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Constitutive Theory in General Relativity and Einstein-Cartan Theory: Spin Balances, Energy-Momentum Balances and Weyssenhoff Fluid

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It is shown, that the usually considered spin balances are too restrictive and only valid for pointlike particles. Furthermore, we will derive the full spin balance and discuss the Weyssenhoff-Fluid.

1 Introduction

We are interested in constitutive theory under the influence of gravitation. General Relativity Theory (GRT) using Riemann geometry for geometrization of the gravitation force is currently assumed to be the most correct geometrization. But from a constitutive point of view some problems arise, which are generated by different types of coupling between geometric and constitutive properties in different geometrizations.
In the Einstein-Cartan theory (ECT) (Schouten, 1924, 1954; Hehl et al., 1995) with non-vanishing torsion and curvature, the spin is coupled with torsion, and the energy-momentum tensor, which is spin-dependent, non-symmetric, and not divergence-free, is coupled with curvature. If the torsion vanishes, also the spin tensor and the skew-symmetric part of the energy-momentum tensor vanish.
In the Einstein theory with vanishing torsion and non-vanishing curvature, the spin appears as in the Einstein-Cartan theory in the non-symmetric and not divergence-free energy-momentum tensor, which is split into its symmetric and skew-symmetric part. The divergence-free symmetric part couples by the Einstein equations to curvature, whereas the skew-symmetric part does not couple to any geometric quantity. It represents the source in spin balance.
In the Minkowski theory, being flat and torsion-free, there are no geometric objects to which spin and energy-momentum tensor are coupled. If we regard Minkowski- and Einstein theories as special cases of the Einstein-Cartan theory, and presupposing they all have the same type of coupling, consequently in this view Einstein theory has to be spin-free and Minkowski theory is only valid in vacuum.

Until now, concerning cosmological models and materials like the Weyssenhoff fluid in GRT and ECT, only reduced spin balances are used. Here we consider in the framework of GRT and ECT the general spin balance and its contribution to the energy momentum tensor and its balances. As an example, the Weyssenhoff fluid with regard to the general spin balance is discussed.

2 Lifting the Balance of Spin

In order to give a motivation for the relativistic spin balance, the classical balance of angular momentum is used for deriving the spin balance.

2.1 Nonrelativistic

2.1.1 Classical Balance of Angular Momentum

If the angular momentum is introduced as a tensor of second order

$$M^{ij} = x^{[i}\rho v^{j]} + \rho s^{ij} \tag{1}$$

its balance becomes

$$\frac{1}{2}\partial_t(\rho s^{ij}) + \partial_t x^{[i}\rho v^{j]} + \partial_k\left(\frac{1}{2}\rho s^{ij}v^k - m^{ijk} + x^{[i}\rho v^{j]}v^k - x^{[i}t^{j]k}\right) \quad = \quad l^{ij} + x^{[i}f^{j]} \tag{2}$$

Here m^{ijk} is the couple stress and l^{ij} the external angular moment.

2.1.2 Classical Balance of Spin

Because of $s^{ij} \Leftrightarrow s_k$ in R^3 the balance of angular momentum can be written down as a vector equation. If from this balance the balance of momentum multiplied by $\underline{x}$ is subtracted, we obtain the balance of spin in vector formulation

$$\partial_t(\rho\underline{s}) + \nabla\cdot\left(\underline{v}\rho\underline{s} - \underline{\underline{w}}^\top\right) + \underline{\underline{\underline{\epsilon}}} : \underline{\underline{T}} \quad = \quad \rho\underline{g} \tag{3}$$

2.2 SRT

2.2.1 Balance of Angular Momentum

In special relativity theory the definition of angular momentum becomes

$$M^{\alpha\beta\mu} \quad := \quad x^{[\alpha}T^{\beta]\mu} + S^{\alpha\beta\mu} \tag{4}$$

and the balance results in

$$\partial_\mu M^{\alpha\beta\mu} \quad = \quad L^{\alpha\beta} + x^{[\alpha}f^{\beta]} \tag{5}$$

Here $S^{\alpha\beta\mu}$ is the spin tensor and $L^{\alpha\beta}$ is the external moment.

2.2.2 3+1 Split of $S^{\alpha\beta\gamma}$

By the 4-velocity u^α the spin tensor can be split into the following parts

$$S_{\mu\lambda}^{\cdot\cdot\nu} \quad = \quad s_{\mu\lambda}^{\cdot\cdot\nu} + \tilde{s}_{\mu\lambda}u^\nu + u_\mu\hat{s}_\lambda^{\cdot\nu} + \check{s}_\mu^{\cdot\nu}u_\lambda + \tilde{s}_\mu u_\lambda u^\nu + u_\mu\hat{s}_\lambda u^\nu + u_\mu u_\lambda\check{s}^\nu + s u_\mu u_\lambda u^\nu \tag{6}$$

The spin tensor has to be antisymmetric in its first two indices

$$S_{\mu\lambda}^{\cdot\cdot\nu} \quad = \quad S_{[\mu\lambda]}^{\cdot\cdot\nu} \tag{7}$$

This results in

$$\begin{aligned} S_{[\mu\lambda]}^{\cdot\cdot\nu} \quad &= \quad s_{[\mu\lambda]}^{\cdot\cdot\nu} + \tilde{s}_{[\mu\lambda]}u^\nu + u_{[\mu}\hat{s}_{\lambda]}^{\cdot\nu} + \check{s}_{[\mu|}^{\cdot\nu}u_{|\lambda]} + \tilde{s}_{[\mu}u_{\lambda]}u^\nu + u_{[\mu}\hat{s}_{\lambda]}u^\nu &(8)\\ &= \quad s_{[\mu\lambda]}^{\cdot\cdot\nu} + \tilde{s}_{[\mu\lambda]}u^\nu + u_{[\mu}\Xi_{\lambda]}^\nu + u_{[\mu}\Theta_{\lambda]}u^\nu &(9)\end{aligned}$$

Now one can identify $s_{[\mu\lambda]}^{\cdot\cdot\nu}$ as the couple stress and $\tilde{s}_{[\mu\lambda]}$ as the spin density. As there is up to now no physical interpretation of $u_{[\mu}\Xi_{\lambda]}^\nu$ and $u_{[\mu}\Theta_{\lambda]}u^\nu$, these terms are set to zero, so that (8) results in

$$S_{\mu\lambda}^{\cdot\cdot\nu} \equiv S_{[\mu\lambda]}^{\cdot\cdot\nu} \quad = \quad s_{[\mu\lambda]}^{\cdot\cdot\nu} + \tilde{s}_{[\mu\lambda]}u^\nu \tag{10}$$

2.2.3 Balance of Spin

As (2) follows from the balance of angular momentum (1) we obtain from (4) the balance of spin in SRT

$$\frac{1}{2}\partial_\mu\left(s^{\alpha\beta}u^\mu\right) - \partial_\mu m^{\alpha\beta\mu} \quad = \quad T^{[\alpha\beta]} + L^{\alpha\beta} \tag{11}$$

2.3 Balance of Spin in GRT

Replacing the partial derivatives in (11) by covariant ones, we obtain the balance of spin in GRT

$$\frac{1}{2}\nabla_\mu\left(s^{\alpha\beta}u^\mu\right) - \nabla_\mu m^{\alpha\beta\mu} \quad = \quad T^{[\alpha\beta]} + L^{\alpha\beta} \tag{12}$$

2.4 Balance of Spin in ECT

In ECT the spin tensor is presupposed to couple to the torsion, so that the field equations are

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \quad = \quad \kappa T^{\mu\nu} \quad , \qquad R^{\mu\nu} \neq R^{(\mu\nu)} \tag{13}$$

$$\mathcal{T}_{\alpha\beta}^{\cdot\,\cdot\,\mu} + 3\delta^\mu_{[\alpha}\mathcal{T}_{\beta]} \quad = \quad \kappa S_{\alpha\beta}^{\cdot\,\cdot\,\mu} \tag{14}$$

Then the balance of spin is a result of the field equations. By taking the antisymmetric part of the first field equation, an identity for the Ricci tensor, and the second field equation, one can derive the following balance of spin

$$(\tilde{\nabla}_\alpha - 3\mathcal{T}_\alpha)S_{\mu\lambda}^{\cdot\,\cdot\,\alpha} \quad = \quad T_{[\mu\lambda]} \tag{15}$$

If we want to describe the influence of an external torque, equation (15) has to be modified by a skew-symmetric quantity $L_{\mu\lambda} = L_{[\mu\lambda]}$ and the body torque (see eq. (11))

$$(\tilde{\nabla}_\alpha - 3\mathcal{T}_\alpha)S_{\mu\lambda}^{\cdot\,\cdot\,\alpha} \quad = \quad T_{[\mu\lambda]} + L_{\mu\lambda} \tag{16}$$

By doing this, the first field equation has also to be modified by $L_{\mu\lambda}$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \quad = \quad \kappa T^{\mu\nu} + \kappa L^{\mu\nu} \tag{17}$$

As the energy-momentum balance is symmetric, it is not influenced by this modification.
By comparison with the non-relativistic case one can interprete the components of the spin tensor $S_{\mu\lambda}^{\cdot\,\cdot\,\alpha}$ (see 2.2.2).

3 Geometrical Constraints in GRT and ECT

In GRT and ECT the mixed second order derivatives do not commute. They can be represented by the curvature tensor. Here we write down the formulas for the 4-velocity, the spin density and the spin tensor. Later on we use these relations to rewrite the energy momentum balance due to the constraints given by the relations below.

$$\text{ECT:} \quad u_{\alpha;[\beta\gamma]} \quad = \quad \frac{1}{2}(-\tilde{R}_{\alpha\kappa\gamma\beta}u^\kappa) - \mathcal{T}_{\beta\gamma}^{\cdot\,\cdot\,\kappa}\tilde{\nabla}_\kappa u_\alpha \tag{18}$$

$$\text{GRT:} \quad u_{\alpha;[\beta\gamma]} \quad = \quad \frac{1}{2}(-R_{\alpha\kappa\gamma\beta}u^\kappa) \tag{19}$$

$$\text{ECT:} \quad \tilde{s}_{\alpha\delta;[\beta\gamma]} \quad = \quad \frac{1}{2}(-\tilde{R}_{\alpha\kappa\gamma\beta}\tilde{s}^{\kappa\,\cdot}_{\cdot\,\delta} - \tilde{R}_{\delta\kappa\gamma\beta}\tilde{s}^{\cdot\,\kappa}_{\alpha\,\cdot}) - \mathcal{T}_{\beta\gamma}^{\cdot\,\cdot\,\kappa}\tilde{\nabla}_\kappa\tilde{s}_{\alpha\delta} \tag{20}$$

$$\text{GRT:} \quad \tilde{s}_{\alpha\delta;[\beta\gamma]} \quad = \quad \frac{1}{2}(-R_{\alpha\kappa\gamma\beta}\tilde{s}^{\kappa\,\cdot}_{\cdot\,\delta} - R_{\delta\kappa\gamma\beta}\tilde{s}^{\cdot\,\kappa}_{\alpha\,\cdot})$$

$$\text{ECT:} \quad S^{\kappa\lambda}_{\cdot\,\cdot\,\omega;[\mu\nu]} \quad = \quad \frac{1}{2}(\tilde{R}_{\nu\mu\rho}^{\cdot\,\cdot\,\cdot\,\kappa}S^{\rho\lambda}_{\cdot\,\cdot\,\omega} + \tilde{R}_{\nu\mu\rho}^{\cdot\,\cdot\,\cdot\,\lambda}S^{\kappa\rho}_{\cdot\,\cdot\,\omega} - \tilde{R}_{\nu\mu\omega}^{\cdot\,\cdot\,\cdot\,\rho}S^{\kappa\lambda}_{\cdot\,\cdot\,\rho}) - \mathcal{T}_{\nu\mu}^{\cdot\,\cdot\,\rho}\tilde{\nabla}_\rho S^{\kappa\lambda}_{\cdot\,\cdot\,\omega} \tag{21}$$

$$\text{GRT:} \quad S^{\kappa\lambda}_{\cdot\,\cdot\,\omega;[\mu\nu]} \quad = \quad \frac{1}{2}(R_{\nu\mu\rho}^{\cdot\,\cdot\,\cdot\,\kappa}S^{\rho\lambda}_{\cdot\,\cdot\,\omega} + R_{\nu\mu\rho}^{\cdot\,\cdot\,\cdot\,\lambda}S^{\kappa\rho}_{\cdot\,\cdot\,\omega} - R_{\nu\mu\omega}^{\cdot\,\cdot\,\cdot\,\rho}S^{\kappa\lambda}_{\cdot\,\cdot\,\rho}) \tag{22}$$

According to the space-like character of the spin density and of the couple stress tensor, one finds a relation between the representation of the curvature tensor due to a time-like vector field, like the velocity field, and the representation of the curvature tensor according to the couple stress tensor and the spin density. In order to get the relation one has to contract the spin density and couple stress geometry relations given above with the four-velocity u^α and taking the divergence in this relations. So one obtains in GRT and ECT

$$\begin{aligned} \nabla_{[\beta}\nabla_{\gamma]}[\tilde{s}_{\sigma\cdot}^{\cdot\beta}u^\sigma] &\stackrel{!}{=} 0 \\ \Longleftrightarrow (\nabla_{[\beta}\nabla_{\gamma]}\tilde{s}_{\sigma\cdot}^{\cdot\beta})u^\sigma &= -(\tilde{s}_{\sigma\cdot}^{\cdot\beta})(\nabla_{[\beta}\nabla_{\gamma]}u^\sigma) \end{aligned} \tag{23}$$

and

$$\begin{aligned} \nabla_{[\beta}\nabla_{\gamma]}[s_{\sigma\cdot\cdot}^{\cdot\beta\gamma}u^\sigma] &\stackrel{!}{=} 0 \\ \Longleftrightarrow (\nabla_{[\beta}\nabla_{\gamma]}s_{\sigma\cdot\cdot}^{\cdot\beta\gamma})u^\sigma &= -(s_{\sigma\cdot\cdot}^{\cdot\beta\gamma})(\nabla_{[\beta}\nabla_{\gamma]}u^\sigma). \end{aligned} \tag{24}$$

3.1 Energy Momentum Tensor and its First Covariant Derivatives in GRT and ECT

Independent of the used geometry, the energy momentum tensor can be defined as follows

$$\underline{\underline{T}} = \left(\begin{array}{c|c} \underline{\underline{t}} & \underline{p} \\ \hline \underline{q} & \epsilon \end{array}\right). \tag{25}$$

Now we perform a (3+1)-decomposition of the energy momentum tensor using the projections parallel to the velocity field (u^α) and perpendicular to it, using the projector

$$h_{\gamma\beta} = (g_{\gamma\beta} + \frac{1}{c^2}u_\gamma u_\beta) \tag{26}$$

$$T_{\alpha\beta} = h_{\alpha\cdot}^{\cdot\gamma}t_{\gamma\delta}h^{\delta}_{\cdot\beta} + h_{\alpha\cdot}^{\cdot\gamma}p_\gamma u_\beta + u_\alpha q_\delta h^{\delta}_{\cdot\beta} - \frac{1}{c^2}\epsilon u_\alpha u_\beta = T_{(\alpha\beta)} + T_{[\alpha\beta]} \tag{27}$$

$$\Longleftrightarrow T_{\alpha\beta} = t_{(\alpha\beta)} + u_{(\alpha}Q_{\beta)} - \frac{1}{c^2}\epsilon u_{(\alpha}u_{\beta)} + t_{[\alpha\beta]} + u_{[\alpha}P_{\beta]} + \epsilon\underbrace{u_{[\alpha}u_{\beta]}}_{=0} \tag{28}$$

Using (16) one finally gets a representation for the 4-momentum P_β in the ECT case

$$\tilde{\nabla}_\mu(s_{[\alpha\beta]}^{\cdot\cdot\mu} + \tilde{s}_{[\alpha\beta]}u^\mu) - L_{\alpha\beta} - t_{[\alpha\beta]} - 3\mathcal{T}_\mu S_{\alpha\beta}^{\cdot\cdot\mu} = u_\alpha P_\beta - u_\beta P_\alpha. \tag{29}$$

Contracting (29) with u^α leads to

$$\frac{-1}{c^2}(\epsilon u_\beta + u^\alpha 2\tilde{\nabla}_\mu S_{\beta\alpha}^{\cdot\cdot\mu} - u^\alpha L_{\alpha\beta} - u^\alpha 3\mathcal{T}_\mu S_{\alpha\beta}^{\cdot\cdot\mu}) = P_\beta. \tag{30}$$

Inserting equation (30) back into (27) one obtains

$$\begin{aligned} T_{\alpha\beta} = {} & t_{(\alpha\beta)} - \frac{2}{c^2}\epsilon u_{(\alpha}u_{\beta)} - \frac{1}{c^2}u_{(\alpha|}u^\gamma 2\tilde{\nabla}_\mu S_{|\beta)\gamma}^{\cdot\cdot\mu} + \frac{1}{c^2}u_{(\alpha|}u^\gamma L_{\gamma|\beta)} + \frac{3}{c^2}u^\gamma u_{(\alpha|}\mathcal{T}_\mu S_{\gamma|\beta)}^{\cdot\cdot\mu} + \\ & + t_{[\alpha\beta]} - \frac{1}{c^2}u_{[\alpha|}u^\gamma 2\tilde{\nabla}_\mu S_{|\beta]\gamma}^{\cdot\cdot\mu} + \frac{1}{c^2}u_{[\alpha|}u^\gamma L_{\gamma|\beta]} + \frac{3}{c^2}u^\gamma u_{[\alpha|}\mathcal{T}_\mu S_{\gamma|\beta]}^{\cdot\cdot\mu}. \end{aligned} \tag{31}$$

Crossing out the torsion in the last equation, one gets the analogous representation of the energy momentum tensor in the GRT case.
Taking the divergence of (27) one gets in the ECT case

$$\tilde{\nabla}_\beta T^{\alpha\beta} = \tilde{\nabla}_\beta t^{(\alpha\beta)} - \tilde{\nabla}_\beta\left(\frac{2}{c^2}\epsilon u^{(\alpha}u^{\beta)}\right) + \tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{(\alpha}u^\gamma L_{\gamma\cdot}^{\cdot\beta)}\right) - \tag{32}$$

$$
\begin{aligned}
&-\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{(\alpha|}u^\gamma 4\right)\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{(\alpha|}u^\gamma 4\tilde{\nabla}_\beta\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)+\\
&+\tilde{\nabla}_\beta\left(\frac{3}{c^2}u^\gamma u^{(\alpha|}\mathcal{T}_\mu S_{\gamma\cdot\cdot}^{\cdot|\beta)\mu}\right)+\\
&+\tilde{\nabla}_\beta t^{[\alpha\beta]}+\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{[\alpha|}u^\gamma L_{\gamma\cdot}^{\cdot|\beta]}\right)-\\
&-\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{[\alpha|}u^\gamma 4\right)\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}u^\mu+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{[\alpha|}u^\gamma 4\tilde{\nabla}_\beta\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}u^\mu+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)+\\
&+\tilde{\nabla}_\beta\left(\frac{3}{c^2}u^\gamma u^{[\alpha|}\mathcal{T}_\mu S_{\gamma\cdot\cdot}^{\cdot|\beta]\mu}\right).
\end{aligned}
$$

Now it is possible to replace the skew-symmetric parts of the second covariant derivatives of the spin density and the couple stress in (32) by (23) first and then by (18), one finally gets another possible representation of the balance of the energy momentum tensor including the curvature tensor and no more skew-symmetric parts of the second derivatives of the spin density and the couple stress

$$
\begin{aligned}
\tilde{\nabla}_\beta T^{\alpha\beta} = {} & \tilde{\nabla}_\beta t^{(\alpha\beta)}-\tilde{\nabla}_\beta\left(\frac{1}{c^2}\epsilon u^{(\alpha}u^{\beta)}\right)+\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{(\alpha|}u^\gamma L_{\gamma\cdot}^{\cdot|\beta)}\right)-\\
&-\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{(\alpha|}u^\gamma 4\right)\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{(\alpha|}u^\gamma 4\tilde{\nabla}_{(\beta|}\tilde{\nabla}_{\mu)}\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)+\\
&+\tilde{\nabla}_\beta\left(\frac{3}{c^2}u^\gamma u^{(\alpha|}\mathcal{T}_\mu S_{\gamma\cdot\cdot}^{\cdot|\beta)\mu}\right)+\\
&+\tilde{\nabla}_\beta t^{[\alpha\beta]}+\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{[\alpha|}u^\gamma L_{\gamma\cdot}^{\cdot|\beta]}\right)-\\
&-\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{[\alpha|}u^\gamma 4\right)\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}u^\mu+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{[\alpha|}u^\gamma 4\tilde{\nabla}_{(\beta|}\tilde{\nabla}_{\mu)}\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}u^\mu+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)+\\
&+\tilde{\nabla}_\beta\left(\frac{3}{c^2}u^\gamma u^{[\alpha|}\mathcal{T}_\mu S_{\gamma\cdot\cdot}^{\cdot|\beta]\mu}\right)-\\
&-\frac{4}{c^2}u^{(\alpha|}\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)\left(\tilde{\nabla}_{[\beta|}\tilde{\nabla}_{\mu]}u^\gamma\right)-\\
&-\frac{4}{c^2}u^{[\alpha|}\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}u^\mu+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)\left(\tilde{\nabla}_{[\beta}\tilde{\nabla}_{\mu]}u^\gamma\right).
\end{aligned}
\tag{33}
$$

Inserting now (18) in (33)

$$
\begin{aligned}
\tilde{\nabla}_\beta T^{\alpha\beta} = {} & \tilde{\nabla}_\beta t^{(\alpha\beta)}-\tilde{\nabla}_\beta\left(\frac{2}{c^2}\epsilon u^{(\alpha}u^{\beta)}\right)+\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{(\alpha|}u^\gamma L_{\gamma\cdot}^{\cdot|\beta)}\right)-\\
&-\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{(\alpha|}u^\gamma 4\right)\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|}\beta)\cdot\mu_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{(\alpha|}u^\gamma 4\tilde{\nabla}_{(\beta|}\tilde{\nabla}_{\mu)}\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}u^\mu+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)+\\
&+\tilde{\nabla}_\beta\left(\frac{3}{c^2}u^\gamma u^{(\alpha|}\mathcal{T}_\mu S_{\gamma\cdot\cdot}^{\cdot|\beta)\mu}\right)+\\
&+\tilde{\nabla}_\beta t^{[\alpha\beta]}+\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{[\alpha|}u^\gamma L_{\gamma\cdot}^{\cdot|\beta]}\right)-\\
&-\tilde{\nabla}_\beta\left(\frac{1}{c^2}u^{[\alpha|}u^\gamma 4\right)\tilde{\nabla}_\mu\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}u^\mu+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)-
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&-\frac{1}{c^2}u^{[\alpha|}u^{\gamma}4\tilde{\nabla}_{(\beta}\tilde{\nabla}_{\mu)}\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}\,u^{\mu}+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)+\\
&+\tilde{\nabla}_{\beta}\left(\frac{3}{c^2}u^{\gamma}u^{[\alpha|}\mathcal{T}_{\mu}S_{\gamma\cdot\cdot}^{\cdot|\beta]\mu}\right)+\\
&+\frac{4}{c^2}u^{(\alpha|}\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}\,u^{\mu}+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)\left(\frac{1}{2}\tilde{R}_{\lambda\kappa\mu\beta}u^{\kappa}g^{\lambda\gamma}+\mathcal{T}_{\beta\mu}^{\cdot\cdot\kappa}\tilde{\nabla}_{\kappa}u_{\lambda}g^{\lambda\gamma}\right)+\\
&+\frac{4}{c^2}u^{[\alpha|}\left(\tilde{s}^{|\beta]\cdot}_{\cdot\gamma}\,u^{\mu}+s^{|\beta]\cdot\mu}_{\cdot\gamma\cdot}\right)\left(\frac{1}{2}\tilde{R}_{\lambda\kappa\mu\beta}u^{\kappa}g^{\lambda\gamma}+\mathcal{T}_{\beta\mu}^{\cdot\cdot\kappa}\tilde{\nabla}_{\kappa}u_{\lambda}g^{\lambda\gamma}\right).
\end{aligned}
$$

In the case of GRT and Riemannn geometry one has to satisfy some *additional* constraints according to the used geometry

- $\nabla_{\beta}T^{\alpha\beta}\stackrel{!}{=}0$
- $T^{\alpha\beta}=T^{\beta\alpha}$
- $u_{\alpha;[\beta\gamma]}=\frac{1}{2}(-R_{\alpha\kappa\gamma\beta}u^{\kappa})$
- $R_{\alpha\beta}=R_{\beta\alpha}$
- $\Gamma^{\alpha}_{\beta\delta}=\Gamma^{\alpha}_{\delta\beta}=\{{}^{\,\alpha}_{\beta\delta}\}$

Taking into account these constraints given above leads to the representation of the energy momentum tensor in the GRT case

$$
\begin{aligned}
\nabla_{\beta}T^{(\alpha\beta)} &\stackrel{!}{=} 0\\
0 &= \nabla_{\beta}t^{(\alpha\beta)}-\nabla_{\beta}\left(\frac{2}{c^2}\epsilon u^{(\alpha}u^{\beta)}\right)+\nabla_{\beta}\left(\frac{1}{c^2}u^{(\alpha|}u^{\gamma}L_{\gamma\cdot}^{\cdot|\beta)}\right)-\\
&-\nabla_{\beta}\left(\frac{1}{c^2}u^{(\alpha|}u^{\gamma}4\right)\nabla_{\mu}\left(\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}\,u^{\mu}+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{(\alpha|}u^{\gamma}4\left(\nabla_{(\beta}\nabla_{\mu)}\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}\right)u^{\mu}-\\
&-\frac{1}{c^2}u^{(\alpha|}u^{\gamma}4\nabla_{(\beta}\nabla_{\mu)}s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}+\\
&+\frac{1}{c^2}u^{(\alpha|}4\left(\nabla_{[\beta}\nabla_{\mu]}u^{\gamma}\right)\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}\,u^{\mu}+\\
&+\frac{1}{c^2}u^{(\alpha|}4\left(\nabla_{[\beta}\nabla_{\mu]}u^{\gamma}\right)s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}.
\end{aligned} \tag{35}
$$

Inserting now (19) in (35) leads to the representation of the energy momentum tensor including the curvature tensor in the GRT case

$$
\begin{aligned}
\nabla_{\beta}T^{(\alpha\beta)} &\stackrel{!}{=} 0\\
0 &= \nabla_{\beta}t^{(\alpha\beta)}-\nabla_{\beta}\left(\frac{2}{c^2}\epsilon u^{(\alpha}u^{\beta)}\right)+2\nabla_{\beta}\left(\frac{1}{c^2}u^{(\alpha|}u^{\gamma}L_{\gamma\cdot}^{\cdot|\beta)}\right)-\\
&-\nabla_{\beta}\left(\frac{1}{c^2}u^{(\alpha}u^{\gamma}4\right)\nabla_{\mu}\left(\tilde{s}^{\beta)\cdot}_{\cdot\gamma}\,u^{\mu}+s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}\right)-\\
&-\frac{1}{c^2}u^{(\alpha|}u^{\gamma}4\left(\nabla_{(\beta}\nabla_{\mu)}\tilde{s}^{|\beta)\cdot}_{\cdot\gamma}\right)u^{\mu}-\\
&-\frac{1}{c^2}u^{(\alpha|}u^{\gamma}4\nabla_{(\beta}\nabla_{\mu)}s^{|\beta)\cdot\mu}_{\cdot\gamma\cdot}-\\
&-\frac{1}{c^2}u^{(\alpha|}4\frac{1}{2}R_{\gamma\rho\beta\mu}u^{\rho}\tilde{s}^{|\beta)\gamma}u^{\mu}-\\
&-\frac{1}{c^2}u^{(\alpha|}4\frac{1}{2}R_{\gamma\rho\beta\mu}u^{\rho}s^{|\beta)\gamma\mu}.
\end{aligned} \tag{36}
$$

4 Weyssenhoff-Fluid: Heuristic Description

Now the Weyssenhoff fluid (Weyssenhoff and Raabe, 1947) is discussed as it is worked out by Obukhov and Korotky (Obukhof and Korotky, 1987).
The Weyssenhoff fluid is defined as an ideal spinning fluid.
A spin density is introduced as a skew-symmetric tensor

$$S^{\mu\nu} = -S^{\nu\mu} \tag{37}$$

The spin density is spacelike, which is ensured by the Frenkel condition

$$S^{\mu\nu}u_\nu = 0 \tag{38}$$

The constitutive assumptions (postulates) for a Weyssenhoff fluid are as follows

- The spin tensor is a function of the spin density and the following constitutive equation is assumed

$$S_{\alpha\beta}{}^{\cdot\cdot\mu} = u^\mu \tilde{s}_{\alpha\beta} \tag{39}$$

- The energy-momentum tensor is a function of the energy-momentum density, and is defined as follows

$$T^{\mu}_{\cdot\alpha} = u^\mu P_\alpha \tag{40}$$

- Neglecting the torque $L_{\alpha\beta} \overset{!}{=} 0$.
- Neglecting the couple stress $s_{\alpha\beta\cdot}^{\cdot\cdot\delta} \overset{!}{=} 0$.
- Interaction of the particles as point particles.

Then we have the balance of spin

$$(\tilde{\nabla}_\alpha - 3\mathcal{T}_\alpha)(\kappa S_{\mu\lambda}{}^{\cdot\cdot\alpha}) = \kappa T_{[\mu\lambda]} \tag{41}$$

Next one calculates the explicit form of the energy-momentum-density P_α. This can be done by starting with the anti-symmetric part of the energy-momentum tensor (41) and (40)

$$2T_{[\mu\nu]} = u_\mu P_\nu - u_\nu P_\mu = 2(\tilde{\nabla}_\alpha - 3\mathcal{T}_\alpha)S_{\mu\nu}{}^{\cdot\cdot\alpha} \tag{42}$$

- and with the usual definition of the internal-energy

$$u^\mu P_\mu \overset{!}{=} \epsilon \tag{43}$$

one obtains

$$-c^2 P_\nu = \epsilon u_\nu + 2u^\mu(\tilde{\nabla}_\alpha - 3\mathcal{T}_\alpha)(u^\alpha \tilde{s}_{\mu\nu}) \tag{44}$$

$$T^{\mu}_{\cdot\nu} = -\frac{1}{c^2}\epsilon u^\mu u_\nu - \frac{1}{c^2}2u^\mu u^\alpha \tilde{\nabla}_\beta S_{\alpha\nu}{}^{\cdot\cdot\beta} \tag{45}$$

If it is now assumed that the interaction between the elements of the fluid is given in such a way that

- Pascal's law is valid, one has to modify the equations by an isotropic pressure

$$\hat{T}^{\mu}_{\cdot\nu} = +\frac{1}{c^2}p\delta^\mu_\nu - \frac{1}{c^2}u^\mu(u_\nu(\epsilon + p) + 2u^\alpha \tilde{\nabla}_\beta S_{\alpha\nu}{}^{\cdot\cdot\beta}) \tag{46}$$

5 Concluding Remarks

As we have shown, the usually considered spin balances are too restrictive, as they are only valid for pointlike particles and vanishing couple stress and body torque.
The Weyssenhoff-fluid is a very restricted special case of a spinning fluid: it is considered to consist of pointlike particles, as a consequence of this assumption the couple stress vanishes.

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