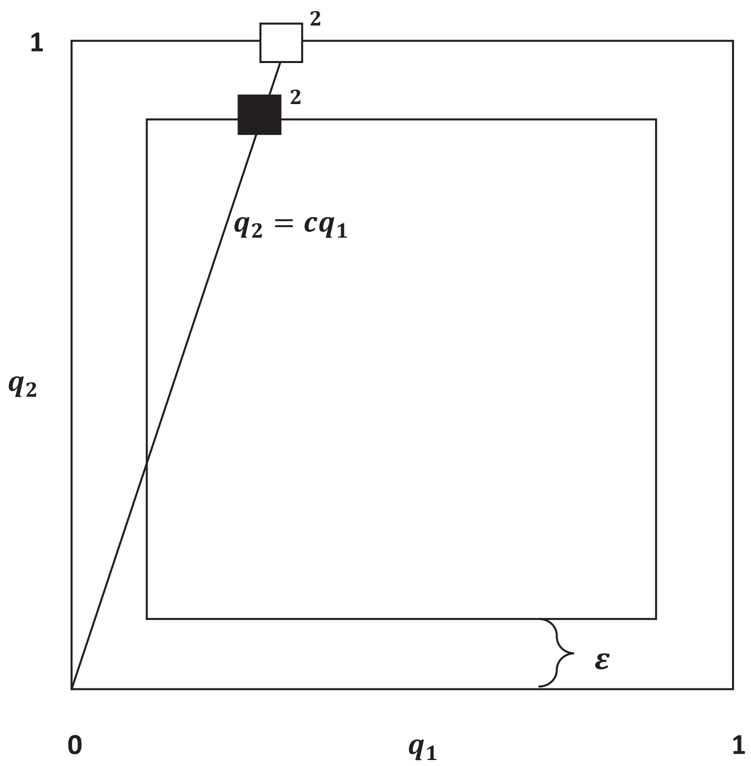


Figure 12: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.2.

5.3. Case $a < b$, $A_1(r_1) = m_2$, $A_2(r_1) = m_1$

The equilibrium points in this case are given as follows (see figure 13):

$$\boldsymbol{q^3 = (0, 1, 1)} \qquad \textbf{(49)}$$

$$q_\varepsilon^3 = (\boldsymbol{\varepsilon}, \mathbf{1}-\boldsymbol{\varepsilon}, \mathbf{1}-\boldsymbol{\varepsilon}) \quad \textbf{(50)}$$

$$q^4 = (\mathbf{1}, \mathbf{0}, \mathbf{0}) \quad \textbf{(51)}$$

$$q_\varepsilon^4 = (\mathbf{1}-\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \quad \textbf{(52)}$$

$$Q^5 = \{(q_1, q_2, q_3) | q_1 = q_2 = \mathbf{0}, a \le q_3 \le b\} \quad \textbf{(53)}$$

$$q_\varepsilon^5 = (\boldsymbol{\varepsilon}, c\boldsymbol{\varepsilon}, b) \quad \textbf{(54)}$$

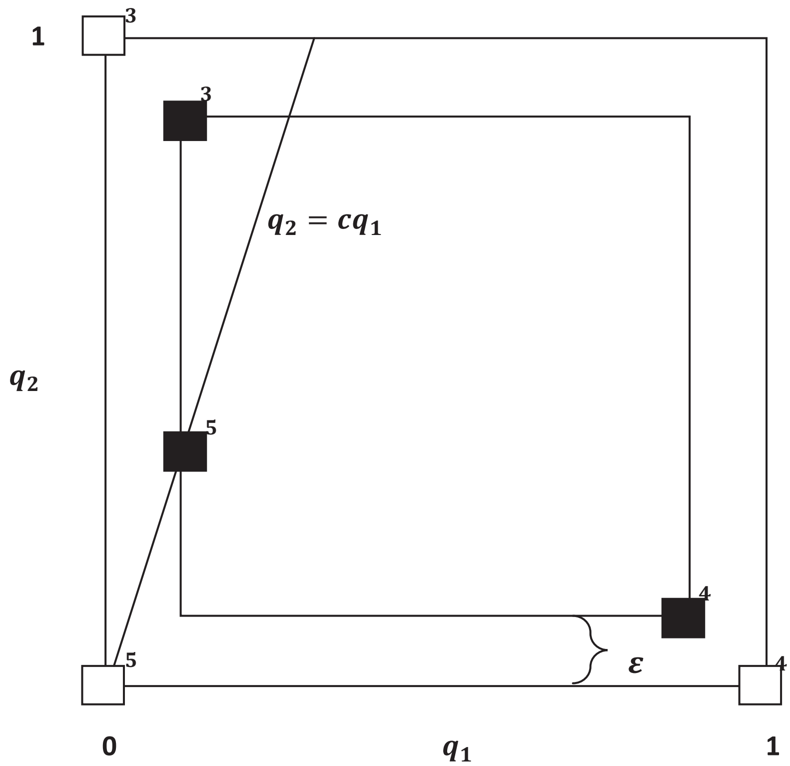


Figure 13: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.3.

Since q_ε^5 is not the solution of a *primitive formation*, the *first candidate set* of the perturbed game consists of q_ε^3 and q_ε^4. There is no *payoff-dominance* between these two equilibrium points because player 1 gets positive payoffs at q_ε^4 and zero payoffs at q_ε^3, whereas player 2 gets positive payoffs at q_ε^3 and zero payoffs at q_ε^4. The *linear tracing procedure* (see subsection 2.3) has to decide which equilibrium point *risk-dominates* the other one (see subsection 2.4).

To analyze the *path of the linear tracing procedure* we start with the determination of the *bicentric priors*. In the following pure strategy symbols with an additional lower index "ε" refer to ε-extreme strategies of the perturbed game. The bicentric priors of the first two players can be calculated easily with the help of the appropriate modification of figure 10. We obtain:

Bicentric prior of player 1:

$$\boldsymbol{p_1(m_{1_\varepsilon}) = \frac{a-\varepsilon}{1-2\varepsilon} \equiv \hat{a}} \tag{55}$$

Bicentric prior of player 2:

$$\boldsymbol{p_2(m_{1_\varepsilon}) = 1 - \frac{b-\varepsilon}{1-2\varepsilon} \equiv 1 - \hat{b}} \tag{56}$$

To compute the bicentric prior of the third player figure 14 is useful.

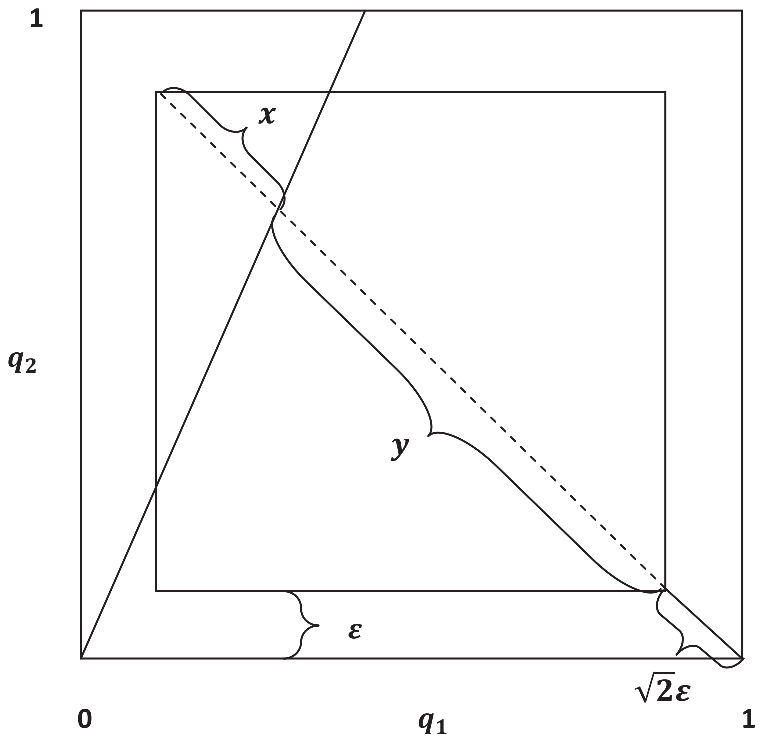


Figure 14: Visualization of player 3's bicentric prior in case 5.3.

Since player 3 assumes that the other player choose either $q^3_{-3_\varepsilon}$ or $q^4_{-3_\varepsilon}$, his expectations are formed along the dashed line in figure 14. x is that part of the whole dashed line at which r_1 is his best reply, y is the rest. Thus his bicentric prior is given by:

$$\boldsymbol{p_3(r_{1_\varepsilon}) = \frac{x}{x+y}} \tag{57}$$

Simple facts of geometry yield:

$$\boldsymbol{\frac{1}{c} = \frac{x+\sqrt{2}\varepsilon}{y+\sqrt{2}\varepsilon}} \tag{58}$$

Furthermore:

$$\boldsymbol{x+y = \sqrt{2}(1-2\varepsilon)} \tag{59}$$

We define $\hat{c}$ as follows:

$$\boldsymbol{\hat{c} \equiv \frac{c-\varepsilon(1+c)}{1-\varepsilon(1+c)}} \tag{60}$$

(57), **(58)**, **(59)** and **(60)** together yield the following result.

Bicentric prior of player 3:

$$\boldsymbol{p_3(r_{1_\varepsilon}) = \frac{1-\varepsilon(1+c)}{1+c-2\varepsilon(1+c)} \equiv \frac{1}{1+\hat{c}}} \tag{61}$$

The "hat" variables $\hat{a}$, $\hat{b}$ and $\hat{c}$, defined in **(55)**, **(56)** and **(60)** converge to a, b and c, respectively, if ε goes to zero. Therefore, $\hat{a} < \hat{b}$ and $\hat{c} > 1$ hold for sufficiently small ε. Let p_{-i} be the i-incomplete bicentric prior resulting from **(55)**, **(56)** and **(61)**. We now analyze what are the players' best replies to the bicentric priors, i.e. the starting point of the linear tracing procedure. Of course, the best replies of the first two players only depend on player 3's bicentric prior.

Best reply to p_{-1} for player 1:

m_{1_ε} if $1/(1+\hat{c}) \leq a$

m_{2_ε} if $1/(1+\hat{c}) \geq a$

Best reply to p_{-2} for player 2:

m_{1_ε} if $1/(1+\hat{c}) \geq b$

m_{2_ε} if $1/(1+\hat{c}) \leq b$

Best reply to p_{-3} for player 3:

r_{1_ε} if $1-\hat{b} \geq \hat{a}c$

r_{2_ε} if $1-\hat{b} \leq \hat{a}c$

Now we examine which combinations of best replies to the bicentric priors are impossible due to parameter restrictions:

i. If $1/(1+\hat{c}) \leq a$ holds then $1/(1+\hat{c}) \geq b$ is impossible because $a < b$.
ii. If $1/(1+\hat{c}) \leq a$ holds then $1-\hat{b} \geq \hat{a}c$ is impossible. The first inequality implies $1-a \leq a\hat{c}$, but this is a contradiction to $1-\hat{b} \geq \hat{a}c$ because $a\hat{c} \cong \hat{a}c$ and $a < \hat{b}$ for sufficiently small ε.
iii. By a similar argument as above we can conclude that $1/(1+\hat{c}) \geq b$ and $1-\hat{b} \leq \hat{a}c$ cannot hold simultaneously.

Next, we want to show the implications of some relations between a, b and c for their corresponding "hat" variables. From **(60)** it is clear that $\hat{c} > c$ holds for each ε. Thus we can conclude:

$$\mathbf{1/(1+\boldsymbol{c}) \leq \boldsymbol{a} \Rightarrow 1/(1+\hat{\boldsymbol{c}})} < a \tag{62}$$

$$\mathbf{1/(1+\boldsymbol{c}) \leq \boldsymbol{b} \Rightarrow 1/(1+\hat{\boldsymbol{c}})} < b \tag{63}$$

Now assume that $1-b = ac$ holds. With the help of **(55)** and **(56)** one can see that this equation is equivalent to the following one:

$$\mathbf{1-\hat{\boldsymbol{b}} = \hat{\boldsymbol{a}}\boldsymbol{c} + (\boldsymbol{c}-1)\frac{\boldsymbol{\varepsilon}}{1-2\boldsymbol{\varepsilon}}} \tag{64}$$

Since $c > 1$ holds we can conclude:

$$\mathbf{1-\boldsymbol{b} \geq \boldsymbol{ac} \Rightarrow 1-\hat{\boldsymbol{b}} > \hat{\boldsymbol{a}}\boldsymbol{c}} \tag{65}$$

After these preparations we can analyze the best replies to the bicentric priors for the four possible relations among a, b and c. The vector of best replies is denoted by q^0. We obtain:

$$\mathbf{1/(1+\boldsymbol{c}) \leq \boldsymbol{a}} < b \wedge 1-b < ac \Rightarrow \boldsymbol{q^0} = (\mathbf{1}-\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \tag{66}$$

$$\boldsymbol{a} < \mathbf{1/(1+\boldsymbol{c}) \leq \boldsymbol{b} \wedge 1-\boldsymbol{b}} < ac \Rightarrow \boldsymbol{q^0} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \tag{67}$$

$$\boldsymbol{a} < \mathbf{1/(1+\boldsymbol{c}) \leq \boldsymbol{b} \wedge 1-\boldsymbol{b} \geq \boldsymbol{ac}} \Rightarrow \boldsymbol{q^0} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}, \mathbf{1}-\boldsymbol{\varepsilon}) \tag{68}$$

$$\boldsymbol{a} < b < \mathbf{1/(1+\boldsymbol{c}) \wedge 1-\boldsymbol{b} \geq \boldsymbol{ac}} \Rightarrow \boldsymbol{q^0} = (\boldsymbol{\varepsilon}, \mathbf{1}-\boldsymbol{\varepsilon}, \mathbf{1}-\boldsymbol{\varepsilon}) \tag{69}$$

No problems arise in the situations given by **(66)** and **(69)** since in both cases the resulting q^0 is one of the two ε-extreme equilibrium points in the first candidate set of the perturbed game and, therefore, no player has an incentive to deviate from this strategy combination along the path of the linear tracing procedure. Thus, in the situation described in **(66)** the limit solution of the game is $q^4 = (1,0,0)$ and in the situation described in **(69)** the limit solution is $q^3 = (0,1,1)$.

The best replies to the bicentric priors in the situations described by **(67)** and **(68)** do not yield an equilibrium point. But in **(67)** the analysis is still simple: Player 2 and player 3 have no incentive to deviate from their initial strategies since they are not only best replies to the bicentric priors but also to q^0. This is not true for player 1, in consequence he must change his strategy if *t*, the tracing parameter, becomes sufficiently large. After he has changed his strategy from m_{2_ε} to m_{1_ε} the ε-extreme equilibrium point $q_\varepsilon^4 = (1-\varepsilon, \varepsilon, \varepsilon)$ is reached and no further change of strategies will occur along the remaining path of the tracing procedure. Thus, the limit solution of the game is $q^4 = (1,0,0)$.

The situation described in **(68)** is more difficult since here two players' (player 2 and player 3) best replies to the bicentric priors are not best replies to q^0. To analyze the path of the linear tracing procedure it must be determined *who is the first to change his strategy*. For this reason we calculate the *destabilization points* (see subsection 2.3) of players 2 and 3. We show that for sufficiently small ε player 2 is the first player to shift to his other strategy.

Player 2's destabilization point t_2 must satisfy the following equation:

$$(\mathbf{1} - \boldsymbol{t}_2)\frac{\mathbf{1}}{\mathbf{1} + \hat{\boldsymbol{c}}} + \boldsymbol{t}_2(\mathbf{1} - \boldsymbol{\varepsilon}) = \boldsymbol{b} \tag{70}$$

Thus t_2 is given as follows:

$$\boldsymbol{t}_2 = \frac{\boldsymbol{b}(\mathbf{1} + \hat{\boldsymbol{c}}) - \mathbf{1}}{(\mathbf{1} - \boldsymbol{\varepsilon})(\mathbf{1} + \hat{\boldsymbol{c}}) - \mathbf{1}} \tag{71}$$

Since $b > 1/(1 + \hat{c})$ holds (see **(63)** and **(68)**), the numerator is positive, and since $b < 1 - \varepsilon$ holds, the numerator is smaller than the denominator. Therefore, $0 < t_2 < 1$ holds for sufficiently small ε. We obtain:

$$\lim_{\varepsilon \to 0} \boldsymbol{t}_2 = \boldsymbol{b} - \frac{\mathbf{1} - \boldsymbol{b}}{\boldsymbol{c}} \tag{72 }$$

Player 2's destabilization point t_3 can be computed with the help of **(42)**:

$$(\mathbf{1} - \boldsymbol{t}_3)(\mathbf{1} - \widehat{\boldsymbol{b}}) + \boldsymbol{t}_3\boldsymbol{\varepsilon} = [(\mathbf{1} - \boldsymbol{t}_3)\widehat{\boldsymbol{a}} + \boldsymbol{t}_3\boldsymbol{\varepsilon}]\boldsymbol{c} \tag{73}$$

Simple computations yield:

$$\boldsymbol{t_3 = \frac{1 - \hat{b} - \hat{a}c}{1 - \hat{b} - \hat{a}c + \varepsilon(c - 1)}} \quad \textbf{(74)}$$

(68) ensures that the numerator is positive for sufficiently small ε and $c > 1$ ensures that the numerator is smaller than the denominator. Hence, $0 < t_3 < 1$ holds. However,

$$\lim_{\varepsilon \to 0} \boldsymbol{t_3 = 1} \quad \textbf{(75)}$$

From **(72)** we know that t_2 is positive but smaller than b for sufficiently small ε. Since $b < 1$ we can conclude that $t_2 < t_3$ holds for sufficiently small ε. This means that player 2 is the first player to shift his strategy. But after his shift (from m_{2_ε} to m_{1_ε}) the equilibrium point q_ε^3 is reached and for t with $t_2 < t \leq 1$ no further strategy changes occur. We have shown that in the situation described by **(68**) the limit solution is q^3.

Now our results for case 5.3 can be summarized. Since q^4 is the limit solution in the situations described by **(66)** and **(67)**, and q^3 is the limit solution in the situations described by **(68)** and **(69)**, we can claim: q^3 is the limit solution if $1 - b \geq ac$ holds. Otherwise q^4 is the limit solution.

5.4. Case $\boldsymbol{a < b,\ A_1(r_1) = A_2(r_1) = m_2}$

Figure 15 indicates the equilibrium points in case 5.4.

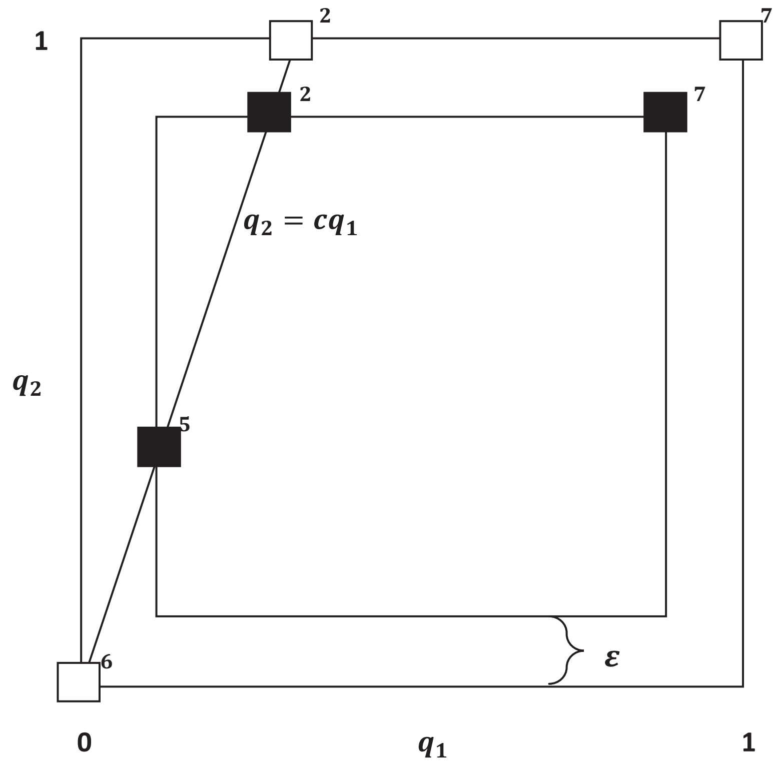


Figure 15: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.4.

The formal equivalents to the square symbols in figure 15 for q^2, q_ε^2 and q_ε^5 are still given by **(47)**, **(48)** and **(54)**, respectively. However, Q^6 is not identical with Q^5 given by **(53)**. Instead, we have:

$$\boldsymbol{Q^6 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, q_3 \geq b\}} \quad \textbf{(76)}$$

Furthermore:

$$\boldsymbol{q^7 = (1, 1, 0)} \quad \textbf{(77)}$$

$$\boldsymbol{q_\varepsilon^7 = (1-\varepsilon, 1-\varepsilon, \varepsilon)} \quad \textbf{(78)}$$

The equilibrium points q_ε^2 and q_ε^5 of the perturbed game are not solutions of *primitive formations*. Consequently the *first candidate set* contains only q_ε^7. For this reason the limit solution of the game is q^7.

5.5. Case $a > b$, $A_1(r_1) = A_2(r_1) = m_1$

Figure 16 illustrates the case considered now.

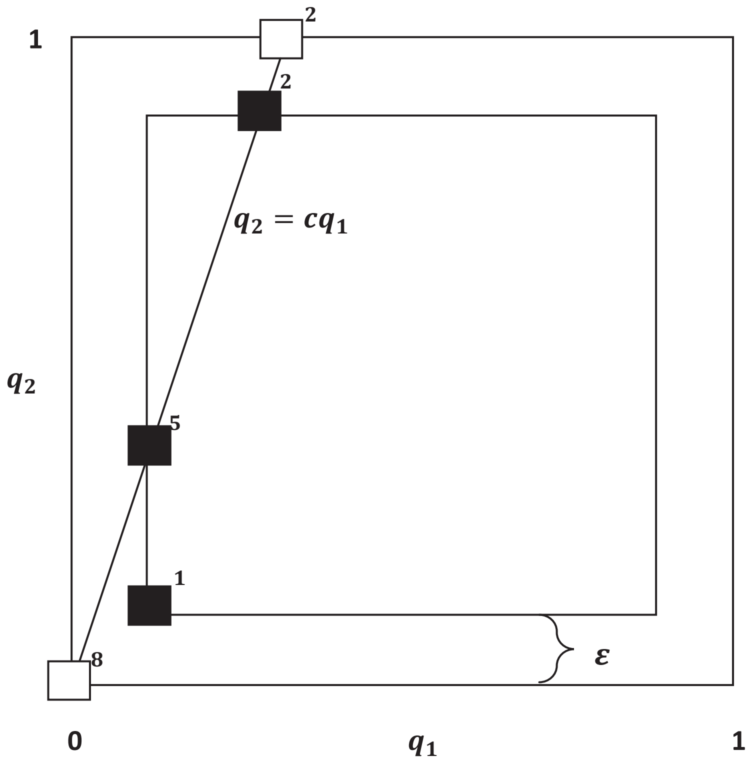


Figure 16: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.5.

q_ε^1, q^2, q_ε^2 and q_ε^5 are still given by **(46), (47)**, **(48)** and **(54)**, respectively. Furthermore:

$$\boldsymbol{Q^8 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, q_3 \leq b\}} \qquad \textbf{(79)}$$

Each perturbed game has three equilibrium points, but only q_ε^1 is the solution of a primitive formation. For this reason, the limit solution of the case 5.5 is $q^1 = (0,0,0)$.

5.6. Case $a > b,\ A_1(r_1) = m_1,\ A_2(r_1) = m_2$

This case is illustrated by figure 17.

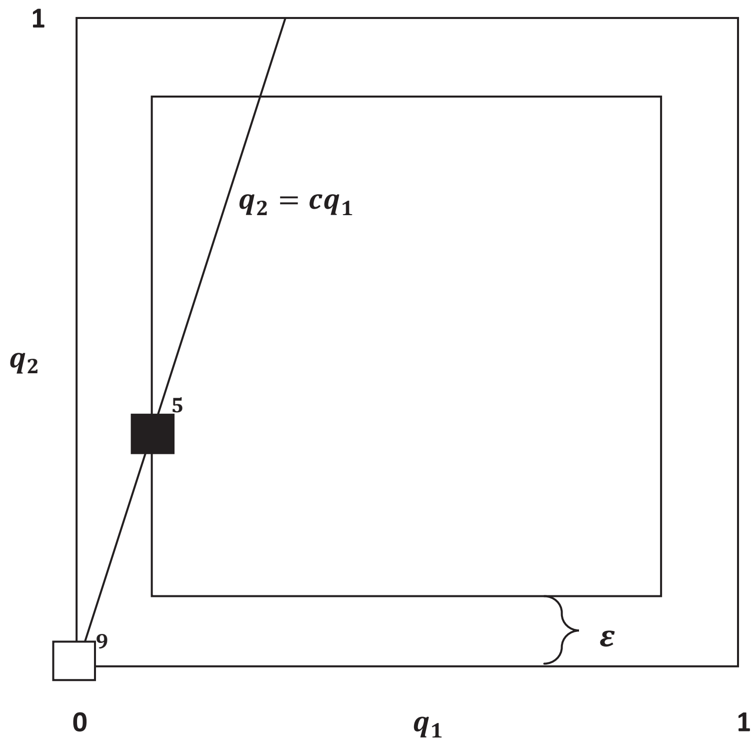


Figure 17: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.6.

The set Q^9 is given as follows:

$$\boldsymbol{Q^9 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, b \leq q_3 \leq a\}} \quad \textbf{(80)}$$

However, the perturbed game has the unique equilibrium point q_ε^5, given by **(54)**. Hence, the limit solution in this case is:

$$\boldsymbol{q^5 = (0, 0, b)} \quad \textbf{(81)}$$

5.7. Case $a > b,\ A_1(r_1) = m_2,\ A_2(r_1) = m_1$

Figure 18 indicates the equilibrium points in case 5.7.

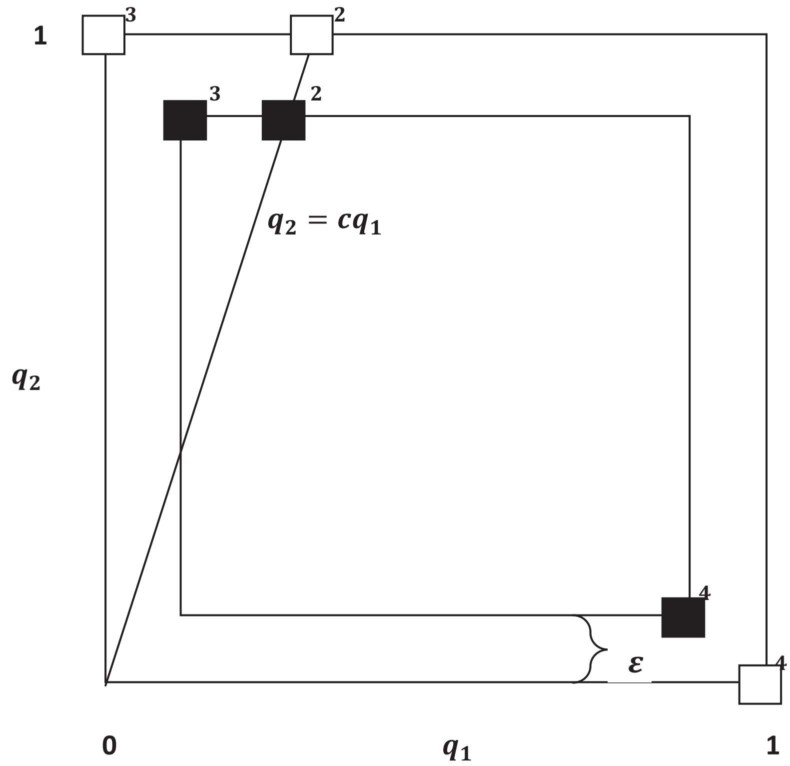


Figure 18: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.7.

$q^2 = (1/c\,, 1, a)$, and the other equilibrium points q^2, q_ε^2, q^3, q_ε^3, q^4 and q_ε^4 are given by **(48)** to **(52)**. Since q_ε^2 is not the solution of a primitive formation of the perturbed game, the first candidate set contains only q_ε^3 and q_ε^4. Obviously, there is payoff-dominance relationship between these two equilibrium points. The risk-dominance by means of the linear tracing procedure has to resolve which of the equilibrium points is the solution of the game.

Notice that case 5.7 is similar to case 5.3 except that $a > b$ holds instead of $a < b$. The *bicentric priors* are equivalent to those given by **(55)**, **(56)** and **(61)**. Moreover, the *best replies to the bicentric priors* are exactly the same as in the analysis of case 5.3, and we omit the repetition of the formulas. However, in the case at hand we can exclude *combinations of best replies to the bicentric priors* different from those in case 5.3. The "hat" variables $\hat{a}$, $\hat{b}$ and $\hat{c}$ are defined as in **(55)**, **(56)** and **(60)**.

i. If $1/(1+\hat{c}) \leq b$ holds then $1/(1+\hat{c}) \geq a$ is impossible because $b < a$.
ii. If $1/(1+\hat{c}) \leq b$ holds then $1-\hat{b} \geq \hat{a}c$ is impossible. The first inequality implies $1-b \leq b\hat{c}$, but this is a contradiction to $1-\hat{b} \geq \hat{a}c$ because $b \cong \hat{b}$ and $\hat{a}c > b\hat{c}$ hold for sufficiently small ε.
iii. By a similar argument as above we can conclude that $1/(1+\hat{c}) \geq a$ and $1-\hat{b} \leq \hat{a}c$ cannot hold simultaneously. The first inequality implies $1-a \geq a\hat{c}$, but this is a contradiction to $1-\hat{b} \leq \hat{a}c$ because $a\hat{c} \cong \hat{a}c$ and $a > \hat{b}$ hold for sufficiently small ε.

Note that the implications for the "hat" variables given by **(62)**, **(63)** and **(65)** still matter. Now we can list the possible relations between a, b and c and the resulting vectors of best replies to the bicentric priors, still denoted by q^0.

$$\boldsymbol{1/(1+c)} \leq \boldsymbol{b} < a \wedge 1-b < ac \Rightarrow \boldsymbol{q^0} = (\boldsymbol{1-\varepsilon}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \tag{82}$$

$$\boldsymbol{b} < \boldsymbol{1/(1+c)} \leq \boldsymbol{a} \wedge \boldsymbol{1-b} < ac \Rightarrow \boldsymbol{q^0} = (\boldsymbol{1-\varepsilon}, \boldsymbol{1-\varepsilon}, \boldsymbol{\varepsilon}) \tag{83}$$

$$\boldsymbol{b} < \boldsymbol{1/(1+c)} \leq \boldsymbol{a} \wedge \boldsymbol{1-b} \geq \boldsymbol{ac} \Rightarrow \boldsymbol{q^0} = (\boldsymbol{1-\varepsilon}, \boldsymbol{1-\varepsilon}, \boldsymbol{1-\varepsilon}) \tag{84}$$

$$\boldsymbol{b} < a < \boldsymbol{1/(1+c)} \wedge \boldsymbol{1-b} \geq \boldsymbol{ac} \Rightarrow \boldsymbol{q^0} = (\boldsymbol{\varepsilon}, \boldsymbol{1-\varepsilon}, \boldsymbol{1-\varepsilon}) \tag{85}$$

In the situation described by **(82)** the best replies to the bicentric priors establish the equilibrium point q_ε^4, and no player has an incentive to shift his strategy along the path of the tracing procedure. $q^4 = (1,0,0)$ is the limit solution of the game. Similarly, $q^3 = (0,1,1)$ is the limit solution in the situation given by **(85)**.

In the situation described by **(83)** player 2 is the only player whose initial strategy is not a best reply to q^0. Hence, if the tracing parameter t becomes sufficiently large player 2 will shift to his other ε-extreme strategy m_{2_ε}. Then the equilibrium point q_ε^4 is reached and no further strategy changes will occur in the remaining course of the linear tracing procedure. Therefore, the limit solution in this situation is $q^4 = (1,0,0)$.

The situation described by **(84)** is similar to that of **(68)**. Now player 1's and player 3's best replies to the bicentric priors are not best replies to q^0. We have to compute their *destabilization points* t_1 and t_3 to decide *who is the first player to shift to his other strategy*.

The computation of *player 1's destabilization point* t_1 is analogously to that of t_2 in **(71)** for the situation of **(68)**. We obtain:

$$\boldsymbol{t_1 = \frac{a(1+\hat{c})-1}{(1-\varepsilon)(1+\hat{c})-1}} \qquad \textbf{(86)}$$

Since $a > 1/(1+\hat{c})$ (see **(62)** and **(84)**) the numerator is positive and since $a < 1-\varepsilon$ the numerator is smaller than the denominator. Therefore, $0 < t_1 < 1$ holds for sufficiently small ε. It follows:

$$\boldsymbol{\lim_{\varepsilon\to 0} t_1 = a - \frac{1-a}{c}} \qquad \textbf{(87)}$$

Player 3's destabilization point t_3 can be obtained by interchanging the ε's by $(1-\varepsilon)$'s in formulas **(73)** and **(74)** because now player 1 and player 2 both choose m_{1_ε} in q^o instead of m_{2_ε} as in the situation of **(68)**. Simple computations yield:

$$\boldsymbol{t_3 = \frac{1-\hat{b}-\hat{a}c}{1-\hat{b}-\hat{a}c+(1-\varepsilon)(c-1)}} \qquad \textbf{(88)}$$

In view of **(65)**, **(84)** and $c > 1$ it is clear that $0 < t_3 < 1$ holds for sufficiently small ε. However, different to **(75)** we obtain now:

$$\boldsymbol{\lim_{\varepsilon\to 0} t_3 = \frac{1-b-ac}{c-b-ac}} \qquad \textbf{(89)}$$

From **(84)** it follows that $0 \le \lim_{\varepsilon\to 0} t_1 < 1$ and that $0 \le \lim_{\varepsilon\to 0} t_3 < 1$ holds. Unfortunately we cannot identify one player who is *always* the first to shift his strategy. For example, let $a = .4,\ b = .1,\ c = 2.0$ (satisfying the conditions of **(84)**). We obtain $\lim_{\varepsilon\to 0} t_1 = 1/10 > \lim_{\varepsilon\to 0} t_3 = 1/11$. Thus, for these parameter values player 3 is the first to shift his strategy. But for $a = .34,\ b = .22,\ c = 2.0$, satisfying also the conditions of **(84)**, we obtain $\lim_{\varepsilon\to 0} t_1 = 1/100 < \lim_{\varepsilon\to 0} t_3 = 1/11$, and player 1 is the first to shift his strategy. Hence, we have to look closer at the parameters.

The condition that $\lim_{\varepsilon\to 0} t_1 = \lim_{\varepsilon\to 0} t_3$ yields the following relation among the parameters *a, b* and *c*:

$$\boldsymbol{b = \frac{2c-ac(c+1)(2-a)}{(c+1)(1-a)}} \qquad \textbf{(90)}$$

If **(90)** holds with "<" instead of "=", we obtain $\lim_{\varepsilon\to 0} t_1 < \lim_{\varepsilon\to 0} t_3$ and, therefore, $t_1 < t_3$ for sufficiently small ε. Otherwise, if **(90)** holds with ">" instead of "=", we obtain $\lim_{\varepsilon\to 0} t_1 > \lim_{\varepsilon\to 0} t_3$ and, therefore, $t_1 > t_3$ for sufficiently small ε. But now consider the case that **(90)** holds strictly. To answer the question who is the first to change his strategy we look directly at t_1 and t_3 as given by **(86)** and **(88)** and substitute the "hat" variables by their definitions in **(55)**, **(56)** and **(60)**. Some tedious definitions show that $t_1 < t_3$ holds for sufficiently small ε if and only if:

$$\boldsymbol{b < \frac{2c - ac(c+1)(2-a) + a(c+1)(3c - 2 - 2\varepsilon(c-1))}{(c+1)(1-a-\varepsilon)}} \tag{91}$$

Obviously, the numerator of the right-hand side of **(90)** is greater than that of **(89)** and the denominator of the right-hand side of **(90)** is smaller than that of **(89).** Because all numerators and denominators are positive for sufficiently small ε it follows that **(89)** implies **(90)**. Hence, if **(89)** holds, player 1 is the first player to shift to his other ε-extreme strategy.

Now we must consider the consequences of a strategy shift of player 1 or player 3 along the path of the linear tracing procedure in the situation described by **(84)**. If player 1 is the first to shift his strategy the strategy combination $(\varepsilon, 1-\varepsilon, 1-\varepsilon)$, i.e. the equilibrium point q_ε^3, is reached and no further strategy changes occur afterwards in the remaining course of the linear tracing procedure.

If player 3 is the first to shift his strategy the strategy combination $(1-\varepsilon, 1-\varepsilon, \varepsilon)$ is reached. This is not an equilibrium point of the perturbed game, but now player 2 is the only player whose momentary strategy is not a best reply to the other players' momentary strategies. So player 2 is the next one who changes his strategy. Then the strategy combination $(1-\varepsilon, \varepsilon, \varepsilon)$, i.e. the equilibrium point q_ε^4, is reached und is sustained until the end of the tracing procedure.

The analysis of the situation given by **(84)** can be summarized as follows. The equilibrium point $q^3 = (0,1,1)$ is the limit solution of the game if the following holds:

$$\boldsymbol{b \leq \frac{2c - ac(c+1)(2-a)}{(c+1)(1-a)}} \tag{92}$$

If **(92)** does not hold, $q^4 = (1,0,0)$ is the limit solution.

Connecting this result for **(84)** with those obtained for **(82)**, **(83)** and **(85)** we can claim:

- If *either* $1 - b \leq ac$ holds *or* if **(84)** holds but **(92)** does not hold, $q^4 = (1,0,0)$ is the limit solution in case 5.7.
- Otherwise, $q^3 = (0,1,1)$ is the limit solution of the game in case 5.7.

5.8. Case $a > b,\ A_1(r_1) = A_2(r_1) = m_2$

The final generic case is illustrated by figure 19.

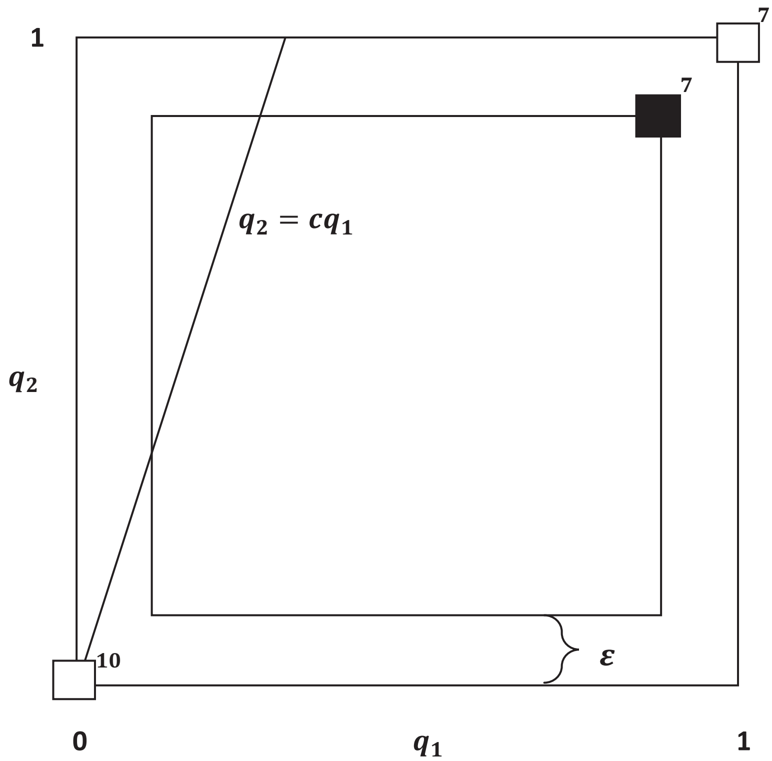


Figure 19: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.8.

The equilibrium points q^7 and q_ε^7 are explained by **(77)** and **(78)**. Q^{10} is given as follows:

$$\boldsymbol{Q^{10}} = \{(\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3})|\boldsymbol{q_1} = \boldsymbol{q_2} = \boldsymbol{0}, \boldsymbol{q_3} \geq \boldsymbol{a}\} \qquad \textbf{(93)}$$

However Q^{10} has no corresponding equilibrium points in the perturbed game. Hence, q_ε^7 is the unique equilibrium point of the perturbed game and $q^7 = (1,1,0)$ is the limit solution of the game.

6. Overview of the Results

In this section we present an overview of the limit solutions for all generic games of the class of signaling games investigated. The solutions were derived in sections 3 and 5. If someone is interested in a special game this overview can be used to pick up quickly its solution.

The first step to find the solution for a particular game is to check whether some strategy sets are semiduplicate classes or whether inferior choices exist. If this is the case the particular player forms an elementary cell and the game is decomposable and reducible (**part A** of this overview reports the results of section 3). **Part B** presents the results of the indecomposable and irreducible games calculated in section 5.

Part A: Solutions of Decomposable and Reducible Games

A1: At least the receiver forms an elementary cell.

After fixing the receiver, both types eventually form cells. When they are fixed, the solution is obtained.

The following case distinctions of part A are concerned with situations where the receiver does not initially form a cell but at least one type does.

A2: Both types form cells.

After fixing the two types, the conditional probabilities that the decision node after player 1's "*inside*" choice is reached are given in table 1. With the help of this table the receiver can easily compute his best reply and the solution is obtained. For convenience we repeat table 1 here.

<table>
<tr><td colspan="2" rowspan="2">Probability for player 3's left node after fixing the types</td><td colspan="3">Player 2</td></tr>
<tr><td>Inferior choice "inside"</td><td>Inferior choice "outside"</td><td>Semiduplicate Class</td></tr>
<tr><td rowspan="3">Player 1</td><td>Inferior choice "inside"</td><td>α</td><td>$\frac{\alpha}{\alpha\varepsilon + (1-\alpha)(1-\varepsilon)}$</td><td>$\frac{2\alpha\varepsilon}{1-(1-2\varepsilon)\alpha}$</td></tr>
<tr><td>Inferior choice "outside"</td><td>$\frac{\alpha(1-\varepsilon)}{\alpha(1-\varepsilon) + (1-\alpha)\varepsilon}$</td><td>$\alpha$</td><td>$\frac{2\alpha(1-\varepsilon)}{1+(1-2\varepsilon)\alpha}$</td></tr>
<tr><td>Semi-duplicate class</td><td>$\frac{\alpha}{\alpha + 2(1-\alpha)\varepsilon}$</td><td>$\frac{\alpha}{\alpha + 2(1-\alpha)(1-\varepsilon)}$</td><td>$\alpha$</td></tr>
</table>

Table 1: Conditional probabilities that the node after player 1's *"inside"* choice is reached, given that the receiver observed an *"inside"* choice.

The remaining case distinctions of part A are concerned with situations where only one type forms a cell. We call him player 1.

A3: Player 1 has the inferior choice "*outside*"

Subcases	**Solution (q_1, q_2, q_3)**
$c \geq 1, b_1 < 0$	(1,1,0)
$c \geq 1, b_1 > 0$	(1,0,0)
$c < 1, b_1 < 0$	(1,c,b)
$c < 1, b_1 > 0, c_3 \geq \alpha(c_2 - c_1)/(1-\alpha)$	(1,1,1)
$c < 1, b_1 > 0, c_3 < \alpha(c_2 - c_1)/(1-\alpha)$, b+c$\leq$ 1	(1,1,1)
$c < 1, b_1 > 0, c_3 < \alpha(c_2 - c_1)/(1-\alpha)$, b+c> 1	(1,0,0)

Table 2: Player 1 has the inferior choice "*outside*".

A4: Player 1 has the inferior choice *"inside"*

Subcases	Solution (q_1, q_2, q_3)
$b_1 > 0$	(0,1,1)
$c > 1, b_1 < 0$	(0,0,b)
$c \leq 1, b_1 < 0$	(0,0,1)

Table 3: Player 1 has the inferior choice "*inside*".

A5: Player 1's choices are semiduplicates

Subcases	Solution (q_1, q_2, q_3)
$c \geq 2, b_1 > 0$	(1/2,1,0)
$c \geq 2, b_1 < 0$	(1,/2,0,0)
$c < 2, b_1 < 0$	(1/2,c/2,b)
$c < 2, b_1 > 0, c_3 > \alpha(c_2 - c_1)/2(1 - \alpha)$	(1/2,1,1)
$c < 2, b_1 > 0, c_3 \leq \alpha(c_2 - c_1)/(1 - \alpha)$, b+$c/2 < 1$	(1/2,1,1)
$c < 2, b_1 > 0, c_3 \leq \alpha(c_2 - c_1)/(1 - \alpha)$, b+$c/2 = 1$	(1/2,c/2,b)
$c < 2, b_1 > 0, c_3 \leq \alpha(c_2 - c_1)/(1 - \alpha)$, b+$c/2 \geq 1$	(1/2,0,0)

Table 4: Player 1's choices are semiduplicates.

Part B: Solutions of Indecomposable and Irreducible Games

Cases	Solution (q_1, q_2, q_3)
B1: $a < b, A_1(r_1) = A_2(r_1) = m_1$	(0,0,0)
B2: $a < b, A_1(r_1) = m_1, A_2(r_1) = m_2$	(1/c,1,a)
B3: : $a < b, A_1(r_1) = m_2, A_2(r_1) = m_1$	
Subcase: $1 - b < ac$	(1,0,0)
Subcase: $1 - b \geq ac$	(0,1,1)
B4: $a < b, A_1(r_1) = A_2(r_1) = m_2$	(1,1,0)
B5: $a > b, A_1(r_1) = A_2(r_1) = m_1$	(0,0,0)
B6: : $a > b, A_1(r_1) = m_1, A_2(r_1) = m_2$	(0,0,b)
B7: : $a > b, A_1(r_1) = m_2, A_2(r_1) = m_1$	
Subcase: $1 - b < ac$	(1,0,0)
Subcase: $1 - b \geq ac, b < \frac{1}{1+c} \leq a,$ $b > (2c - ac(c+)(2 - a)/(c + 1)(1 - a)$	(1,0,0)
Subcase: $1 - b \geq ac, b < \frac{1}{1+c} \leq a,$ $b \leq (2c - ac(c+)(2 - a)/(c + 1)(1 - a)$	(0,1,1)
Subcase: $1 - b \geq ac, a < \frac{1}{1+c}$	(0,1,1)
B8: $a > b, A_1(r_1) = A_2(r_1) = m_2$	(1,1,0)

Table 5: Solutions of indecomposable and irreducible games.

Summary

In this paper we apply the Harsanyi-Selten solution to a class of simple signaling games. Somebody who is not familiar with the theory of Harsanyi and Selten can use this paper as an introduction and can observe different concepts and procedures at work. The overview of the results allows for easy application to economic or other models and for comparisons to the outcomes of alternative equilibrium selection criteria.

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Otto von Guericke University Magdeburg
Faculty of Economics and Management
P.O. Box 4120 | 39016 Magdeburg | Germany

Tel.: +49 (0) 3 91 / 67-1 85 84
Fax: +49 (0) 3 91 / 67-1 21 20

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