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Group Theoretic Approach to Internal and Collective Degrees of Freedom in Mechanics and Field Theory

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Discussed are group-theoretical models of collective degrees of freedom of extended bodies and internal degrees of freedom of point-like objects. We concentrate on the use of groups $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$ and $U(n)$. Relationships with the theory of integrable systems are mentioned.

We have used the terms "collective and internal degrees of freedom". What, roughly, do they mean? Let us suppose that we consider a system with an infinite (including non-denumerable) number of degrees of freedom or with a finite but rather large number of degrees of freedom. Further, let us suppose that it is rather a small number of parameters that is relevant for the system behaviour and that those parameters are non-local in the sense that all degrees of freedom of individual particles enter them, roughly speaking, on the equal footing. If those relevant parameters obey approximately some autonomous evolution equations, we say they are collective variables and the corresponding evolution equations give an account of the behaviour of the system. Formally this means that we are dealing with some quotient manifold or with submanifold of the configuration space Q (the submanifold consists of representatives of cosets). Internal degrees of freedom of objects which are essentially non-extended or which are so small that details of their spatial structure are hidden, are described in such a way that conversely, their configuration spaces Q are fibre bundles over the space or space-time manifold M. The base points describe their spatial localization, whereas the fibre points are internal variables.

Usually the typical models of collective and internal degrees of freedom have to do with Lie groups or their homogeneous spaces. Because of this quite often some rigorous solutions may be found at least in the form of series; this is due to the analycity of Lie groups.

On the classical level the use of Lie groups as configuration spaces has an additional advantage, namely their tangent and cotangent bundles are canonically trivialized in two ways

$$TG \simeq G \times G', \qquad T^*G \simeq G \times G'^*. \tag{1}$$

G' denoting the Lie algebra and G'^* – its dual. For example, if G is a matrix group, then the elements of TG are identified with pairs

$$(g, \omega) \qquad \text{or} \qquad (g, \omega'), \tag{2}$$

where

$$\omega = \dot{g}\, g^{-1}, \qquad \omega' = g^{-1}\, \dot{g}, \qquad \omega = g\, \omega'\, g^{-1}. \tag{3}$$

The canonical example is the rigid body with the configuration space $SO(n, \mathbb{R})$. ω is the angular velocity in laboratory representation, whereas the matrix elements of ω' are co-moving components of the angular velocity. Similarly, the elements of T^*G are then represented by pairs

$$(g, \sigma) \qquad \text{or} \qquad (g, \sigma'), \tag{4}$$

where $\sigma \in SO(n, \mathbb{R}')^* \simeq SO(n, \mathbb{R})$ is a laboratory representation of the internal canonical angular momentum (spin), and $\sigma' \in SO(n, \mathbb{R}')^* \simeq SO(n, \mathbb{R})$ is its co-moving representation; $\sigma = g\sigma' g^{-1}$. The quantities σ, σ' are Hamiltonian generators (momentum maps) of left and right regular translations in $SO(n, \mathbb{R})$, respectively.

Our subject here is an affine model of collective or internal degrees of freedom, i.e., an affine top, affinely-rigid body. This is a natural extension of gyroscopic degrees of freedom, because affine geometry is more primary, more fundamental than the Euclidean one.

Of course, the dominant role in physics is played by orthogonal and pseudo-orthogonal groups which are symmetries of the Euclidean space or pseudo-Euclidean space–time. Similarly, in typical physical theories except the gauge gravitation theory, internal symmetries are described by unitary groups. Nevertheless, the amorphous affine symmetry was also used in various branches of physics, for example in the collective models of nuclei. Linear groups occur also as non-invariance groups in certain quantum multi-body problems. There were also attempts to base the gauge approach to gravitation on the linear group $GL(4, \mathbb{R})$ as a gauge group.

Affinely-rigid body is a system, the behaviour of which is confined in such a way that all affine geometric relationships between its constituents remain invariant, similarly as in the usual rigid body all metrical relationships are frozen. This is a compromise between rigid body mechanics and the general theory of deformable objects, which may be considered as dynamical systems on the infinite-dimensional Lie group of all diffeomorphisms in $\mathbb{R}^n$ (physically $n = 3$) or volume preserving diffeomorphisms (ideal fluids) (Arnold, 1978; Binz, 1971). Such a model and other discretization-based approaches provide a convenient and reasonable simplification, because it is rather difficult to be rigorous when dealing with infinite-dimensional groups (Slawianowski, 1982).

In an affinely-rigid motion the material and spatial Cartesian coordinates of material points a^K, x^i are related to each other by the formula

$$x^i(t,a) = \varphi^i{}_K(t)\, a^K + r^i(t), \tag{5}$$

where φ is a time-dependent non-singular matrix and r is a time-dependent vector. They describe the relative motion and the centre of mass motion, respectively. Metric tensors of the physical and material spaces will be denoted by g and η, respectively. Obviously in Cartesian coordinates both are represented by the Kronecker matrix. The mass distribution in material representation is described by the fixed positive measure μ, and the density of forces per unit mass by the vector-valued function Φ. Inertial properties of the body are described by the total mass M and the constant tensor of internal inertia

$$M = \int d\mu(a), \qquad \mathcal{J}^{KL} = \int a^K\, a^L\, d\mu(a). \tag{6}$$

The total force and dipole moment of forces are given by

$$F^i = \int \Phi^i\,(t, r, \varphi, \dot{r}, \dot{\varphi}, a)\, d\mu(a), \tag{7}$$

$$N^{ij} = \int \left(x^i(t,a) - r^i\right)\, \Phi^j\,(t, r, \varphi, \dot{r}, \dot{\varphi}, a)\, d\mu(a). \tag{8}$$

N^{ij} is related to the centre of mass and both F, N depend on r, φ, $\dot{r}$, $\dot{\varphi}$ and perhaps explicitely on t. In a more explicit form

$$N^{ij} = \varphi^{iK} \int a^K\, \Phi^j\, d\mu(a). \tag{9}$$

The doubled skew-symmetric part of N equals the usual moment of forces (torque). D'Alembert principle implies that equations of motion have the form

$$M\,\frac{d^2 r^i}{dt^2} = F^i, \qquad \varphi^i{}_K\,\frac{d^2\varphi^j{}_L}{dt^2}\,\mathcal{J}^{KL} = N^{ij}. \tag{10}$$

The non-holonomic affine velocity in laboratory and co-moving representations is given by

$$\Omega^i{}_j = \frac{d\varphi^i{}_A}{dt}\,\varphi^{-1\,A}{}_j, \qquad \hat{\Omega}^A{}_B = \varphi^{-1\,A}{}_i\,\frac{d\varphi^i{}_B}{dt}. \tag{11}$$

If the motion is rigid

$$\eta_{AB} = g_{ij}\,\varphi^i{}_A\,\varphi^j{}_B, \tag{12}$$

then Ω^{ij}, $\hat{\Omega}^{AB}$ are skew-symmetric (indices moved respectively by g and η) and become the usual angular velocities.

Introducing the dipole moment of the linear momentum distribution

$$K^{ij} = \int \left(x^i(t,a) - r^i\right)\,\left(\dot{x}^j(t,a) - \dot{r}^j\right)\,d\mu(a) = \varphi^i{}_K\,\frac{d\varphi^j{}_L}{dt}\,\mathcal{J}^{KL}, \tag{13}$$

and its co-moving representation

$$K^{AB} = \varphi^{-1\,A}{}_i\,\varphi^{-1\,B}{}_j\,K^{ij}, \tag{14}$$

one can write our equations of motions as balance laws, obtaining in particular the affine counterpart of the rigid body Euler equations

$$\frac{dP^i}{dt}\;F^i, \qquad \frac{dK^{ij}}{dt} = \frac{d\varphi^i{}_A}{dt}\,\frac{d\varphi^j{}_B}{dt}\,\mathcal{J}^{AB} + N^{ij}. \tag{15}$$

Obviously, $P^i = M\,\dfrac{dr^i}{dt}$ is the total linear momentum.

Let us quote also other equivalent forms of the above system

$$\frac{dP^i}{dt} = F^i, \qquad \frac{dK^{ij}}{dt} = \Omega^i{}_m K^{mj} + N^{ij}, \tag{16}$$

$$\frac{dP^A}{dt} = -P^B\,\tilde{\mathcal{J}}_{BC}\,K^{CA} + F^A, \qquad \frac{dK^{AB}}{dt} = -K^{AC}\,\tilde{\mathcal{J}}_{CD}\,K^{DB} + N^{AB}, \tag{17}$$

$$M\,\frac{dv^A}{dt} = -M\hat{\Omega}^A{}_B v^B + F^A, \qquad \frac{d\hat{\Omega}^B{}_C}{dt}\,\tilde{\mathcal{J}}^{CA} = -\hat{\Omega}^B{}_D\,\hat{\Omega}^D{}_C\mathcal{J}^{CA} + N^{AB}, \tag{18}$$

where quantities with capital indices refer to the comoving system, and

$$\tilde{\mathcal{J}}^{AC}\,\mathcal{J}_{CB} = \delta^A{}_B. \tag{19}$$

If the system is non-dissipative and has the Lagrangian $L = T - V(r,\varphi)$, then

$$F^i = -g^{ik}\,\frac{\partial V}{\partial r^k}, \qquad N^{ij} = -g^{ik}\,\frac{\partial V}{\partial \varphi^j{}_A}\,\varphi^k{}_A. \tag{20}$$

A Legendre transformation expresses the conjugate momenta through generalized velocities

$$p_i = \frac{\partial L}{\partial \dot{r}^i} = M g_{ij}\,\frac{dr^j}{dt}, \qquad P^A{}_i = \frac{\partial L}{\partial \dot{\varphi}^i_A} = g_{ij}\,\frac{d\varphi^j{}_B}{dt}\,\mathcal{J}^{BA}. \tag{21}$$

The quantities

$$P^i{}_j := \varphi^i{}_A\,P^A{}_j = g_{jk}K^{ki}, \qquad \hat{P}^A_B := P^A{}_i\varphi^i{}_B \tag{22}$$

are Hamiltonian generators of left and right regular translations

$$\left[\varphi^i{}_A\right] \mapsto \left[U^i{}_j\,\varphi^j{}_A\right], \qquad \left[\varphi^i{}_A\right] \mapsto \left[\varphi^i{}_B\,V^B{}_A\right], \tag{23}$$

where U, V are nonsingular matrices. The kinetic energy

$$T = T_{\mathrm{tr}} + T_{\mathrm{rel}} = \frac{M}{2}\,g_{ij}\,\frac{dr^i}{dt}\,\frac{dr^j}{dt} + \frac{1}{2}\,g_{ij}\,\frac{d\varphi^i{}_A}{dt}\,\frac{d\varphi^j{}_B}{dt}\,\mathcal{J}^{AB} \tag{24}$$

is never invariant under left or right regular translations. Thus $P^i{}_j$, $P^A{}_B$ fail to be constants of motion. Therefore, even if there is no potential, the above dynamical model, based on the d'Alembert principle, is not a system on a Lie group in the Hermann–Arnold sense, therefore its structure essentially differs from that of the usual rigid body (where T is left-invariant and, in the case of spherical top, also right-invariant). However, there are no doubts concerning its physical applicability. Let us mention, e.g., astrophysical problems (vibration of stars, shape of earth) (Bogoyavlenski, 1985), macroscopic elasticity (when the size of the body is comparable with the length of excited waves), micromorphic continua, vibrations of molecules, collective modes of nuclei. Nevertheless, the study of left and right invariant kinetic energies, when equations of motion are not derivable from the usual d'Alembert principle, is at least of academic interest. One can also expect some non-standard applications in the collective model of nuclei, and it turns out, also in the theory of integrable one-dimensional lattices.

It is obvious that any left-invariant kinetic energy of the internal motion must have the form

$$T_{\mathrm{rel}} = \frac{1}{2}\,\mathcal{L}_A{}^B{}_C{}^D\,\Omega^A{}_B\,\Omega^C{}_D, \tag{25}$$

where $\mathcal{L}$ is constant and $\mathcal{L}_A{}^B{}_C{}^D = \mathcal{L}_C{}^D{}_A{}^B$.

The only possibility for the translational part is

$$T_{\mathrm{tr}} = \frac{M}{2}\,C_{ij}\,\frac{dr^i}{dt}\,\frac{dr^j}{dt} = \frac{1}{2}\,\eta_{AB}\,\varphi^{-1\,A}{}_i\,\varphi^{-1\,B}{}_j\,\frac{dr^i}{dt}\,\frac{dr^j}{dt} = \frac{M}{2}\,\eta_{AB}\,V^A V^B, \tag{26}$$

where C is the inverse Cauchy deformation tensor. Similarly, for the right-invariant models we have

$$T_{\mathrm{rel}} = \frac{1}{2}\,\mathcal{R}_i{}^j{}_k{}^l\,\Omega^i{}_j\,\Omega^k{}_l, \tag{27}$$

where $\mathcal{R}_i{}^j{}_k{}^l = \mathcal{R}_k{}^l{}_i{}^j$ is constant. As previously (24),

$$T_{\mathrm{tr}} = \frac{M}{2}\,G_{AB}\,v^A v^B = \frac{M}{2}\,g_{ij}\,\frac{dr^i}{dt}\,\frac{dr^j}{dt}, \tag{28}$$

where G denotes the Green deformation tensor,

$$G_{AB} = g_{ij}\,\varphi^i{}_A\,\varphi^j{}_B. \tag{29}$$

Although the left invariant translational energy (26) with metric coefficients built of internal degrees of freedom looks rather exotic, one should mention that models of this type were used in defect theory (Zorski, 1968[1]; Zorski, 1968[2]).

The balance forms of equations of motion corresponding to (25), (27) read, respectively

$$\frac{dP^i{}_j}{dt} = N^i{}_j, \qquad \frac{d\hat{P}^A_B}{dt} = \hat{N}^A_B, \tag{30}$$

$P^i{}_j$, $\hat{P}^A_B$ given by (22). Unlike (16), (17), (18) they become conservation laws for the interaction-free case (due to affine invariance), i.e., when $N^i{}_j = 0$, $\hat{N}^A_B = 0$.

Let us concentrate now on the models for relative motion. Those suggested above (24), (25), (26) are too general to be effectively treated. Therefore, we assume that the usual one (24) is isotropic in the material space, i.e., $\mathcal{J}^{AB} = \mu\eta^{AB}$, the same will be assumed about the left invariant one (25), and let the right invariant one (27) be isotropic in the physical space. The corresponding expressions for affinely-invariant models are given by

$$T_{\mathrm{rel}} = \frac{A}{2}\,\mathrm{Tr}(\hat{\Omega}^2) + \frac{B}{2}\,(\mathrm{Tr}\hat{\Omega})^2 + \frac{C}{2}\,(\mathrm{Tr}\hat{\Omega}^T\,\hat{\Omega}), \tag{31}$$

$$T_{\mathrm{rel}} = \frac{A}{2}\,\mathrm{Tr}(\Omega^2) + \frac{B}{2}\,(\mathrm{Tr}\Omega)^2 + \frac{C}{2}\,(\mathrm{Tr}\Omega^T\,\Omega), \tag{32}$$

where A, B, C are constant, and transpositions are meant in the sense of the metrics η, g. By an appropriate choice of A, B, C one can achieve the positive definitness of these expressions. The doubly-invariant model (left and right under linear group)

$$T^{\mathrm{rel}} = \frac{A}{2}\,\mathrm{Tr}(\Omega^2) + \frac{B}{2}\,(\mathrm{Tr}\,\Omega)^2 = \frac{A}{2}\,\mathrm{Tr}\,(\hat{\Omega})^2 + \frac{B}{2}\,(\mathrm{Tr}\,\hat{\Omega})^2 \tag{33}$$

is never positively definite, because $SL(n, \mathbb{R})$ is a non-compact semi-simple group. However, expressions (31), (32) differ from (33) by constants of motion in such a way that their discussion may be in a sense reduced to that of (33). It is convenient and in principle not less general to put $B = 0$ in (33).

Another possibility of collective modes consists in compactifying deformative degrees of freedom, i.e., replacing in an appropriate way the group $GL(n, \mathbb{R})$ by $U(n)$.

If the motion of internal degrees of freedom is described by time-dependent unitary matrices $U(t) \in U(n)$ then we introduce, as in the $GL(n, \mathbb{R})$-case, the Lie algebraic objects

$$\Omega = \frac{dU}{dt}\,U^{-1}, \qquad \hat{\Omega} = U^{-1}\,\frac{dU}{dt}. \tag{34}$$

They are anti-Hermitian, and the collective kinetic energy of internal motion may be postulated as the following positive expression

$$T = -\frac{A}{2}\,\mathrm{Tr}(\Omega^2) = -\frac{A}{2}\,\mathrm{Tr}(\hat{\Omega}^2), \qquad A > 0. \tag{35}$$

It is convenient to use the two-polar decomposition of $\varphi \in GL(N, \mathbb{R})$, namely

$$\varphi = L\,D\,R^T, \tag{36}$$

where L, $R \in SO(n, \mathbb{R})$ are proper-orthogonal, and D is diagonal

$$D = \mathrm{Diag}\ \left(\exp\left(q^1\right), \ldots, \exp\left(q^n\right)\right), \qquad q^a \in \mathbb{R}. \tag{37}$$

Compatification to $U(n)$ consists in putting

$$D = \mathrm{Diag}\ \left(\exp\left(iq^1\right), \ldots, \exp\left(iq^n\right)\right), \qquad q^a \in \mathbb{R}. \tag{38}$$

In the case of $GL(n, \mathbb{R})$ there is a very natural interpretation: q^a-s are deformation invariants, and L, R describe the orientations of principal axes of the Cauchy and Green deformation tensors, respectively. The $U(n)$-description replaces straight-lines of invariants by circles.

Performing the Legendre transformations on the kinetic energies (24), (33) (this one with $B = 0$), (35), we obtain the following Hamiltonians

$$H = \frac{1}{2A}\sum_i {P_i}^2 + \frac{1}{8A}\sum_{ij}\frac{M_{ij}^2}{(Q^i - Q^j)^2} + \frac{1}{8A}\sum_{ij}\frac{N_{ij}^2}{(Q^i + Q^j)^2}, \tag{39}$$

$$H = \frac{1}{2A}\sum_i {p_i}^2 + \frac{1}{32A}\sum_{ij}\frac{M_{ij}^2}{\mathrm{sh}^2\,\frac{q^i - q^j}{2}} - \frac{1}{32A}\sum_{ij}\frac{N_{ij}^2}{\mathrm{ch}^2\,\frac{q^i - q^j}{2}}, \tag{40}$$

$$H = \frac{1}{2A}\sum_i {p_i}^2 + \frac{1}{32A}\sum_{ij}\frac{M_{ij}^2}{\sin^2\,\frac{q^i - q^j}{2}} + \frac{1}{32A}\sum_{ij}\frac{N_{ij}^2}{\cos^2\,\frac{q^i - q^j}{2}}. \tag{41}$$

The meaning of symbols is as follows: $Q^a = \exp\left(q^a\right)$, P_a is its canonical conjugate momentum, p_a is a canonical momentum conjugate to q^a,

$$M_{ab} = -V_{ab} - S_{ab}, \qquad N_{ab} = V_{ab} + S_{ab}, \tag{42}$$

and S_{ab}, V_{ab} are, respectively, canonical angular momenta (Hamiltonian generators of right regular translations by $SO(n, \mathbb{R})$ acting on L and left ones acting on R). They are non-holonomic (Poisson non-commuting) canonical momenta conjugate to co-moving angular velocities of the L- and R-rigid tops,

$$l = L^{-1}\,\frac{dL}{dt}, \qquad r = R^{-1}\,\frac{dR}{dt}. \tag{43}$$

It is surprising and promising that the models (39), (40), (41) are, as seen, strongly related to the integrable lattices studied by Calogero–Moser and Sutherland (Calogero and Marchioro, 1974; Guillemin and Sternberg, 1984; Moser, 1975).

The above analysis is performed for arbitrary n, and this generality is interesting for studying the mentioned relationship with lattices. Obviously, for applications to deformable bodies, only the special cases $n = 3$, 2 are relevant.

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