TECHNISCHE MECHANIK, Band 22, Heft 1, (2002), 36-42
Manuskripteingang: 13. Dezember 2001

On the Evolution of Simple Material Structures [1]

M. Elżanowski, E. Binz

The evolution of a distribution of material inhomogeneities is investigated by analyzing the evolution of the corresponding material connections. Some general geometric relations governing such evolutions are derived. These relations are then analyzed by looking at the restrictions imposed by the material symmetry group.

1 Introduction

Laws of evolution are integral part of theories such as plasticity and visco-plasticity. Since plasticity is often viewed as a process of re-arrangement of patterns of defects it seems natural to discuss the issue of evolution of inhomogeneities within the framework of the geometric theory of uniformity, *cf.*, Elżanowski (1995), Epstein & de Leon (1998), Wang & Truesdell (1973).

The structure of the law of material evolution has been already investigated within this realm both in the context of simple materials as well as those of the second-order, see e.g., Epstein & Maugin (1996), Maugin & Epstein (1998), and Epstein (1999). In the approach presented there material evolution was modeled by a first order differential equation for the uniformity maps. Postulating the principles of *covariance* and *actual evolution*, and assuming the uniformity of evolution (laws of evolution are material point independent), the geometric methods were used to investigate the form and structure of such a differential equation.

In this note we look at the evolution of material from yet another perspective by investigating how the time dependent deformations (evolutions) of a uniform reference configuration show through the evolution of the corresponding material connection. Our objective is to determine if there may be point-wise evolutions of the uniformity maps, which although non-trivial and constitutively admissible, produce no measurable change of the underlying pattern of inhomogeneities. We believe that such evolutions of material structures my account for these non-elastic deformations which do not change "defectiveness" of the material body as measured by the torsion of its material connection. A somewhat similar problem was investigated by Parry (see Parry (2001) and references therein) within the context of the structurally based theory of defects. Using purely kinematic considerations he was able to show that there exists a non-trivial class of inelastic deformations between states possessing the same elastic invariants. Such deformations are akin to the classical slip mechanism of the phenomenological plasticity and represent rearrangement of material points while preserving the local lattice structure.

Our presentation is divided into a number of short sections. After a brief review of the concepts of material uniformity, homogeneity and that of a material connection in Section 2 the mathematical aspects of the evolution of connections are discussed in Section 3. This is followed in Section 4 by the discussion of the role of the Principle of covariance and the Principle of actual evolution as pertaining to the choice of a particular material symmetry group.

2 Uniformity and Material Connections

The material body B is a continuum having the structure of an orientable differentiable manifold. We assume that it can be covered by a single (global) coordinate chart. In other words, we postulate that the body B possesses a global *reference configuration.* Although fairly general theory can be, and has been, developed without this simplifying assumption such considerations are beyond the scope of this note. The interested reader may consult Wang & Truesdell (1973) and Elżanowski (1995).

[1]This paper is dedicated to Professor Wolfgang Muschik on the occasion of his 65th birthday.

The mechanical response of a *simple material body* is determined point-wise by the value of the deformation gradient at each material point $x \in \mathrm{B}$. Adopting an Euclidean vector space $\mathbf{V}$ as a *reference crystal* (an archetype of a material element) the local configuration of the body at the point occupied by x is given by a linear map $\mathbf{K}(x)$ from $\mathbf{V}$ into the tangent space of B at x. Specifically, selecting a frame $\mathbf{e}_A$ in $\mathbf{V}$ the local configuration at x is represented by the induced basis

$$\mathbf{f}_j(x) = K_j^A(x)\mathbf{e}_A \tag{1}$$

where the matrix valued functions $K_j^A(x)$ represent the linear transformation $\mathbf{K}(x)$. The deformation gradient $\mathbf{F}$ can now be viewed as a linear automorphism of the tangent space of B at x. Namely,

$$\hat{\mathbf{f}}_i(x) = F_i^j\mathbf{f}_j(x) \tag{2}$$

where the frame $\hat{\mathbf{f}}_i$ represents the deformed configuration at x. The density of the stored energy per unit reference volume is given in pure elasticity by a function $W(\mathbf{F}; x)$. We say that the body B is *materially uniform* if there exists a selection of linear isomorphisms $\mathbf{P}(x)$ from $\mathbf{V}$ into the tangent space of B and a function $\hat{W}$ such that

$$W(\mathbf{F}; x) = \hat{W}(\mathbf{F}\mathbf{P}(x)) \tag{3}$$

at any x and for all deformation gradients $\mathbf{F}$. $\hat{W}$ represents here the density of stored elastic energy of B per unit volume of the reference crystal while the collection of the induced bases $\mathbf{u}_j(x) = P_j^A(x)\mathbf{e}_A$ defines the *uniform reference configuration*.

An automorphism $\mathbf{G}$ of the reference crystal $\mathbf{V}$ is said to be a *material symmetry* (of a reference crystal) if given a uniformity map $\mathbf{P}(x)$

$$\mathbf{H}(x) = \mathbf{P}(x)\mathbf{G} \tag{4}$$

is also a uniformity map, i.e.,

$$W(\mathbf{F}; x) = \hat{W}(\mathbf{F}\mathbf{P}(x)) = \hat{W}(\mathbf{F}\mathbf{H}(x)) \tag{5}$$

for all deformation gradients $\mathbf{F}$. If $\hat{\mathbf{e}}_A$ is a basis of the reference crystal $\mathbf{V}$ such that

$$\mathbf{e}_A = G_A^B\hat{\mathbf{e}}_B, \tag{6}$$

where the matrix G_A^B represents the symmetry element $\mathbf{G}$, then

$$\hat{\mathbf{u}}_j = H_j^B\hat{\mathbf{e}}_B = P_j^A G_A^B\mathbf{e}_B \tag{7}$$

defines yet another uniform reference configuration. Two different uniform frames, say $\hat{\mathbf{u}}_j$ and $\mathbf{u}_j$, are then related by

$$\hat{\mathbf{u}}_l = P_l^A G_A^B (P^{-1})_B^j \mathbf{u}_j = G_l^j \mathbf{u}j \tag{8}$$

where G_l^j represents an element of the *symmetry group of the material point* x corresponding to G_A^B symmetry of the reference crystal.

A smooth collection of uniform configurations over the body B represents a hypothetical re-arrangement of local configurations of material points so that the relative mechanical response becomes point independent. Being materially uniform is the mathematical way of saying that the body B is made of the same material at all points. The uniform configuration does not necessarily represent any true physical state of the body B as a whole as it may not necessarily come from any global configuration, even if globally defined.

A uniform configuration may or may not come from a global configuration of the body B. However, if it does, the material body B is said to be *homogeneous*. In other words, the materially uniform body B is homogeneous if among all its uniform configurations there exists at least one which is also a global (truly physical) configuration.

Given the uniform reference configuration $\mathbf{u}_j$ or equivalently the corresponding *uniformity maps* $P_j^A(x)$ one is able to define the concept of *material parallelism.* Indeed, consider a vector field $\mathbf{w}$ defined at least in some open neighborhood of the material point $x \in \mathrm{B}$. Let $\mathbf{w} = \tilde{w}^j \mathbf{u}_j$ where, in general, both $\tilde{w}^j$ and $\mathbf{u}_j$ are material point dependent. We say that the vector field $\mathbf{w}$ is *parallel* relative to the uniform frame $\mathbf{u}_j$ if its coordinates $\tilde{w}^j$, are constant functions of position. In other words, if for any direction, say $\mathbf{f}_\alpha$, the directional derivative $\tilde{w}^j_{,\alpha} = 0$. To this end let $w^A := \tilde{w}^j P_j^A$. Then

$$\tilde{w}^j_{,\alpha} = w^A_{,\alpha}(P^{-1})^j_A - w^A (P^{-1})^l_A P^B_{l,\alpha}(P^{-1})^j_B. \tag{9}$$

where the relation

$$(P^{-1})^j_{A,\alpha} = -(P^{-1})^l_A P^B_{l,\alpha}(P^{-1})^j_B \tag{10}$$

has been utilized. It is now easy to observe that the vector field $\mathbf{w} = \tilde{w}^j \mathbf{u}_j = \tilde{w}^j P_j^A \mathbf{e}_A = w^A \mathbf{e}_A$ is parallel in $\mathbf{u}_j$ if and only if

$$w^A_{,\alpha} = w^D (P^{-1})^l_D P^C_{l,\alpha}. \tag{11}$$

Note also that

$$\Gamma^C_{D\alpha}(P) := (P^{-1})^l_C P^C_{l,\alpha} \tag{12}$$

are, as shown in Wang & Truesdell (1973), the Christoffel symbols of a linear connection on B pulled back to the reference crystal. In the reference configuration of the body B these functions become

$$\Gamma^k_{l\alpha}(P) = P^C_{l,\alpha}(P^{-1})^k_C, \tag{13}$$

and

$$\tilde{w}^j_{,\alpha} = w^A (P^{-1})^j_{A,\alpha} + \tilde{w}^l \Gamma^j_{l\alpha}(P). \tag{14}$$

If another uniform reference configuration, say as given by (7), is considered where gauging (modifying) by the symmetry group of the reference crystal may as well be material point dependent the corresponding Christoffel symbols are

$$\Gamma^C_{A\alpha}(H) = (G^{-1})^D_A \Gamma^B_{D\alpha}(P) G^C_B + (G^{-1})^B_A G^C_{B,\alpha} \tag{15}$$

as it can easily be seen from (12) and (7). A vector field $\mathbf{w}$ which is parallel relative to $\mathbf{u}_j$ may not be parallel relative to $\hat{\mathbf{u}}_j$. However, we say that $\mathbf{w}$ is *materially uniform* (or *materially parallel*) if there exists a uniform reference configuration $\mathbf{w}$ is parallel in.

Summarizing the above presentation we may say that any uniform reference configuration of the material body B, given by some assignment of the uniformity maps $\mathbf{P}(x)$, defines a parallelism and the corresponding material connection represented by the Christoffel symbols $\Gamma^A_{D\alpha}(P)$. A smooth gauging by the symmetry elements of the reference crystal induces in turn the whole family of material connections. It can be shown by a straightforward calculation that all material connections have zero curvature, Elżanowski at al (1990). They may, however, have a nonzero torsion. On the other hand, if a given material connection, which already has no curvature, is symmetric then there exists a reference configuration such that the corresponding Christoffel symbols vanish, and the underlying uniform reference configuration $\mathbf{u}_i$ is integrable i.e., it comes from a global placement of the body B. Inversely, if any global configuration is a uniform reference configuration then the corresponding uniformity maps $P_k^A(x)$ can be selected as point independent making the corresponding Christoffel symbols vanish. In short, we say that the body B is homogeneous if and only if there exists a torsion free material connection, *cf.*, Wang & Truesdell (1973).

3 Time Evolution of Material Connections

We shall investigate now how the material connections evolve under the gauging by the elements of the group of linear automorphisms of the reference crystal $GL(\mathbf{V})$. As any material connection is uniquely defined by a uniform reference configuration we look first at the gauging of these configurations. Hence, let us consider some uniform reference configuration $\mathbf{u}_j(x) = P_j^A(x)\mathbf{e}_A$ and let $G_A^B(x,t)$ represent a one-parameter family of linear automorphisms of the reference crystal $\mathbf{V}$, and such that $G_A^B(x,0) = \delta_A^B$ for any x. Superposing one operation onto the other we get a one-parameter family of reference configurations

$$\mathbf{u}_j(x,t) := P_j^B(x,t)\mathbf{e}_B = P_j^A(x)G_A^B(x,t)\mathbf{e}_B. \tag{16}$$

These configurations are not necessarily uniform configurations unless the automorphisms $G_A^B(x,t)$ are the symmetries of the reference crystal $\mathbf{V}$. Note also that given any other *evolution* of reference configurations, say $\hat{P}_j^B(x,t)$, there always exists a smooth one-parameter family of linear automorphisms $\hat{G}_A^B(x,t)$ such that $\hat{P}_j^B(x,t) = P_j^A(x,t)\hat{G}_A^B(x,t)$.

Let us now restrict our analysis to a single material point and consider two different evolutions, say $P_j^A(t)$ and $H_j^A(t)$. We say that these evolutions are *parallel* if there exists a non-trivial automorphism G_A^B such that

$$H_j^B(t) = P_j^A(t)G_A^B. \tag{17}$$

It seems natural to expect that parallel evolutions are somewhat "equivalent". To this end let us compare the "time" derivatives of the corresponding induced frames $\mathbf{u}_j(t) := P_j^A(t)\mathbf{e}_A$ and $\mathbf{w}_i(t) := H_i^A(t)\mathbf{e}_A$. First

$$\dot{\mathbf{w}}_i = \dot{H}_i^A\mathbf{e}_A = \dot{H}_i^B(H^{-1})_B^l\mathbf{w}_l \qquad = \dot{P}_i^C G_C^B(G^{-1})_B^D(P^{-1})_D^k\mathbf{w}_k = \dot{P}_i^C(P^{-1})_C^k\mathbf{w}_k \tag{18}$$

while

$$\dot{\mathbf{u}}_j = \dot{P}_j^A\mathbf{e}_A = \dot{P}_j^B(P^{-1})_B^k\mathbf{u}_k. \tag{19}$$

Let $L(\mathbf{P}) := \dot{\mathbf{P}}\mathbf{P}^{-1}$, or in coordinates, $L_j^k(P) := \dot{P}_j^B(P^{-1})_B^k$. Therefore, given a family of uniformity maps, say $P_j^A(t)$, the time rate of change of the family of the induced uniform frames $\mathbf{u}_j(t) = P_j^A(t)\mathbf{e}_A$ is

$$\dot{\mathbf{u}}_j = L_j^k(P)\mathbf{u}_k. \tag{20}$$

In fact, as easily confirmed by equations (18) and (19) the following is true:

Proposition 1 *Two evolutions $P_j^A(t)$ and $H_k^B(t)$ of the uniformity maps are parallel if and only if the corresponding mappings $L(\mathbf{P})$ and $L(\mathbf{H})$ are identical.*

Following Epstein & Maugin (1996) we shall call $L(\mathbf{P}) = \dot{\mathbf{P}}\mathbf{P}^{-1}$ the *inhomogeneity velocity gradient.*

As we have argued earlier any uniform reference configuration defines a material connection. Also, any two material connections differ by a point-wise action of the symmetry group (a collection of all symmetries) of the reference crystal. In fact, it is easy to show that any two zero-curvature linear connections on B differ by the deformation (gauging) of the elements of $GL_3(I\!R)$, isomorphic to $GL(\mathbf{V})$, *cf.*, Kobayashi & Nomizu (1963).

Let us therefore consider two parallel evolutions of the uniformity maps, namely $H_k^B(t) = P_k^A(t)G_A^B$. As the automorphism G_A^B is time independent the time derivatives of the corresponding material connections are related by

$$\dot{\Gamma}_{A\alpha}^C(H) = (G^{-1})_A^D\dot{\Gamma}_{D\alpha}^B(P)G_B^C \tag{21}$$

as implied by (15). Moreover, a straightforward computations show that

$$H_j^A(t)\dot{\Gamma}_{A\alpha}^C(H)(H^{-1})_C^p(t) = P_j^D(t)\dot{\Gamma}_{D\alpha}^B(P)(P^{-1})_B^p(t). \tag{22}$$

The *induced connection velocity*

$$\pounds_{j\alpha}^p(P) := P_j^D(t)\dot{\Gamma}_{D\alpha}^B(P)(P^{-1})_B^p(t) \tag{23}$$

becomes the connection counterpart of the inhomogeneity velocity gradient $L^p_j(P)$.

Proposition 2 *For any two parallel evolutions of the uniform reference configurations the corresponding induced connection velocities are identical.*

In contrast to Proposition 1 the converse to Proposition 2 is not obvious at all. Hence let us look at the gauging of material connections not only by the symmetry group of the reference crystal but rather the whole group of isomorphisms $GL(\mathbf{V})$. To this end let $G^B_A(x,t)$ represent a deformation of the reference crystal. Let us also assume that the material connection $\Gamma^A_{B\alpha}$ - generated by the uniformity maps $P^A_j(x)$ - is given. Applying the family $G^B_A(x,t)$ introduces the family of reference configurations (not necessarily uniform) $H^A_j(x,t)\mathbf{e}_A = P^B_j(x)G^A_B(x,t)\mathbf{e}_A$ and the family of linear connections $\Gamma^C_{A\alpha}(H)(t)$. Evaluating the time derivative of these Christoffel symbols one gets that

$$\dot{\Gamma}^C_{A\alpha}(H) = (G^{-1})^D_A\Gamma^B_{D\alpha}(P)\dot{G}^C_B - (G^{-1})^F_A\dot{G}^E_F(G^{-1})^D_E\Gamma^B_{D\alpha}(P)G^C_B + \dot{\Gamma}^C_{A\alpha}(G). \tag{24}$$

Moreover, for any collection of uniformity maps $P^B_j(x,t)$

$$L^l_{j,\alpha}(P) = \dot{P}^B_{j,\alpha}(P^{-1})^l_B + \dot{P}^B_j(P^{-1})^l_{B,\alpha} = \dot{P}^B_{j,\alpha}(P^{-1})^l_B - \dot{P}^B_j(P^{-1})^r_B P^C_{r,\alpha}(P^{-1})^l_C \tag{25}$$

which is simply identical to $\pounds^l_{j\alpha}(P)$. Consequently, the definition of the induced connection velocity and (24) imply that

$$(P^{-1})^k_C\dot{\Gamma}^C_{A\alpha}(H)P^A_j = \pounds^k_{j\alpha}(\mathbf{G}(x,t)) + [\Gamma^B_{D\alpha}(P), L^k_j(\mathbf{G}(x,t))] \tag{26}$$

where $[\cdot,\cdot]$ denotes the Lie algebra commutator. This finally leads to the following conclusion:

Proposition 3 *Given material connection $\Gamma^B_{D\alpha}(P)$ and the family of gauge transformations $G^A_B(x,t)$ the family of connection forms $\Gamma^B_{D\alpha}(H^A_j(x,t))$, where $H^A_j(x,t) = P^B_j(x)G^A_B(x,t)$, will not evolve if and only if*

$$\pounds^k_{j\alpha}(G^A_B(x,t)) = [L^k_j(G^A_B(x,t)), \Gamma^B_{D\alpha}(P)]. \tag{27}$$

When the gauge transformations $G^A_B(x,t)$ are material point independent the relation (27) reduces to

$$[L^k_j(G^A_B(t)), \Gamma^B_{D\alpha}(P)] = 0. \tag{28}$$

4 Material Evolution

As long as a (uniform) body remains elastic its material structure (a collection of all uniform reference configurations) , as determined by the density of its stored energy function W, remains unchanged. However, if we allow the body to experience other than elastic deformations while assuming that the strain energy is still measurable the underlying geometric structure may change. For example, it is traditionally accepted that plasticity involves a mechanism which modifies the distribution of inhomogeneities, defects in particular. Mathematically, such a re-arrangement of defect patterns can only be observed if the underlying material structure evolves, i.e., the set of uniform reference configurations and the corresponding material connections evolve outside of the symmetry group.

The exact form of the law of evolution of any particular material can only be determined through constitutive modeling. There are, however, some general principles we would like any "reasonable" law of evolution to satisfy. In particular, we postulate that any such law satisfies the following two fundamental principles:

- **Principle of covariance:** *A law of evolution must be independent of the particular reference configuration chosen.*

- **Principle of actual evolution:** *A law of evolution must at all times select the inhomogeneity velocity gradient $L(\mathbf{P})$ outside of the algebra of the instantaneous symmetry group.*

These principles were originally postulated by Epstein & Maugin (1996) where it was also suggested that the evolution of a material is governed by a first order differential equation for the uniformity maps with the Eshelby tensor as the driving force. In this work, consistent with our view on the evolution of structures, we assume the evolution law of the form:

$$\dot{\Gamma}^A_{B\alpha} = f^A_{B\alpha}(\Gamma^D_{C\alpha}, P^D_j, \cdots) \tag{29}$$

where the functionals $f^A_{B\alpha}$ may still depend on other objects like for example the Eshelby tensor or the deformation gradient. Note that the law of evolution is taken material point independent to parallel the uniformity of the body. According to the *Principle of covariance* such evolution law must be invariant under the change of the global reference configuration. In fact it can be shown, see Binz & Elżanowski (2001), that this postulate prohibits any functional $f^A_{B\alpha}$ from being dependent explicitly on the uniformity maps P^A_j.

We proceed now to investigate the role of the material symmetry group in the context of the *Principle of actual evolution*. In contrast to what was done in Epstein (1999) we shall not investigate the form of the evolution law. Rather, looking at different symmetry groups and their algebras, we shall try to determine the restrictions which the *Principle of actual evolution* imposes on the choice of the allowed evolutions (gauging). In other words, what is a proper evolution i.e., the evolution changing the essential characteristics of a distribution of material inhomogeneities. Aided by Proposition 3 we shall try to determine the sets of solutions to the relations (27) and (28). We assume that any allowable gauge transformation $G^A_B(x,t)$ is unimodular at any x and any t, and that we only consider unimodular material symmetries. We also assume that the symmetry group remains unchanged during the evolution. The much more difficult case of the evolution process in which not only the uniform reference configuration but also the structure group may change is left for future research.

To start our analysis let us look closer at $sl_3(\mathbb{R})$, the Lie algebra of the special linear group $\mathrm{SL}_3(\mathrm{R})$, that is the space of all trace-less 3×3 matrices. Let so_3 denote the algebra of the special orthogonal group SO_3, namely the set of all skew-symmetric 3×3 matrices. Furthermore let sym_3 be the space of all trace-less symmetric 3×3 matrices while $so_{2,3}$ stands for the Lie algebra of the group of all rotations about a fixed axis. This is a one-dimensional subalgebra of so_3. Thus we have

$$sl_3(\mathbb{R}) = so_3 \oplus sym_3. \tag{30}$$

and it is elementary to observe that so_3, $so_{2,3}$, and the set of all trace-less diagonal 3×3 matrices $\mathrm{diag}\{a, b, -(a+b)\}$, are all abelian subalgebras of $sl_3(\mathbb{R})$ while sym_3 is only a vector subspace.

First, let us consider the relation (28) where the gauge transformations $G^A_B(x,t)$ are assumed material point independent. Supposing that the connection form $\Gamma^A_{B\alpha}(P)$ takes value in a non-trivial subalgebra $h\subset sl_3(\mathbb{R})$ and accepting the *Principle of actual evolution* we look for the deformations $G^A_B(t)\in \mathrm{SL}_3(\mathbf{R})$ such that $L(\mathbf{G})\notin h$ and $[L(\mathbf{G}), X] = 0$ for every $X\in h$. In other words, given the subalgebra $h\subset sl_3(\mathbb{R})$, we look for the set

$$c(h) := \{\, Y\in sl_3(\mathbb{R})/h : [\,Y, X] = 0 \quad \text{for all} \quad X\in h\} \tag{31}$$

where $sl_3(\mathbb{R})/h$ denotes the complement of h in $sl_3(\mathbb{R})$. Note that, in general, $sl_3(\mathbb{R})/h$ is not a Lie algebra. Consequently, $c(h)$ is not a Lie algebra either, *cf.*, Carter *at al.* (1995). It is now a matter of simple calculations to show that:

- *full isotropy:*

$$c(so_3) = \{0\}. \tag{32}$$

- *transversal isotropy:*

$$c(so_{2,3}) = \{c_{ij}\} \quad \text{where} \quad c_{12} = c_{22} = -\frac{1}{2}c_{33}, c_{12} = -c_{21}, c_{13} = c_{23} = c_{32} = c_{31} = 0. \tag{33}$$

We may also add that in the case of the simple elastic fluid, where the material symmetries are all unimodular transformations, every evolution is trivial. On the other hand every evolution of the triclinic crystal - which has no symmetries - is non-trivial.

In conclusion; we have shown that as far as the material point independent deformations of material structures are concerned every proper deformation (i.e., obeying the principle of actual evolution) of the isotropic material structure yields a change in the material connection, see (32). However, there are some nontrivial proper evolutions of transversely isotropic structure which while deforming the structure will not alter the corresponding material connection, (33). More detailed analysis of this as well as the material point dependent case will be presented in Binz & Elżanowski (2001).

Acknowledgments: The work presented in this note was partially supported be grants from the Ministry of Education of Baden-Württemberg and the University of Mannheim. The writing of this paper was completed during the second author's sabbatical stay at the University of Calgary. This paper was presented at CIMRF-2001 held at Technical University of Berlin, September 3-6, 2001. Travel support for both visits was provided by the Faculty Development Fund of Portland State University.

Literature

1. Binz, E., and Elżanowski, M., *Another look at the evolution of material structures*, (2001), preprint.

2. Carter, R., Segal, G., and Macdonald, I.G., Lectures on Lie Groups and Lie Algebras, London Mathematical Society Student Text **32**, Cambridge University Press, Cambridge, 1995.

3. Elżanowski, M., Mathematical Theory of Uniform Material Structures, Kielce University od Technology (Politechnika Świetokrzyska), Kielce, 1995.

4. Elżanowski, M., Epstein, M., and Śniatycki, J., *G-Structures and Material Homogeneity*, J. Elasticity, **23**(2-3), (1990), 167–180.

5. Epstein, M., *Towards a complete second-order evolution laws*, Math. Mech. Solids, **4**(2), (1999), 251–266.

6. Epstein, M., and de Leon, M., *Geometrical theory of uniform Cosserat media*, J. Geometry and Physics, **26**, (1998), 127–170.

7. Epstein, M., and Maugin, G.A., *On the geometrical material structure of anelasticity*, Acta Mechanica, **115**, (1996), 119–134.

8. Kobayashi, S., and Nomizu, K., Foundations of Differential Geometry, Vol.I, Wiley, New York, 1963.

9. Maugin, G.A., and Epstein, M., *Geometrical material structure of elastoplasticity*, International J. PLasticity, **14**, 1-3, (1998), 109–115.

10. Parry, G.P., *The moving frame and defects in crystals*, Int.J. Solids Struct. **38**, (2001), 1071–1087.

11. Wang, C.-C., and Truesdell, C., Introduction to Rational Elasticity, Nordhoff, Leyden, 1973.

Address: Ernst Binz, Department of Mathematics, University of Mannheim, Mannheim, Germany, e-mail: binz@euler.math.uni-mannheim.de. Marek Z. Elżanowski, Department of Mathematical Sciences, Portland State University, Portland, Oregon, U.S.A., e-mail: marek@mth.pdx.edu.