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Evaluation of Conjugate Stresses to Seth's Strain Tensors

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An explicit expression providing the symmetric stress tensor $\mathbf{T}^{(m)}$ conjugate to the Seth's strain measure $\mathbf{E}^{(m)}$ for each integer $m \neq 0$ is presented. The result is obtained by exploiting an original approach for the solution of a tensor equation in the unknown $\mathbf{T}^{(m)}$ expressed as function of the powers of the right stretch tensor $\mathbf{U}$. The proposed approach is based upon the spectral decomposition of $\mathbf{U}$ and exploits some peculiar features of the set of fourth-order tensors obtained as linear combination of dyadic and square tensor products of the eigenprojectors of $\mathbf{U}$. On the basis of such properties it is shown that the unknown $\mathbf{T}^{(m)}$ can be expressed in the given reference frame as linear combination of six fourth-order tensors scaled through coefficients which are rational functions of the eigenvalues of $\mathbf{U}$.

1 Introduction

The class of symmetric strain measures known as Seth's strains is defined by the expression (Seth, 1964):

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^{(m)} - \mathbf{1}) = \frac{1}{m}\sum_{i=1}^{N}(\lambda_i^m - 1)\mathbf{u}_i \otimes \mathbf{u}_i \tag{1}$$

where m is an arbitrary integer, λ_i and $\mathbf{u}_i$ are in turn the eigenvalues and eigenvectors of $\mathbf{U}$ while $\mathbf{1}$ is the rank-two identity tensor. Unless differently stated, we shall use in the sequel bold-faced lowercase (uppercase) symbols to denote vectors (second-order tensors) on a $N-$dimensional ($N = 2$ or $N = 3$) inner product space $\mathcal{V}$ over the real field.

The class of strain tensors defined by equation (1) embodies several well-known strain measures such as the logarithmic strain tensor, obtained by setting $m = 0$, the Biot strain ($m = 1$), and the Green-Lagrange tensor, which corresponds to the choice $m = 2$. More general strain measures have been subsequently introduced by Hill (1968) as

$$\mathbf{E} = \mathbf{E}(\mathbf{U}) = \sum_{i=1}^{N} f(\lambda_i)\mathbf{u}_i \otimes \mathbf{u}_i \tag{2}$$

where $f(\cdot)$ is a strictly-increasing smooth scalar function satisfying the conditions $f(1) = 0$ and $f'(1) = 1$. In particular, the class of Seth's strain tensors is obtained from equation (2) by setting

$$f(\lambda_i) = \begin{cases} \dfrac{1}{m}(\lambda_i^m - 1) & \text{if} \quad m \neq 0 \\ \ln(\lambda_i) & \text{if} \quad m = 0 \end{cases} \tag{3}$$

The stress measures associated with strains belonging to the Seth's class can be derived by invoking the classical notion of work-conjugacy, see e.g. Hill (1968). Specifically, a symmetric second-order tensor $\mathbf{T}^{(m)}$ is said to be work-conjugate of the strain measure $\mathbf{E}^{(m)}$ if it fulfills the condition

$$\mathbf{T}^{(m)} \cdot \dot{\overline{\mathbf{E}^{(m)}}} = J\boldsymbol{\sigma} \cdot \mathbf{d} \tag{4}$$

where $\dot{\overline{\mathbf{E}^{(m)}}}$ denotes the material time derivative of $\mathbf{E}^{(m)}$, $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{d}$ the rate of deformation tensor and J is the determinant of the deformation gradient $\mathbf{F}$, i.e., the third principal invariant of $\mathbf{U}$. The stresses conjugate to $\mathbf{E}^{(2)}$, and $\mathbf{E}^{(1)}$ are well-known, see e.g. Guo (1984), Hill (1978), Ogden (1984), Šilhavý (1997), Truesdell and Noll (1965). They are the second Piola-Kirchhoff tensor

$$\mathbf{T}^{(2)} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-t} \tag{5}$$

and the Biot stress tensor:

$$\mathbf{T}^{(1)} = \frac{1}{2}\left[\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}\right] \tag{6}$$

The expression of the stress $\mathbf{T}^{(0)}$ conjugate to $\ln(\mathbf{U})$ has been first worked out by Hoger (1987) as

$$\begin{aligned}\mathbf{T}^{(0)} &= J\mathbf{R}^t[\Lambda_1\boldsymbol{\sigma} + \Lambda_2(\mathbf{V}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{V}) + \Lambda_3(\mathbf{V}^2\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{V}^2) + \Lambda_4\mathbf{V}\boldsymbol{\sigma}\mathbf{V} + \\ &+\Lambda_5(\mathbf{V}^2\boldsymbol{\sigma}\mathbf{V} + \mathbf{V}\boldsymbol{\sigma}\mathbf{V}^2) + \Lambda_6\mathbf{V}^2\boldsymbol{\sigma}\mathbf{V}^2]\mathbf{R}\end{aligned} \tag{7}$$

where $\mathbf{R}$ is the rotation tensor, $\mathbf{V}$ the left stretch tensor and $\Lambda_1 \ldots \Lambda_6$ are functions of the eigenvalues of $\mathbf{U}$, assumed to be distinct. Finally, the expression for $\mathbf{T}^{(-1)}$

$$\mathbf{T}^{(-1)} = \frac{1}{2}\left[\mathbf{T}^{(-2)}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T}^{(-2)}\right] \tag{8}$$

has been contributed by Guo and Man (1992).

Moreover, by using Hill's principal axis method and Rivlin's representation formula for isotropic tensor functions (Rivlin, 1955), Guo and Man were the first ones to present a systematic approach to the derivation of the stress $\mathbf{T}^{(m)}$ conjugate to the Seth's strain $\mathbf{E}^{(m)}$ for arbitrary integers $m \neq 0$. In particular, they showed that $\mathbf{T}^{(m)}$ can be characterized as the solution of the tensor equation in the unknown $\mathbf{X}$

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{H} \tag{9}$$

the explicit form of which is

$$\begin{aligned} m = 2 &\quad \mathbf{U}\mathbf{X} + \mathbf{X}\mathbf{U} = \mathbf{H} \\ m = 3 &\quad \mathbf{U}^2\mathbf{X} + \mathbf{U}\mathbf{X}\mathbf{U} + \mathbf{X}\mathbf{U}^2 = \mathbf{H} \\ m = 4 &\quad \mathbf{U}^3\mathbf{X} + \mathbf{U}^2\mathbf{X}\mathbf{U} + \mathbf{U}\mathbf{X}\mathbf{U}^2 + \mathbf{X}\mathbf{U}^3 = \mathbf{H} \\ \cdots\cdots\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{10}$$

with

$$\mathbf{H} = m\mathbf{T}^{(1)} \qquad \text{for } m > 0 \tag{11}$$

and

$$\mathbf{H} = m\mathbf{U}^{m-1}\mathbf{T}^{(-1)}\mathbf{U}^{m-1} \qquad \text{for } m < 0 \tag{12}$$

The derivation of (9) is quite standard, see Guo and Man (1992) for the details.

It is worth noting that the tensor equation (9) may also used to provide the relationships between different stress measures conjugate to the Seth's strain tensors, see also Farahani and Naghdabadi (2000). Actually, for any pair of positive integers m and n, use of the work-conjugacy identity

$$\mathbf{T}^{(m)} \cdot \dot{\overline{\mathbf{E}^{(m)}}} = \mathbf{T}^{(n)} \cdot \dot{\overline{\mathbf{E}^{(n)}}} \tag{13}$$

allows one to characterize $\mathbf{T}^{(m)}$ as the solution of the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \frac{m}{n}\sum_{r=1}^{n} \mathbf{U}^{n-r}\mathbf{T}^{(n)}\mathbf{U}^{r-1} \tag{14}$$

The relationship between $\mathbf{T}^{(-m)}$ and $\mathbf{T}^{(n)}$ or $\mathbf{T}^{(-n)}$ can be obtained by noting that

$$\mathbf{T}^{(m)} \cdot \dot{\overline{\mathbf{U}^{(m)}}} = \mathbf{U}^{(-m)}\mathbf{T}^{(-m)}\mathbf{U}^{(-m)} \cdot \dot{\overline{\mathbf{U}^{(m)}}} \tag{15}$$

so that from (14) one has

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \frac{m}{n}\sum_{r=1}^{n} \mathbf{U}^{m+n-r}\mathbf{T}^{(n)}\mathbf{U}^{m+r-1} \tag{16}$$

and

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \frac{m}{n}\sum_{r=1}^{n} \mathbf{U}^{m-r}\mathbf{T}^{(-n)}\mathbf{U}^{m-n+r-1} \tag{17}$$

for $\mathbf{X} = \mathbf{T}^{(-m)}$.

Relative to any principal basis for $\mathbf{U}$, the solution of (9) can be obtained by the simple component formula

$$\mathbf{X}_{ij} = \frac{\mathbf{H}_{ij}}{\sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1}} \qquad i,j = 1,\dots,N \tag{18}$$

which immediately follows from (9) by decomposing the tensors $\mathbf{X}$ and $\mathbf{H}$ under the principal frame for $\mathbf{U}$, see also Guo and Man (1992).

Several researchers have derived expressions of $\mathbf{X}$ in tensor (intrinsic) form. Besides the one provided in Guo and Man (1992), the solution of (9) has been pursued in Guo et al. (1994), while different approaches to the evaluation of $\mathbf{T}^{(m)}$ have been presented in Guansuo and Quingwen (1999), Guansuo et al. (2000); more elaborate results on conjugate stresses and rates of arbitrary Seth-Hill strain measures can be found in Alfano et al. (2002), Guansuo et al. (1999), Xiao (1995).

We here present a novel approach to the evaluation of $\mathbf{T}^{(m)}$, which is based upon some recent results contributed by the authors (Rosati and Valoroso, 2001a). Namely, in order to obtain the solution of equation (9), we provide hereafter a generalization of the methodology illustrated in Rosati and Valoroso (2001b) for the solution of the tensor equation

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{H} \tag{19}$$

which represents the specialization of (9) for $m = 2$.

Such a generalization is carried out by exploiting the properties of the square tensor product between second-order tensors first introduced by Del Piero (1979), which allows one to express the equation (9) in the equivalent form

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{H} \quad \Leftrightarrow \quad \mathbb{U}\mathbf{X} = \mathbf{H} \tag{20}$$

where:

$$\mathbb{U} = \sum_{r=1}^{m} \mathbf{U}^{m-r} \boxtimes \mathbf{U}^{r-1} \tag{21}$$

Specifically, by suitably enhancing the treatment developed in Xiao (1995), it is shown how to express $\mathbb{U}$ as linear combination of the $N(N+1)/2$ tensors $\mathbb{U}_{ij}^{\otimes}$=2sym$(\mathbf{U}_i \otimes \mathbf{U}_j)$, $(i,j = 1,\dots,N)$, and of the $N(N-1)/2$ tensors $\mathbb{U}_{ij}^{\boxtimes} = \mathbf{U}_i \boxtimes \mathbf{U}_j + \mathbf{U}_j \boxtimes \mathbf{U}_i$, $(i \neq j)$, where $\mathbf{U}_k$ is the generic eigenprojector of $\mathbf{U}$. It is then proved that the coefficients multiplying the terms $\mathbb{U}_{ij}^{\otimes}$ can be arranged in a matrix $[\mathbf{U}^{\otimes}]$ which is nonsingular if $\mathbb{U}$ does share the same property; moreover, under this last hypothesis, the coefficients $u_{ij}^{\boxtimes}$ multiplying the tensors $\mathbb{U}_{ij}^{\boxtimes}$ turn out to be non-zero. By virtue of a further result contributed by the authors in Rosati and Valoroso (2001a), the inverse of $\mathbb{U}$ is then expressed as $\mathbb{U}^{-1} = [\mathbf{U}^{\otimes}]_{ij}^{-1}\mathbb{U}_{ij}^{\otimes} + (1/u_{ij}^{\boxtimes})\mathbb{U}_{ij}^{\boxtimes}$, thus providing an amazingly simple expression for $\mathbf{X}$ in the given reference frame in terms of $\mathbf{H}$ and of the eigenvalues and eigenprojectors of $\mathbf{U}$.

2 Some Algebra of Fourth- and Second-order Tensors

Let Lin be the space of all linear transformations (tensors) on $\mathcal{V}$ and $\mathbb{L}$in the space of all tensors on Lin. The dyadic $\otimes$ and square $\boxtimes$ tensor product between two elements $\mathbf{A}, \mathbf{B} \in$ Lin are defined by

$$(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \qquad (\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^t \qquad \forall\, \mathbf{C} \in \mathrm{Lin} \tag{22}$$

where the superscript t stands for transpose. Their cartesian components are given by

$$(\mathbf{A} \otimes \mathbf{B})_{ijkl} = \mathbf{A}_{ij}\mathbf{B}_{kl} \qquad (\mathbf{A} \boxtimes \mathbf{B})_{ijkl} = \mathbf{A}_{ik}\mathbf{B}_{jl} \qquad \forall\, \mathbf{A}, \mathbf{B} \in \mathrm{Lin} \tag{23}$$

The definition of square tensor product, first introduced in Del Piero (1979), is nowadays widespread in both theoretical (Podio-Guidugli and Virga, 1987; Rosati, 2000) and computational mechanics (Palazzo et al., 2001; Rosati and Valoroso, 2001a).

An interesting result, which relates the dyadic and square tensor product, is

$$(\mathbf{a} \otimes \mathbf{b}) \boxtimes (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \otimes \mathbf{c}) \otimes (\mathbf{b} \otimes \mathbf{d}) \qquad \forall\, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{V} \tag{24}$$

which can be used to infer that the dyadic and square tensor product of projection operators, i.e. tensors defined as $\mathbf{P} = \mathbf{e} \otimes \mathbf{e}, (\mathbf{e} \in \mathcal{V}; |\mathbf{e}| = 1)$, coincide

$$\mathbf{P} \otimes \mathbf{P} = \mathbf{P} \boxtimes \mathbf{P} \tag{25}$$

Given $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{V}$ and $\mathbf{A} \in$ Lin, the following relations can be proved

$$\begin{aligned} &(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \qquad (\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d}) \\ &\mathbf{A} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{A} = \mathbf{A}\mathbf{b} \cdot \mathbf{a} \qquad \mathbf{A}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{A}\mathbf{a} \otimes \mathbf{b} \qquad (\mathbf{a} \otimes \mathbf{b})\mathbf{A} = \mathbf{a} \otimes \mathbf{A}^t\mathbf{b} \end{aligned} \tag{26}$$

while composition formulas between elements of $\mathbb{L}$in and Lin are

$$\begin{aligned} &(\mathbf{A} \otimes \mathbf{B})(\mathbf{c} \otimes \mathbf{d}) = [\mathbf{B} \cdot (\mathbf{c} \otimes \mathbf{d})]\mathbf{A} = (\mathbf{B}^t\mathbf{c} \cdot \mathbf{d})\mathbf{A} \\ &(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{c} \otimes \mathbf{d}) = \mathbf{A}(\mathbf{c} \otimes \mathbf{d})\mathbf{B}^t = \mathbf{A}\mathbf{c} \otimes \mathbf{B}\mathbf{d} \end{aligned} \tag{27}$$

for every $\mathbf{B} \in$ Lin.

2.1 Eigenprojectors of Second-order Symmetric Tensors

Let Sym $\subseteq$ Lin denote the linear subspace of second-order symmetric tensors and $\mathbf{A} \in$ Sym. According to the spectral theorem (Halmos, 1958) $\mathbf{A}$ is amenable to the representation

$$\mathbf{A} = \sum_{i=1}^{N} a_i \mathbf{a}_i \otimes \mathbf{a}_i = \sum_{i=1}^{N} a_i \mathbf{A}_i \tag{28}$$

where a_i $(i = 1, \ldots, N)$ are the eigenvalues of $\mathbf{A}$, supposed to be distinct, and $\mathbf{a}_i$ the associated unit eigenvectors.

The eigenprojectors $\mathbf{A}_i$ form a set of N mutually orthogonal tensors having unit norm; they further fulfill the following properties

$$\mathbf{A}_i\mathbf{A}_j = \mathbf{A}_j\mathbf{A}_i = \begin{cases} \mathbf{A}_i & \text{if } i = j \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \sum_{i=1}^{N} \mathbf{A}_i = \mathbf{1} \tag{29}$$

Remark. It can be shown that if $p < N$ eigenvalues of $\mathbf{A}$ do coincide, it is always possible to define N orthogonal eigenprojectors so as to fulfill relations (28)-(29).

The eigenprojectors can be expressed as function of $\mathbf{A}$ and of its eigenvalues through Sylvester's formula. For $N=3$ this result can be deduced by solving the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{A} \\ \mathbf{A}^2 \end{bmatrix} \tag{30}$$

which is obtained by exploiting the properties (28) and (29). This yields (Luher and Rubin, 1990)

$$\mathbf{A}_i = \frac{(\mathbf{A}-a_j\mathbf{1})(\mathbf{A}-a_k\mathbf{1})}{(a_i-a_j)(a_i-a_k)} \qquad i,j,k=1,\ldots,N \quad j\neq k\neq i \tag{31}$$

provided that the eigenvalues of $\mathbf{A}$ are distinct.

If two eigenvalues do coincide the previous formula specializes to

$$\mathbf{A}_1 = \frac{(\mathbf{A}-a_2\mathbf{1})}{(a_1-a_2)} \qquad \mathbf{A}_2 = \frac{(\mathbf{A}-a_1\mathbf{1})}{(a_2-a_1)} \tag{32}$$

3 Uniqueness of the Solution of the Tensor Equation $\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{H}$

The tensor equation (9) admits a unique solution if and only if the tensor $\mathbb{U}$ defined in (21) is nonsingular or, equivalently, if all its eigenvalues are non-zero. The eigenvalues of $\mathbb{U}$ can be expressed in terms of the eigenvalues of $\mathbf{U}$ through the following:

Lemma 1. The eigenvalues of the rank-four tensor $\mathbb{U}$, see equation (21), are the $N(N+1)/2$ scalars $\sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1}$ $(i,j=1,\ldots,N)$ where λ_k is the generic (eventually coalescent) eigenvalue of $\mathbf{U}$. The relevant eigenvectors are $\mathbf{u}_i \otimes \mathbf{u}_j$ where $\mathbf{u}_k$ is the eigenvector of $\mathbf{U}$ associated with λ_k.

Proof. Invoking relationships $(22)_2$ and $(27)_2$ one has:

$$\left(\sum_{r=1}^{m} \mathbf{U}^{m-r} \boxtimes \mathbf{U}^{r-1}\right)(\mathbf{u}_i \otimes \mathbf{u}_j) = \sum_{r=1}^{m} (\mathbf{U}^{m-r}\mathbf{u}_i) \otimes (\mathbf{U}^{r-1}\mathbf{u}_j) = \sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1} \mathbf{u}_i \otimes \mathbf{u}_j \tag{33}$$

so that $\mathbf{u}_i \otimes \mathbf{u}_j$ is an eigenvector of $\mathbb{U}$ having $\sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1}$ as associated eigenvalue.

Since the eigenvalues λ_i of $\mathbf{U}$ are positive, such are those of $\mathbb{U}$. Accordingly, we can state the following:

Proposition 1. The tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{H} \tag{34}$$

does always admit a unique solution.

4 Representation of Symmetric Fourth-order Tensors in Principal Space

Let us consider a symmetric fourth-order tensor $\mathbb{A}$ expressed in the form:

$$\mathbb{A} = \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{N-1} \left[c_{\alpha\beta}^{\otimes} (\mathbf{A}^{\alpha} \otimes \mathbf{A}^{\beta} + \mathbf{A}^{\beta} \otimes \mathbf{A}^{\alpha}) + c_{\alpha\beta}^{\boxtimes} (\mathbf{A}^{\alpha} \boxtimes \mathbf{A}^{\beta} + \mathbf{A}^{\beta} \boxtimes \mathbf{A}^{\alpha}) \right] \tag{35}$$

where $c_{\alpha\beta}^{\otimes}$ and $c_{\alpha\beta}^{\boxtimes}$ are arbitrary scalars. As shown in Palazzo et al. (2001), the above expression represents a general class of nonlinear tensor function of a tensor argument $\mathbf{A}$.

We are interested to investigate on the expression of $\mathbb{A}$ resulting from the spectral representation of $\mathbf{A}$; to be specific we shall set $N = 3$. Recalling (25) and (28), one has

$$
\begin{aligned}
\mathbb{A} &= \sum_{i,j=1}^{3} a_{ij}^{\otimes} \mathbf{A}_i \otimes \mathbf{A}_j + \sum_{\substack{i,j=1\\ i\neq j}}^{3} a_{ij}^{\boxtimes} \mathbf{A}_i \boxtimes \mathbf{A}_j = [\mathbf{A}^{\otimes}] \cdot [\mathbb{A}^{\otimes}] + [\mathbf{A}^{\boxtimes}] \cdot [\mathbb{A}^{\boxtimes}] = \\
&= \begin{bmatrix} a_{11}^{\otimes} & a_{12}^{\otimes} & a_{13}^{\otimes} \\ a_{12}^{\otimes} & a_{22}^{\otimes} & a_{23}^{\otimes} \\ a_{13}^{\otimes} & a_{23}^{\otimes} & a_{33}^{\otimes} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_1 \otimes \mathbf{A}_1 & \mathbf{A}_1 \otimes \mathbf{A}_2 & \mathbf{A}_1 \otimes \mathbf{A}_3 \\ \mathbf{A}_2 \otimes \mathbf{A}_1 & \mathbf{A}_2 \otimes \mathbf{A}_2 & \mathbf{A}_2 \otimes \mathbf{A}_3 \\ \mathbf{A}_3 \otimes \mathbf{A}_1 & \mathbf{A}_3 \otimes \mathbf{A}_2 & \mathbf{A}_3 \otimes \mathbf{A}_3 \end{bmatrix} + \\
&+ \begin{bmatrix} 0 & a_{12}^{\boxtimes} & a_{31}^{\boxtimes} \\ a_{12}^{\boxtimes} & 0 & a_{23}^{\boxtimes} \\ a_{31}^{\boxtimes} & a_{23}^{\boxtimes} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbb{O} & \mathbf{A}_1 \boxtimes \mathbf{A}_2 & \mathbf{A}_1 \boxtimes \mathbf{A}_3 \\ \mathbf{A}_2 \boxtimes \mathbf{A}_1 & \mathbb{O} & \mathbf{A}_2 \boxtimes \mathbf{A}_3 \\ \mathbf{A}_3 \boxtimes \mathbf{A}_1 & \mathbf{A}_3 \boxtimes \mathbf{A}_2 & \mathbb{O} \end{bmatrix}
\end{aligned} \tag{36}
$$

the entries $a_{ij}^{\otimes}$ and $a_{ij}^{\boxtimes}$ being polynomial expressions of the eigenvalues of $\mathbf{A}$ and of the coefficients $c_{\alpha\beta}^{\otimes}$ and $c_{\alpha\beta}^{\boxtimes}$. The dots between arrays $[\,]\cdot[\,]$ used in the previous formula indicate sum of the products of the elements having the same position.

As an example of the previous representation formula of fourth-order tensors we consider the identity $\mathbb{I}$ in $\mathbb{L}\mathrm{in}$. Observing that $\mathbb{I} = \mathbf{1} \boxtimes \mathbf{1}$ and recalling $(29)_2$ we have

$$
\begin{aligned}
\mathbb{I} &= \mathbf{1} \boxtimes \mathbf{1} = \mathbf{A}_1 \otimes \mathbf{A}_1 + \mathbf{A}_2 \otimes \mathbf{A}_2 + \mathbf{A}_3 \otimes \mathbf{A}_3 + \\
&+ \mathbf{A}_1 \boxtimes \mathbf{A}_2 + \mathbf{A}_2 \boxtimes \mathbf{A}_1 + \mathbf{A}_2 \boxtimes \mathbf{A}_3 + \mathbf{A}_3 \boxtimes \mathbf{A}_2 + \mathbf{A}_3 \boxtimes \mathbf{A}_1 + \mathbf{A}_1 \boxtimes \mathbf{A}_3
\end{aligned} \tag{37}
$$

Hence:

$$
\begin{aligned}
\mathbb{I} &= [\mathbf{1}^{\otimes}] \cdot [\mathbb{A}^{\otimes}] + [\mathbf{1}^{\boxtimes}] \cdot [\mathbb{A}^{\boxtimes}] = \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_1 \otimes \mathbf{A}_1 & \mathbf{A}_1 \otimes \mathbf{A}_2 & \mathbf{A}_1 \otimes \mathbf{A}_3 \\ \mathbf{A}_2 \otimes \mathbf{A}_1 & \mathbf{A}_2 \otimes \mathbf{A}_2 & \mathbf{A}_2 \otimes \mathbf{A}_3 \\ \mathbf{A}_3 \otimes \mathbf{A}_1 & \mathbf{A}_3 \otimes \mathbf{A}_2 & \mathbf{A}_3 \otimes \mathbf{A}_3 \end{bmatrix} + \\
&+ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbb{O} & \mathbf{A}_1 \boxtimes \mathbf{A}_2 & \mathbf{A}_1 \boxtimes \mathbf{A}_3 \\ \mathbf{A}_2 \boxtimes \mathbf{A}_1 & \mathbb{O} & \mathbf{A}_2 \boxtimes \mathbf{A}_3 \\ \mathbf{A}_3 \boxtimes \mathbf{A}_1 & \mathbf{A}_3 \boxtimes \mathbf{A}_2 & \mathbb{O} \end{bmatrix}
\end{aligned} \tag{38}
$$

The main result exploited in the paper is provided by the following

Lemma 2. Let $\mathbb{A} = [\mathbf{A}^{\otimes}] \cdot [\mathbb{A}^{\otimes}] + [\mathbf{A}^{\boxtimes}] \cdot [\mathbb{A}^{\boxtimes}]$ be a rank-four tensor. Then $\mathbb{A}$ is nonsingular if and only if $[\mathbf{A}^{\otimes}]$ is nonsingular and each off-diagonal entry of $[\mathbf{A}^{\boxtimes}]$ is non-zero.

Proof. By definition:

$$
\mathbb{A} \text{ nonsingular} \quad \Leftrightarrow \quad \mathrm{Ker}\,\mathbb{A} = \{\mathbf{B} \in \mathrm{Lin} \ : \ \mathbb{A}\mathbf{B} = \mathbf{0}\} = \{\mathbf{0}\} \tag{39}
$$

Stated equivalently, $\mathbb{A}\mathbf{B} \neq \mathbf{0}$ if and only if $\mathbf{B} \neq \mathbf{0}$.

Let $\mathbb{A}$ be nonsingular. Expressing $\mathbf{B}$ in the cartesian frame represented by the eigenvectors of $\mathbf{A}$

$$
\mathbf{B} = \sum_{k,l=1}^{3} b_{kl} \mathbf{a}_k \otimes \mathbf{a}_l \tag{40}
$$

and observing that the following composition rules

$$
\begin{aligned}
(\mathbf{A}_i \otimes \mathbf{A}_j)(\mathbf{a}_k \otimes \mathbf{a}_l) &= \begin{cases} \mathbf{A}_i & \text{if } j = k = l \\ \mathbf{0} & \text{otherwise} \end{cases} \\
(\mathbf{A}_i \boxtimes \mathbf{A}_j)(\mathbf{a}_k \otimes \mathbf{a}_l) &= \begin{cases} \mathbf{a}_i \otimes \mathbf{a}_j & \text{if } i = k \text{ and } j = l \\ \mathbf{0} & \text{otherwise} \end{cases}
\end{aligned}
\tag{41}
$$

hold true on account of (27), it can be shown that

$$
\mathbb{A}\mathbf{B} = \begin{bmatrix} a_{11}^{\otimes} & a_{12}^{\otimes} & a_{13}^{\otimes} \\ a_{12}^{\otimes} & a_{22}^{\otimes} & a_{23}^{\otimes} \\ a_{13}^{\otimes} & a_{23}^{\otimes} & a_{33}^{\otimes} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{22} \\ b_{33} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix} + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} a_{ij}^{\boxtimes} b_{ij} (\mathbf{a}_i \otimes \mathbf{a}_j) \tag{42}
$$

Choosing as $\mathbf{B}$ the tensor whose associated matrix $[\mathbf{B}]$ in the principal reference frame for $\mathbf{A}$ has only one non-zero off-diagonal term $b_{ij} \neq 0\,(i \neq j)$, yields

$$
\mathbb{A}\mathbf{B} = a_{ij}^{\boxtimes} b_{ij} (\mathbf{a}_i \otimes \mathbf{a}_j) \qquad \text{(no sum on } i \text{ nor } j) \tag{43}
$$

Hence, to ensure $\mathbb{A}\mathbf{B} \neq \mathbf{0}$, it must necessarily be $a_{ij}^{\boxtimes} \neq 0$, $(i,j = 1, \ldots, N;\ i \neq j)$. Consider now a diagonal matrix for $[\mathbf{B}]$ with non-zero entries; then:

$$
\mathbb{A}\mathbf{B} = [\mathbf{A}^{\otimes}][\mathbf{b}] \cdot [\mathbf{A}_i] \tag{44}
$$

where it has been set $[\mathbf{b}] = [b_{11}, b_{22}, b_{33}]^t$ and $[\mathbf{A}_i] = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3]^t$. Since $[\mathbf{b}] \neq \mathbf{o}$ by hypothesis, $\mathbb{A}\mathbf{B} \neq \mathbf{0}$ if and only if $\operatorname{Ker}[\mathbf{A}^{\otimes}] = \{\mathbf{o}\}$, i.e. if $[\mathbf{A}^{\otimes}]$ is non-singular.

Let us now prove the if part of the lemma. Notice that the N tensors $\mathbf{A}_i$, $(i = 1, \ldots, N)$, and $\mathbf{a}_j \otimes \mathbf{a}_k$, $(j, k = 1, \ldots, N)$ are linearly independent since they represent a basis for Lin. Hence, their unique linear combination yielding the null tensor is the one with null coefficients. Since, by hypothesis

$$
\begin{aligned}
&[\mathbf{A}^{\otimes}][\mathbf{b}] = \mathbf{o} &&\Leftrightarrow &&[\mathbf{b}] = \mathbf{o} \\
&a_{ij}^{\boxtimes} b_{ij} = 0 &&\Leftrightarrow &&b_{ij} = 0 \qquad \text{no sum on } i \text{ nor } j
\end{aligned}
\tag{45}
$$

we infer from (42) that $\mathbb{A}\mathbf{B} = \mathbf{0}$ if and only if $\mathbf{B} = \mathbf{0}$, i.e. if $\mathbb{A}$ is invertible. □

The second result which plays a paramount role in the sequel is contained in the following lemma whose detailed proof can be found in Rosati and Valoroso (2001a)

Lemma 3. Given a nonsingular fourth-order tensor $\mathbb{A}$ of the form

$$
\mathbb{A} = [\mathbf{A}^{\otimes}] \cdot [\mathbb{A}^{\otimes}] + [\mathbf{A}^{\boxtimes}] \cdot [\mathbb{A}^{\boxtimes}] \tag{46}
$$

it turns out to be:

$$
\mathbb{A}^{-1} = [\mathbf{A}^{\otimes}]^{-1} \cdot [\mathbb{A}^{\otimes}] + [\mathbf{A}^{\boxtimes}]^{-1} \cdot [\mathbb{A}^{\boxtimes}] \tag{47}
$$

where $[\mathbf{A}^{\otimes}]^{-1}$ is the inverse matrix of $[\mathbf{A}^{\otimes}]$ and $[\mathbf{A}^{\boxtimes}]^{-1}$ is the matrix whose components are the reciprocals of the non-zero entries of $[\mathbf{A}^{\boxtimes}]$.

5 Evaluation of the Conjugate Stress $\mathbf{T}^{(m)}$

Formula (10) shows that the tensor equation (9) represents a generalization of the more classical tensor equation $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ widely encountered in continuum mechanics, see e.g. Scheidler (1994) and references therein.

At least in principle, one may apply to (9) the general approach presented in Rosati (2000). This would require in turn to represent $\mathbb{U}^{-1}$ either in the particular form

$$\begin{aligned}\mathbb{U}^{-1} &= a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{U} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{U}) + c(\mathbf{U} \boxtimes \mathbf{U}) + d(\mathbf{U}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{U}^2) + \\ &+ e(\mathbf{U}^2 \boxtimes \mathbf{U} + \mathbf{U} \boxtimes \mathbf{U}^2) + f(\mathbf{U}^2 \boxtimes \mathbf{U}^2)\end{aligned} \tag{48}$$

suggested from the condition

$$\mathbb{U}^{-1}\mathbb{U} = \mathbb{U}\mathbb{U}^{-1} = \mathbb{I} = \mathbf{1} \boxtimes \mathbf{1} \tag{49}$$

or in the form

$$\begin{aligned}\mathbb{U}^{-1} &= a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{U} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{U}) + c(\mathbf{U}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{U}^2) + \\ &+ d(\mathbf{1} \otimes \mathbf{1}) + e(\mathbf{U} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{U}) + f(\mathbf{U}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{U}^2) + \\ &+ g(\mathbf{U} \otimes \mathbf{U}) + h(\mathbf{U}^2 \otimes \mathbf{U} + \mathbf{U} \otimes \mathbf{U}^2) + i(\mathbf{U}^2 \otimes \mathbf{U}^2)\end{aligned} \tag{50}$$

stemming from the property which characterizes the solution of (9) as an isotropic tensor function of $\mathbf{U}$ and $\mathbf{H}$, linear in $\mathbf{H}$, see Sidoroff (1978). However, in order to arrive at a solution for $\mathbf{X}$ expressed solely in terms of $\mathbf{U}$ and $\mathbf{H}$, it would be necessary to dispose of a general formula providing for any integer $q > 4$, $\mathbf{U}^q$ as function of the invariants of $\mathbf{U}$ and of $\mathbf{1}$, $\mathbf{U}$ and $\mathbf{U}^2$ through repeated applications of the Cayley-Hamilton theorem. Unfortunately, this formula, which is required for determining the coefficients of the representations (48) or (50) upon enforcement of (49), is well-known to be still lacking in the specialized literature (Taher and Rachidi, 2001).

For this reason we here exploit an alternative original approach to the solution of the tensor equation (9) which is based on the results presented in the previous section. According to (36) we can express the tensor $\mathbb{U}$ defined by (21) as

$$\begin{aligned}\mathbb{U} &= \sum_{r=1}^{m} \mathbf{U}^{m-r} \boxtimes \mathbf{U}^{r-1} = [\mathbf{U}^{\otimes}] \cdot [\mathbb{U}^{\otimes}] + [\mathbf{U}^{\boxtimes}] \cdot [\mathbb{U}^{\boxtimes}] = \\ &= \begin{bmatrix} m\lambda_1^{m-1} & 0 & 0 \\ 0 & m\lambda_2^{m-1} & 0 \\ 0 & 0 & m\lambda_3^{m-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{U}_1 \otimes \mathbf{U}_1 & \mathbf{U}_1 \otimes \mathbf{U}_2 & \mathbf{U}_1 \otimes \mathbf{U}_3 \\ \mathbf{U}_2 \otimes \mathbf{U}_1 & \mathbf{U}_2 \otimes \mathbf{U}_2 & \mathbf{U}_2 \otimes \mathbf{U}_3 \\ \mathbf{U}_3 \otimes \mathbf{U}_1 & \mathbf{U}_3 \otimes \mathbf{U}_2 & \mathbf{U}_3 \otimes \mathbf{U}_3 \end{bmatrix} + \\ &+ \begin{bmatrix} 0 & \sum_{r=1}^{m} \lambda_1^{m-r}\lambda_2^{r-1} & \sum_{r=1}^{m} \lambda_1^{m-r}\lambda_3^{r-1} \\ \sum_{r=1}^{m} \lambda_1^{m-r}\lambda_2^{r-1} & 0 & \sum_{r=1}^{m} \lambda_2^{m-r}\lambda_3^{r-1} \\ \sum_{r=1}^{m} \lambda_1^{m-r}\lambda_3^{r-1} & \sum_{r=1}^{m} \lambda_2^{m-r}\lambda_3^{r-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbb{O} & \mathbf{U}_1 \boxtimes \mathbf{U}_2 & \mathbf{U}_1 \boxtimes \mathbf{U}_3 \\ \mathbf{U}_2 \boxtimes \mathbf{U}_1 & \mathbb{O} & \mathbf{U}_2 \boxtimes \mathbf{U}_3 \\ \mathbf{U}_3 \boxtimes \mathbf{U}_1 & \mathbf{U}_3 \boxtimes \mathbf{U}_2 & \mathbb{O} \end{bmatrix}\end{aligned} \tag{51}$$

It is worth noting that the conditions stated in Lemma 1 ensuring that $\mathbb{U}$ is nonsingular can be immediately deduced from the previous representation formula and Lemma 2. Therefore, we ultimately infer

$$\mathbb{U}^{-1} = [\mathbf{U}^{\otimes}]^{-1} \cdot [\mathbb{U}^{\otimes}] + [\mathbf{U}^{\boxtimes}]^{-1} \cdot [\mathbb{U}^{\boxtimes}] \tag{52}$$

by virtue of Lemma 3. Hence, one obtains the expression of $\mathbf{X}$ as:

$$\mathbf{X} = \left[\sum_{i=1}^{N} \frac{1}{m} \lambda_i^{1-m} \mathbf{U}_i \otimes \mathbf{U}_i + \sum_{j=1}^{N(N-1)/2} \frac{1}{\sum_{r=1}^{m} \lambda_j^{m-r}\lambda_k^{r-1}} (\mathbf{U}_j \boxtimes \mathbf{U}_k + \mathbf{U}_k \boxtimes \mathbf{U}_j) \right] \mathbf{H} \tag{53}$$

where $k = 1 + MOD(j, N)$ and the function MOD is defined as $MOD(j, N) := j - [INT(j/N)] * N$, being INT the standard truncate-to-integer function.

Straighforward calculations show that the components of the previous expression relative to any principal basis for $\mathbf{U}$ do coincide with the ones provided by (18) and that $\mathbf{X}$ and $\mathbf{H}$ are simultaneously symmetric or skew-symmetric.

Since for any pair of distinct eigenvalues λ_j, λ_k it turns out to be

$$\sum_{r=1}^{m} \lambda_j^{m-r} \lambda_k^{r-1} = \frac{\lambda_j^m - \lambda_k^m}{\lambda_j - \lambda_k}; \tag{54}$$

by invoking (25) and recalling that $\mathbf{X} = \mathbf{T}^{(m)}$ we finally get the particularly compact formula

$$\mathbf{T}^{(m)} = \left[\sum_{i=1}^{N} \frac{1}{m} \lambda_i^{1-m} \mathbf{U}_i \odot \mathbf{U}_i + \sum_{j=1}^{N(N-1)/2} \frac{\lambda_j - \lambda_k}{\lambda_j^m - \lambda_k^m} (\mathbf{U}_j \boxtimes \mathbf{U}_k + \mathbf{U}_k \boxtimes \mathbf{U}_j) \right] \mathbf{H} \tag{55}$$

where the symbol $\odot$ stands either for the dyadic or the square product.

6 Concluding Remarks

It has been provided an explicit intrinsic expression for the stress $\mathbf{T}^{(m)}$ conjugate to the Seth's strain measure $\mathbf{E}^{(m)}$ valid for arbitrary integers $m \neq 0$. The contributed expression has several distinguished features with respect to existing formulas.

First, it provides the solution directly in the given reference frame. If, for some reason, the solution needs to be expressed as function of $\mathbf{U}$ and its powers, proper use can be made of Sylvester's formula (31). Second, no special distinction needs to be made between the three- and two-dimensional case as it happens for other direct formulas contributed in the literature. Third, the case of coincident eigenvalues can be trivially dealt with provided that, for each eigenvalue with multiplicity $s > 1$, one associates s suitably defined eigenprojectors.

As a final remark we note that, by following an approach similar to the one developed in the present work, more complex tensor equations such as $\sum_{r,s=1}^{m} g_{rs} \mathbf{A}^r \mathbf{X} \mathbf{B}^s = \mathbf{H}$ can be solved with reasonable effort. This issue will be addressed in a forthcoming paper.

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