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Spinning Fluids in Relativistic Hydrodynamics

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We study the well known propagation and constraint equations in General Relativity for the case where the matter source is an ideal Weyssenhoff fluid. Moreover we derive these equations for the Einstein-Cartan theory of gravitation for the same matter source. We discuss the different couplings of the matter content in detail in both theories and consider especially the behavior of the spin, angular and total angular momentum.

1 Introduction

The present paper deals with the relation of spinning fluids and several relativistic gravitational theories, which was a part of the discussion between thermodynamicists and relativists in the research group of Muschik.

Spinning fluids are of special interest in gravitational physics (Obukhov and Korotkii, 1987; Obukhov and Piskareva, 1989). In General Relativity Theory (GRT) spinning fluids are a complex matter source for the gravitational field and a number of additional effects may occur; for instance, in the context of cosmological models they can eliminate the cosmological singularity (Korotkii and Obukhov, 1992). Moreover it is possible to test the stability of the obtained results which are mostly depending on simple matter sources like dust or ideal fluid.

Furthermore they enable the description of complex matter sources of the gravitational field in distinguished gravitational theories and allow one to find out the special behavior of such complex sources in different theories. These different couplings can now be compared with respect to the GRT case. So we may hope to find out estimates which probably are measurable. Therefore we try to contribute to answer the question, whether the general theory of relativity is the correct theory of gravity or it is necessary to prefer a more general theory.

Here we like to focus on the behavior of spinning fluids in General Relativity and in the Einstein-Cartan Theory of Gravitation (ECT) (for a comprehensive overview, cf. Hehl et al., 1976)). This is one of the simplest generalizations of the GRT containing torsion which is directly coupled to the spin density. We describe the influence of spinning fluids on the propagation and constraint equations in both theories.

The paper is organized as follows. In Sec. 2 we briefly summarize the kinematical description of relativistic fluids, the model of spinning fluids and basic properties of GRT and ECT. In Sec. 3 we derive the propagation and constraint equations and compare the results obtained in both theories.

2 Theoretical Foundations

2.1 Kinematical Description of the Fluid

We describe the matter as an one-component fluid with the four-velocity u^α of each volume element. The four-velocity fulfills the normalization condition $u^\alpha u_\alpha = c^2 = 1$ (c is the velocity of light). The signature is in this case -2.

The gradient of the velocity field is decomposed in the following way (e. g. Ellis, 1973)

$$\nabla_\mu u_\nu = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} + u_\mu a_\nu \tag{1}$$

Here the antisymmetric space-like part $\omega_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu \nabla_{[\alpha} u_{\beta]}$ denotes the rotation of the flow of the fluid, the space-like symmetric traceless part $\sigma_{\mu\nu} = h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \nabla_{(\alpha} u_{\beta)} - \frac{1}{3}\Theta h_{\mu\nu}$ the shear of the fluid (the volume of the element is conserved, but the shape is distorted), $\Theta = \nabla_{\lambda} u^{\lambda}$ the expansion (change of the volume), $a^{\mu} = u^{\nu}\nabla_{\nu} u^{\mu}$ the acceleration of the velocity field and $h^{\alpha}{}_{\mu} = g^{\alpha}{}_{\mu} - u^{\alpha} u_{\mu}$ the projector on the space orthogonal to u^{α}. These quantities are defined for a comoving observer, who measures this quantities with respect to a Fermi-Walker transported reference frame (for a detailed discussion e. g. (Stephani, 1990)). It is sometimes useful to introduce the vector of rotation as $\omega^{\alpha} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta} u_{\beta} \omega_{\gamma\delta}$ and the values of shear $\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ and rotation $\omega^2 = \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta}$.

2.2 Weyssenhoff Model

The matter content is described by the Weyssenhoff fluid model, which is a continuous medium with internal angular momentum (spin). The spin density is described by the second rank skew-symmetric tensor

$$Q_{\mu\nu} = -Q_{\nu\mu}, \tag{2}$$

whose spatial components coincide in the rest frame with the density of the 3-vector of spin of a matter element. This is provided by imposing the covariant constraint, the so-called Frenkel condition

$$Q_{\mu\nu} u^{\nu} = 0 \tag{3}$$

So we may write the spin vector also in the form

$$P^{\alpha} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta} u_{\beta} Q_{\gamma\delta} \tag{4}$$

The rotational equations of motion are given by the motion of spin

$$\nabla_{\nu}(u^{\nu} Q_{\kappa\lambda}) = u_{\kappa} u^{\rho} \nabla_{\nu}(u^{\nu} Q_{\rho\lambda}) - u_{\lambda} u^{\rho} \nabla_{\nu}(u^{\nu} Q_{\rho\kappa}) = u_{\kappa}\, Q_{\lambda\rho}\, a^{\rho} - u_{\lambda}\, Q_{\kappa\rho}\, a^{\rho} \tag{5}$$

and the translational equations of motion follows as usual from the contracted Bianchi identities which depend on the chosen spacetime (Chrobok et al., 2001).

2.3 General Relativity

GRT describes the gravitational field with the help of a 4-dimensional Riemannian spacetime. In this case the field equations for the metric field $g_{\mu\nu}$ are

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{6}$$

The general energy-momentum tensor can be written in the following way

$$T_{\mu\nu} = \rho_{\mathrm{eff}} u_{\mu} u_{\nu} - p_{\mathrm{eff}} h_{\mu\nu} + 2u_{(\mu} q_{\nu)} + \pi_{\mu\nu}, \tag{7}$$

where ρ_{eff} is the effective energy density, p_{eff} is the effective pressure, q^{ν} is the energy flux and $\pi_{\nu\mu}$ anisotropic pressure (cf., e.g. Ellis, 1973). Besides one has $q^{\nu} u_{\nu} = 0$, $\pi^{\nu\mu} u_{\mu} = 0$ and $\pi^{\nu}{}_{\nu} = 0$. The energy-momentum tensor of the Weyssenhoff fluid reads

$$T_{\alpha\beta} = \epsilon\, u_{\alpha} u_{\beta} - p\, h_{\alpha\beta} - 2(g^{\kappa\lambda} + u^{\kappa} u^{\lambda})\, \nabla_{\kappa}(u_{(\alpha} Q_{\beta)\lambda}) \tag{8}$$

as was shown in (Obukhov and Piskareva, 1989) in a modern treatment of the curved spacetime variational theory. One can decompose this energy-momentum tensor into effective quantities as follows (Chrobok et al., 2001)

$$\rho_{\mathrm{eff}} = \epsilon - 2\omega_{\kappa\lambda} Q^{\kappa\lambda}, \qquad p_{\mathrm{eff}} = p - \frac{2}{3}\omega_{\kappa\lambda} Q^{\kappa\lambda}, \qquad q_{\nu} = -\,(h^{\kappa}{}_{\nu} \nabla_{\lambda}\, Q_{\kappa}{}^{\lambda} + a^{\lambda}\, Q_{\lambda\nu}), \tag{9}$$
$$\pi_{\nu\mu} = 2\omega_{(\nu}{}^{\kappa}\, Q_{\mu)\kappa} - 2\sigma_{(\nu}{}^{\kappa}\, Q_{\mu)\kappa} - \frac{2}{3}\, h_{\nu\mu}\, \omega_{\kappa\lambda}\, Q^{\kappa\lambda}$$

We emphasize that this fluid is a complex anisotropic fluid. Especially the appearing energy density and the pressure depend on the relative orientations of the rotation (the orbital angular momentum) and the spin momentum. The propagation and constraint equations can be derived from the Ricci-identity

$$\nabla_{[\nu}\nabla_{\mu]}u_\lambda = -\frac{1}{2}R_{\nu\mu\lambda}{}^{\kappa}u_\kappa \tag{10}$$

using the field equations, making several contractions and projections and applying the symmetry relations of the geometric quantities.

2.4 Einstein-Cartan Theory

The Einstein-Cartan theory of gravitation describes the gravitational field with the help of curvature and torsion in a 4-dimensional Riemann-Cartan spacetime. The geometrical properties and their identities can be found in (Schouten, 1954). The field equations become now more complicated (Hehl et al., 1976). The first one

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{11}$$

describes the coupling between curvature, torsion and the energy- momentum tensor. The second one appearing now, couples the third-rank torsion tensor $S_{\alpha\beta}{}^{\gamma}$ to the third-rank spin density tensor $Q_{\alpha\beta}{}^{\gamma}$

$$S_{\alpha\beta}{}^{\gamma} + 2\delta^{\gamma}_{[\alpha}S_{\beta]} = \kappa Q_{\alpha\beta}{}^{\gamma}. \tag{12}$$

Once again we assume the energy-momentum tensor in the following way

$$T_{\mu\nu} = \rho_{\text{eff}}u_\mu u_\nu - p_{\text{eff}}h_{\mu\nu} + u_\mu q_\nu + l_\mu u_\nu + \pi_{\mu\nu}, \tag{13}$$

but due to the fact that it is no longer symmetric we have to distinguish between the energy flux l_μ and the momentum flux q_μ and the anisotropic pressure is no longer symmetric. Again we have to fulfill the conditions $q^\nu u_\nu = 0$, $l^\mu u_\mu = 0$, $\pi^{\nu\mu}u_\mu = 0$ and $\pi^\nu{}_\nu = 0$.

In this theory the energy-momentum tensor of the Weyssenhoff fluid is

$$T^\mu{}_\nu = \epsilon u^\mu u_\nu - ph^\mu{}_\nu + 2u^\mu u^\alpha \nabla_\beta Q_{\alpha\nu}u^\beta \tag{14}$$

which follows from the variational method in Riemann-Cartan space (Obukhov and Korotkii, 1987). The decomposition of this energy- momentum tensor in effective quantities gives

$$\rho_{\text{eff}} = \epsilon\ , \qquad p_{\text{eff}} = p\ , \qquad l^\mu = 0\ , \qquad q^\nu = -2a_\alpha Q^{\alpha\nu} = -2a_\alpha \eta^{\alpha\nu\rho\sigma}P_\rho u_\sigma, \qquad \pi^\mu{}_\nu = 0 \tag{15}$$

It should be stressed, in comparison to GRT, that the momentum flux is the only new quantity. Moreover, the energy density and the pressure are unaffected by the spin density. The third rank tensor of the spin density takes in this case the form

$$Q_{\mu\nu}{}^{\rho} = Q_{\mu\nu}u^\rho \tag{16}$$

The derivation of the propagation and constraint equations now uses the Ricci-identity in Riemann-Cartan space

$$\nabla_{[\nu}\nabla_{\mu]}u_\lambda = -\frac{1}{2}R_{\nu\mu\lambda}{}^{\kappa}u_\kappa - S_{\nu\mu}{}^{\rho}\nabla_\rho u_\lambda, \tag{17}$$

and, as above, we will use the two field equations and the symmetry relations of the Riemann-Cartan curvature tensor.

3 Derivation of the Propagation and Constraint Equations

The comprehensive description of the propagation and constraint equations, e.g., is included in (Ellis, 1973) for the GRT case (in the presence of spinning fluid see (Obukhov, private communication)). The

Raychaudhuri equation (20) (discussed below) in the ECT case is derived in the work of Tafel (Tafel, 1973). Furthermore almost all equations for the ECT case are obtained by Palle (Palle, 1999), but we obtain partly different results. We shall comment their results later.

If one contracts equation (10) with $g^{\mu\lambda}u^{\nu}$ and using the field equations (6) one is led to the propagation equation of the expansion, the so-called Raychaudhuri equation. In the case of GRT we obtain

$$\dot{\Theta} = 2(\omega^2 - \sigma^2) + a^{\nu}{}_{;\nu} - \frac{1}{3}\Theta^2 - \kappa\frac{1}{2}(3p_{\text{eff}} + \rho_{\text{eff}}) \tag{18}$$

The $\dot{\Theta} = u^{\nu}\nabla_{\nu}\Theta$ denotes here and in the following the proper time derivative. To consider the case of the Weyssenhoff fluid in GRT we have to use the decomposition (9) and receive

$$\dot{\Theta} = 2(\omega^2 - \sigma^2) + a^{\nu}{}_{;\nu} - \frac{1}{3}\Theta^2 - \kappa\frac{1}{2}(3p + \epsilon) + 2\kappa\omega_{\alpha\beta}Q^{\alpha\beta} \tag{19}$$

Here the ; denotes the covariant derivative with respect to the Christoffel symbol which is here also the full connection. Moreover the kinematical quantities include also only the Christoffel symbol. We derive the analogous equation in ECT with the same operation but now one has to use equation (17) and take the field equations (11, 12) into account

$$\dot{\Theta} = 2(\omega^2 - \sigma^2) + \nabla_{\nu}a^{\nu} - \frac{1}{3}\Theta^2 - \kappa\frac{1}{2}(3p_{\text{eff}} + \rho_{\text{eff}}) + \text{sources from spin density} \tag{20}$$

The occuring additional sources are zero in the case of the Weyssenhoff fluid. So equation (20) reduces formally to the GRT case equation (19), except for the now vanishing coupling term between spin and angular momentum. But the kinematical quantities (also of course ∇) are defined with respect to the full connection.

The projection of equation (10) on $h^{\nu}{}_{(\alpha}h^{\mu}{}_{\beta)}u^{\lambda}$ gives the space-like symmetric part of equation (10) and leads to the propagation equation of shear

$$\begin{aligned} h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}\dot{\sigma}_{\alpha\beta} = h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}a_{(\alpha;\beta)} - a_{\nu}a_{\mu} - \omega_{\nu\rho}\omega^{\rho}{}_{\mu} - \sigma_{\nu\alpha}\sigma_{\mu}{}^{\alpha} - \frac{2}{3}\Theta\sigma_{\nu\mu} - \\ -\frac{1}{3}h_{\nu\mu}\left(2(\omega^2 - \sigma^2) + a^{\alpha}{}_{;\alpha}\right) - \frac{1}{2}\pi_{\nu\mu} + E_{\nu\mu} \end{aligned} \tag{21}$$

Here $E_{\nu\mu} = C_{\nu\alpha\mu\beta}u^{\alpha}u^{\beta}$ denotes the "electric" part of the Weyl tensor (conformal curvature tensor). We like to stress that the time development of shear includes besides the kinematical terms, material sources $\pi_{\nu\mu}$ and geometrical sources (tidal forces) $E_{\nu\mu}$. The same procedure as above, now using equation (17), gives the propagation equation of shear in the ECT case

$$\begin{aligned} h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}\dot{\sigma}_{\alpha\beta} = h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}\nabla_{(\alpha}a_{\beta)} - a_{\nu}a_{\mu} - \omega_{\nu\rho}\omega^{\rho}{}_{\mu} - \sigma_{\nu\alpha}\sigma_{\mu}{}^{\alpha} - \\ -\frac{2}{3}\Theta\sigma_{\nu\mu} - \frac{1}{3}h_{\nu\mu}\left(2(\omega^2 - \sigma^2) + \nabla_{\alpha}a^{\alpha}\right) - \frac{1}{2}\pi_{(\nu\mu)} + E_{(\nu\mu)} \end{aligned} \tag{22}$$

The equations (21) and (22) are formally identical, but in the ECT case we have to consider the symmetric parts of the anisotropic pressure and the electric part of the Weyltensor. Once again and in the following the kinematical quantities are defined with respect to different connections.

If we consider the Weyssenhoff fluid as source in GRT we obtain with the help of equation (9)

$$\begin{aligned} h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}\dot{\sigma}_{\alpha\beta} = h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}a_{(\alpha;\beta)} - a_{\nu}a_{\mu} + \omega_{(\nu}{}^{\rho}(\omega_{\mu)\rho} - \kappa Q_{\mu)\rho}) - \frac{2}{3}\Theta\sigma_{\nu\mu} - \\ -\sigma_{(\nu}{}^{\rho}(\sigma_{\mu)\rho} + \kappa Q_{\mu)\rho}) - \frac{1}{3}h_{\nu\mu}\left(2(\omega^2 - \sigma^2) + \kappa\omega_{\alpha\beta}Q^{\alpha\beta} + a^{\alpha}{}_{;\alpha}\right) + E_{\nu\mu} \end{aligned} \tag{23}$$

and observing some additional source containing the spin density. In the case of ECT no such additional sources occur for the Weyssenhoff model, because of the equation (15) the anisotropic pressure part vanishes.

We obtain the propagation equation of rotation by projection of equation (10) onto $h^{\nu}{}_{[\alpha}h^{\mu}{}_{\beta]}u^{\lambda}$. Thus it gives the antisymmetric space-like part

$$h_{\nu}{}^{\alpha}h_{\mu}{}^{\beta}\dot{\omega}_{\alpha\beta} = -\frac{2}{3}\Theta\omega_{\nu\mu} + 2\omega_{\rho[\nu}\sigma_{\mu]}{}^{\rho} + h_{\nu}{}^{\gamma}h_{\mu}{}^{\delta}a_{[\delta;\gamma]} \tag{24}$$

It is necessary to notice that, in contrast to equation (21), in this propagation equation no material or geometric sources occur. The propagation of rotation is only influenced by the kinematical quantities. In the case of ECT this situation changes once again. We consider the projection of equation (17) onto $h^{\nu}{}_{[\alpha}h^{\mu}{}_{\beta]}u^{\lambda}$ and obtain

$$h_{\tau}{}^{\nu}h_{\sigma}{}^{\lambda}\dot{\omega}_{\nu\lambda} = -\frac{2}{3}\Theta\omega_{\tau\sigma} + 2\omega_{\mu[\tau}\sigma_{\sigma]}{}^{\mu} + h_{\tau}{}^{\nu}h_{\sigma}{}^{\lambda}\nabla_{[\nu}a_{\lambda]} - E_{[\tau\sigma]} + \frac{1}{2}\pi_{[\tau\sigma]} + 2u^{\mu}S_{[\tau|\mu}{}^{\rho}\nabla_{\rho|}u_{\sigma]} \tag{25}$$

Now additional source terms for the time development of the rotation appear, namely the antisymmetric electric part of the Weyltensor which in general is zero in GRT and a coupling to the antisymmetric part anisotropic pressure occurs. If we consider the antisymmetric electric part of the Weyltensor

$$E_{[\tau\sigma]} = \frac{1}{2}\pi_{[\tau\sigma]} - u_{\mu}u_{\kappa}(2\nabla_{[\tau}S_{\sigma]}{}^{\kappa\mu} + \nabla^{\kappa}S_{\tau\sigma}{}^{\mu} - 2S_{\sigma\tau}{}^{\rho}S_{\rho}{}^{\kappa\mu} - 4S^{\kappa}{}_{[\tau|\rho|}S_{\sigma]}{}^{|\rho|\mu}) \tag{26}$$

we observe that this part is induced by the torsion and the anisotropic pressure. This equation can be simplified for the Weyssenhoff fluid in ECT

$$\begin{aligned} h_{\tau}{}^{\nu}h_{\sigma}{}^{\lambda}(\dot{\omega}_{\nu\lambda} + \kappa\dot{Q}_{\nu\lambda}) = -\frac{2}{3}\Theta(\omega_{\tau\sigma} - \kappa Q_{\tau\sigma}) + 2(\omega_{\mu[\tau} - \kappa Q_{\mu[\tau})\sigma_{\sigma]}{}^{\mu} + h_{\tau}{}^{\nu}h_{\sigma}{}^{\lambda}\nabla_{[\nu}a_{\lambda]} + \\ +2\kappa Q_{[\tau}{}^{\lambda}\omega_{\sigma]\lambda} \end{aligned} \tag{27}$$

The left-hand side is the change of the total (orbital plus spin) angular momentum with respect to the proper time of the observer. The right-hand side shows the coupling between the total angular momentum and the other quantities and an orbital and spin momentum coupling term. Using the equation of motion for the spin (5) one obtains the propagation equation for the rotation

$$h_{\tau}{}^{\nu}h_{\sigma}{}^{\lambda}\dot{\omega}_{\nu\lambda} = -\frac{2}{3}\Theta(\omega_{\tau\sigma} - \frac{5}{2}\kappa Q_{\tau\sigma}) + 2(\omega_{\mu[\tau} - \kappa Q_{\mu[\tau})\sigma_{\sigma]}{}^{\mu} + h_{\tau}{}^{\nu}h_{\sigma}{}^{\lambda}\nabla_{[\nu}a_{\lambda]} + 2\kappa Q_{[\tau}{}^{\lambda}\omega_{\sigma]\lambda} \tag{28}$$

Additionally to the propagation equations one obtains constraint equations. The first one describes the coupling between rotation and acceleration in GRT and follows from equation (10) and the antisymmetry relations of the curvature tensor by contraction with $\eta^{\nu\mu\lambda\rho}u_{\rho}$

$$\nabla_{\alpha}\omega^{\alpha} = -2\omega^{\alpha}a_{\alpha} \tag{29}$$

Thus it describes a balance of the rotation of the velocity field. This equation is independent of the matter sources. This changes in the ECT case. Now using equation (17) and equations (11,12) the same procedure leads to a coupling with the spin density

$$\nabla_{\nu}\omega^{\nu} = -2a_{\nu}\omega^{\nu} + \text{coupling with spin density} \tag{30}$$

In the case of the Weyssenhoff fluid this can be written as

$$\nabla_{\nu}(\omega^{\nu} + \kappa P^{\nu}) = -2a_{\nu}(\omega^{\nu} + \kappa P^{\nu}), \tag{31}$$

which is now a balance equation for the total angular momentum.

Another constraint equation connects the kinematical quantities with the energy flux of matter. We obtain this equation with the help of the equations (10, 7) by contracting it with $g^{\mu\lambda}$ and projecting onto $h^{\nu}{}_{\rho}$ as

$$\kappa q_{\tau} = h_{\tau\beta}(\omega^{\beta\alpha}{}_{;\alpha} + \sigma^{\beta\alpha}{}_{;\alpha} - \frac{2}{3}\Theta^{;\beta}) + a^{\mu}(\omega_{\mu\tau} + \sigma_{\mu\tau}) \tag{32}$$

With the help of the decomposition (9) we achieve for the Weyssenhoff fluid

$$0 = h_{\tau\beta}(\omega^{\beta\alpha}{}_{;\alpha} + \kappa Q^{\beta\alpha}{}_{;\alpha} + \sigma^{\beta\alpha}{}_{;\alpha} - \frac{2}{3}\Theta^{;\beta}) + a^{\mu}(\omega_{\mu\tau} + \kappa Q_{\mu\tau} + \sigma_{\mu\tau}) \tag{33}$$

Now in the GRT case we observe that the spin and orbital momentum form the total angular momentum. The same technique leads to the constraint equation in ECT

$$\kappa l_{\tau} = h_{\tau}{}^{\nu}(\nabla^{\lambda}(\omega_{\nu\lambda} + \sigma_{\nu\lambda}) - \frac{2}{3}\nabla_{\nu}\Theta) + a^{\lambda}(\omega_{\lambda\tau} + \sigma_{\lambda\tau}) + \text{coupling terms with spin density.} \tag{34}$$

For the Weyssenhoff fluid this simplifies to

$$0 = h_{\tau}{}^{\nu}(\nabla^{\lambda}(\omega_{\nu\lambda} + \sigma_{\nu\lambda}) - \frac{2}{3}\nabla_{\nu}\Theta) + a^{\lambda}(\omega_{\lambda\tau} + 2\kappa Q_{\lambda\tau} + \sigma_{\lambda\tau}). \tag{35}$$

The fact that the total angular momentum appears is not realized in this equation. The last constraint equation follows in GRT from (10) projecting onto $\eta_{\beta}{}^{\pi\nu\mu}u_{\pi}$ symmetrization and projection onto the three-space orthogonal to u^{α} as

$$H_{\alpha\beta} = \left(\omega_{\mu(\alpha|;\nu} + \sigma_{\mu(\alpha|;\nu}\right)\eta_{|\beta)}{}^{\pi\nu\mu}u_{\pi} + 2\omega_{(\alpha}a_{\beta)} \tag{36}$$

In this equation $H_{\kappa\lambda} := -\frac{1}{2}\eta_{\kappa}{}^{\pi\nu\mu}C_{\nu\mu\lambda}{}^{\sigma}u_{\pi}u_{\sigma}$ defines the "magnetic" part of the Weyl tensor. We like to emphasize that this part of the Weyl tensor is purely induced by the kinematical quantities. The calculation of this constraint equation in ECT using (17, 12) shows that

$$H_{\alpha\beta} = \nabla_{\nu}\left(\omega_{\mu(\alpha|} + \sigma_{\mu(\alpha|}\right)\eta_{|\beta)}{}^{\pi\nu\mu}u_{\pi} + 2\left(\omega_{(\alpha} + \kappa P_{(\alpha}\right)a_{\beta)} \tag{37}$$

Once again an additional source term occurs containing the spin density, so that this part of the equation describes the coupling between the total angular momentum and the acceleration. But like in the case of the energy flow constraint this appears only in the second part, the other term on the r.h.s. of (37) doesn't show this property.

The equations which we have derived are the complete set of propagation and constraint equations in GRT and ECT. The Raychaudhuri equation (20) for ECT shows the same properties as the result of Tafel (Tafel, 1973) if we consider that we can use the transformation $\overset{ECT}{\omega_{\alpha\beta}} \longrightarrow \overset{GRT}{\omega_{\alpha\beta}} - \kappa Q_{\alpha\beta}$ to achieve the corresponding quantities with respect to the rotation in GRT. Palle (Palle, 1999) criticizes in this equation the cross-term $\kappa\omega_{\alpha\beta}Q^{\alpha\beta}$, which however is correct. Besides in this work the transformation to GRT quantities is extensively used with the disadvantage that one has to deal with derivatives from different spaces in the same equation. This makes the results inscrutable. Moreover some interesting coupling terms seem to be lost.

4 Conclusions

The derivation of the propagation and constraint equations in ECT shows, in contrast to GRT, that they include some additional couplings. In some equations they look quite natural as combining spin and orbital momentum to the total angular momentum. Especially matter sources for the rotation of the fluid appear and are related to the spin density of matter. One obtains also a balance equation for the total angular momentum. But the behavior of the total angular momentum seems to be also described sometimes better in the GRT than in ECT. From this point of view one can not prefer one of the theories. So one has to look at special cases and their physical realizations and try to make observations which could give some predications.

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