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# The Slow March towards an Analytical Mechanics of Dissipative Materials

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*The concern of this work is the derivation of material balance laws for the Green–Naghdi (G–N) theory of dissipationless thermoelasticity. The lack of dissipation allows for a physically meaningful variational formulation which is used for the application of Noether's theorem. The balance laws on the material manifold are derived and the exact conditions under which they hold are rigorously studied.*

## 1 Introduction

This work is devoted to the canonical formulation of thermoelasticity theory without dissipation (Green and Naghdi, 1993). For this purpose, we use the celebrated theorem of Noether according to which it is possible to obtain a conservation law for every given variational symmetry. We also use what is called (Lovelock and Rund, 1975) "invariance identity" to obtain the non-homogeneous terms of the material balance laws, i.e., the material forces or some kind of moment of such forces. In this manner we obtain all equations of interest, that is, the balance of linear momentum, the equation of entropy, the balance of canonical momentum, the balance of scalar moment of canonical momentum, and the energy equation, all in the apparently "dissipationless form".

Dascalu and Maugin (1995) used the G–N theory to formulate the corresponding *canonical balance laws of momentum and energy* – of interest in the design of fracture criteria – which, contrary to the expressions of the classical theory, indeed present *no* source of dissipation and canonical momentum (Epstein and Maugin, 1995). In recent works (Maugin, 1997, 2000) the consistency between the expressions of intrinsic dissipation and source of canonical momentum in dissipative continua has been shown. This is developed within the framework of *material* or *configurational forces*, that world of forces which, for instance, drive structural rearrangements and material defects of different types on the material manifold (Maugin, 1995; Gurtin, 2000; Kienzler and Herrmann, 2000).

We alternatively use throughout the paper the vectorial as well as the index notation to represent Cartesian vectors and tensors, thus rectangular coordinate systems are adopted in all cases. The motion of a thermoelastic body is described by the smooth mapping $x_\beta = x_\beta(X_B)$, where $B, \beta = 1, 2, 3, 4$, $X_4 = t$, $x_4 = \alpha$, $\alpha = \alpha(\mathbf{X}, t)$ is the thermal displacement scalar field, a primitive concept according to G–N theory. The temperature field is defined to be the time derivative of $\alpha$, thus $\Theta(\mathbf{X}, t) := \dot{\alpha}(\mathbf{X}, t)$. Also, we use the notation $\mathbf{X}$ to denote the material space variable, and $\mathbf{x}$ for the spatial position of the particle $\mathbf{X}$ at time $t$. In a coordinate system these variables will be written as $X_L, L = 1, 2, 3$ and $x_i, i = 1, 2, 3$, respectively. Generally, Greek indices will range from 1 to 4, while the lower-case Latin ones will range from 1 to 3. Also, the capital letters K, L, M,...will range from 1 to 3 and A, B,... from 1 to 4. We use two distinct differential operators $\frac{\partial}{\partial X_A}$ and $\frac{D}{DX_A}$. The former is the usual partial derivative operator while the latter denotes the partial derivative which accounts for the underlying function composition. Also, the usual notations $GradF = \nabla_R F = DF/DX_L$, $Div\mathbf{F} = DF_L/DX_L$ and $\dot{F} = DF/Dt$ for gradient, divergence and material time derivative, respectively, are used. All the results presented in this paper are given without any proof due to the lack of space. The interested reader can find the omitted proofs as well as additional results in a forthcoming paper of the authors.

## 2 Preliminaries

According to the G–N theory the field equations of thermoelasticity of type II (Green and Naghdi, 1993) i.e., the momentum and energy equations are given as follows

$$\frac{D\mathbf{p}}{Dt} - Div\mathbf{T} = \mathbf{0}, \quad -(\frac{D\Psi}{Dt} + \frac{D\Theta}{Dt}\eta) + tr(\mathbf{T}\dot{\mathbf{F}}) - \mathbf{S}\cdot\nabla_R\Theta = 0 \tag{1}$$

where $\mathbf{p} = \rho_R\mathbf{v}$ is the physical momentum, $\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}$ is the velocity field, $\Psi$ is the free energy function per unit volume, $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor, $\mathbf{F}$ is the deformation gradient tensor, $\mathbf{S}$ is the entropy flux vector and $\eta$ is the entropy density per unit volume. Also, the constitutive equations are given in the form

$$\mathbf{T} = \frac{\partial\Psi}{\partial\mathbf{F}}, \quad \mathbf{S} = -\frac{\partial\Psi}{\partial\beta}, \quad \eta = -\frac{\partial\Psi}{\partial\dot{\alpha}}, \tag{2}$$

where $\beta = \nabla_R\alpha$, $\alpha = \alpha(\mathbf{X}, t)$.

In the sequel, we give some fundamental elements related to variational symmetries and Noether's theorem. Let a $C^2$ function be

$$L = L(X_A, x_\alpha, x_{\alpha,A}), \quad A = 1, 2, ...n, \alpha = 1, 2, ...m,$$

where $X_A \in G$, $G$ is a smooth domain of $R^n$ and $x_\alpha(X_A)$ is a sufficiently smooth function. Consider the functional $I : C^2(G) \to R$ given as follows

$$I(x_\alpha) = \int_G L(X_A, x_\alpha, x_{\alpha,A})dV. \tag{3}$$

The necessary condition for the functional $I$ to attain an extremum is given by the well–known Euler–Lagrange equations

$$\frac{\partial L}{\partial x_\gamma} - \frac{D}{DX_A}(\frac{\partial L}{\partial x_{\gamma,A}}) = 0, \quad \forall X_A \in G \tag{4}$$

where the summation convention is used when repeated indices appear.

Consider now the $n+m$–dimensional Euclidean space $E^{m+n}$ made up by the dependent and independent variables. A group of infinitesimal transformations will be of the form

$$\tilde{X}_A = X_A + \epsilon_w Z_A^w, \qquad \tilde{x}_\alpha = x_\alpha + \epsilon_w\zeta_\alpha^w, \quad w = 1, 2, ..., \mu \tag{5}$$

where $Z_A^w$ and $\zeta_\alpha^w$ are $C^\infty$ with respect to $X_B, x_\beta$ and $\epsilon_w$ are reals.

The following *invariance identity* is a necessary and sufficient condition for a group of infinitesimal transformations to be a variational symmetry (Lovelock and Rund, 1975; Olver, 1993).

$$\frac{\partial L}{\partial X_A}Z_A^w + \frac{\partial L}{\partial x_\alpha}\zeta_\alpha^w + \frac{\partial L}{\partial x_{\alpha,A}}(\frac{D\zeta_\alpha^w}{DX_A} - x_{\alpha,B}\frac{DZ_B^w}{DX_A}) + L\frac{DZ_A^w}{DX_A} = 0. \tag{6}$$

Now, we give a version of Noether's theorem (Lovelock and Rund, 1975) convenient to our purpose

**Theorem of Noether** *If the functional* (3) *is invariant under the* $\mu$*–parameter group of transformations given by equations* (5), *then there exist* $\mu$ *conservation laws of Euler–Lagrange equations* (4) *given by*

$$-\frac{D\theta_A^w}{DX_A} = [\frac{\partial L}{\partial x_\gamma} - \frac{D}{DX_A}(\frac{\partial L}{\partial x_{\gamma,A}})](\zeta_\gamma^w - x_{\gamma,B}Z_B^w) = 0, \tag{7}$$

*where*

$$\theta_A^w = -(LZ_A^w - \frac{\partial L}{\partial x_{\alpha,A}}x_{\alpha,B}Z_B^w + \frac{\partial L}{\partial x_{\alpha,A}}\zeta_\alpha^w). \tag{8}$$

## 3 Variational Symmetries and Conservation Laws

**Definition** *The Lagrangian function of a thermoelastic body without dissipation is defined to be of the form*

$$L(X_L,\dot{x}_i,\dot{\alpha},\frac{\partial x_i}{\partial X_L},\frac{\partial \alpha}{\partial X_L})=\frac{1}{2}\rho_R(X_L)\dot{x}_i\dot{x}_i-\Psi(X_L,\frac{\partial x_i}{\partial X_L},\dot{\alpha},\frac{\partial \alpha}{\partial X_L}). \tag{9}$$

Thus, the functional $I$ for the case under consideration will take the form

$$I(x_i,\alpha)=\int_{t_1}^{t_2}\int_{\Omega}L(X_L,\dot{x}_i,\dot{\alpha},x_{i,L},\alpha_{,L})dV dt, \tag{10}$$

where $\Omega$ is a smooth domain of $R^3$, and $[t_1,t_2]$ an interval of $R$. Notice that $L$ is not an explicit function of $x_i$ by virtue of Galilean invariance (translations in physical space of placements). Neither is it an explicit function of $\alpha$ itself, implying a sort of *gauge invariance* very similar to that of electrostatics for the electric potential. The variational statement can be written as follows:

**Proposition 1** *Let the constitutive relations* (2) *hold. Among all admissible functions of motion and thermal displacement for a thermoelastic body without dissipation, those ones which afford an extreme value for the action integral defined by* (9–10), *will satisfy the equations*

$$\frac{DT_{Li}}{DX_L}-\frac{D}{Dt}(\rho_R\dot{x}_i)=0,\quad \frac{DS_L}{DX_L}+\frac{D\eta}{Dt}=0. \tag{11}$$

Thus the variational statement provides two equations. The first of them is the equation of motion and coincides with the corresponding one of G–N theory, i.e. equation (1a). The second one is an equation for the balance of entropy. Thus, as far as the variational principle is considered, the field equations for thermoelasticity without dissipation are equations (11) instead of (1) of G–N theory. The other required equation (1b) is an energy equation and can not directly stem from a variational principle. What can be expected is that it appears as a consequence of Noether's theorem considering invariance in time translations. More on the structure of variational principles in the presence of dissipative processes in (Van and Muschik, 1995; Van and Nyiri, 1999).

**Invariance under Translations**
First, consider invariance in material space and time translations.

**Lemma 1** *If the action integral of a thermoelastic body without dissipation is invariant under the group of space and time translations, then it is a homogeneous body.*

Next, we give the main result concerning the group of translations.

**Proposition 2** *Let the motion and the temperature functions $x_i$ and $\Theta$ satisfy the field equations* (11) *for a homogeneous thermoelastic body without dissipation through constitutive relations* (2) *on the domain $\Omega\times[t_1,t_2]$. Then the following conservation laws also hold on $\Omega\times[t_1,t_2]$*

$$\frac{D}{DX_L}(L\delta_{KL}+T_{Li}x_{i,K}-S_L\beta_K)-\frac{D}{Dt}(\rho_R\dot{x}_i x_{i,K}+\eta\beta_K)=0, \tag{12}$$

$$\frac{D}{DX_L}(T_{Li}\dot{x}_i-S_L\Theta)+\frac{D}{Dt}(L-\rho_R\dot{x}_i\dot{x}_i-\eta\Theta)=0. \tag{13}$$

The second of the above conservation laws, equation (13), corresponds to time translations, thus it is related to the conservation of energy. After some calculation and taking into account the equation of motion (11a), it can written as

$$-(\dot{\Psi}+\dot{\Theta}\eta)+T_{Li}v_{i,L}-S_L\Theta_{,L}=0, \tag{14}$$

which coincides with equation (1b).

**Invariance under the Scaling Group**
In this case we will use the following one–parameter group of scalings in material and physical space

$$\tilde{X}_A = X_A + \epsilon X_A \qquad \tilde{x}_\alpha = x_\alpha - \epsilon x_\alpha, \tag{15}$$

In this case the invariance identity becomes

$$\frac{\partial L}{\partial X_A} X_A - 2(\frac{\partial L}{\partial x_{\alpha,A}}) x_{\alpha,A} + 4L = 0. \tag{16}$$

The most interesting result for the scaling group concerns the linear case for which the following identity holds

**Lemma 2** *Let us assume that a thermoelastic body without dissipation admits an $L$ homogeneous of degree two in the components $x_{\alpha,A}$, then its Lagrangian satisfies the following Euler identity*

$$\frac{\partial L}{\partial x_{\alpha,A}} x_{\alpha,A} = 2L. \tag{17}$$

**Proposition 3** *Let the motion and the temperature functions $x_i$ and $\Theta$ satisfy the field equations* (11) *for a homogeneous thermoelastic body without dissipation through linear constitutive relations on the domain $\Omega \times [t_1, t_2]$. Then the following conservation law also holds on $\Omega \times [t_1, t_2]$*

$$\begin{aligned} &\frac{D}{DX_L}[(L\delta_{KL} + T_{Li}x_{i,K} - S_L\beta_K)X_K + (T_{Li}\dot{x}_i - S_L\Theta)t + T_{Li}x_i - S_L\alpha] + \\ &\frac{D}{Dt}[-(\frac{1}{2}\rho_R\dot{x}_i\dot{x}_i + e)t - (\rho_R\dot{x}_i x_{i,K} + \eta\beta_K)X_K - \rho_R\dot{x}_i x_i - \eta\alpha] = 0, \end{aligned} \tag{18}$$

*where $e = e(\mathbf{X}, t)$ is the internal density function per unit volume.*

## 4 Material Balance Laws

So far, we have presented conservation laws of the G–N equations of thermoelasticity. From the point of view of material mechanics, it is interesting to focus on what can be called *material balance laws.* To obtain such equations we must allow for the presence of sources in the already derived equations. This, in turn, can be done by relaxing the assumptions we have posed in order to obtain them. By this method, for every conservation law we can obtain a balance (non-conservation) law. Applying this procedure to conservation laws (12) and (18) we obtain

$$\frac{D}{DX_L}(L\delta_{KL} + T_{Li}x_{i,K} - S_L\beta_K) - \frac{D}{Dt}(\rho_R\dot{x}_i x_{i,K} + \eta\beta_K) = \frac{\partial L}{\partial X_K}, \tag{19}$$

$$\begin{aligned} &\frac{D}{DX_L}[(L\delta_{KL} + T_{Li}x_{i,K} - S_L\beta_K)X_K + (T_{Li}\dot{x}_i - S_L\Theta)t + T_{Li}x_i - S_L\alpha] + \\ &\frac{D}{Dt}[-(\frac{1}{2}\rho_R\dot{x}_i\dot{x}_i + e)t - (\rho_R\dot{x}_i x_{i,K} + \eta\beta_K)X_K - \rho_R\dot{x}_i x_i - \eta\alpha] = \frac{\partial L}{\partial X_L}X_L. \end{aligned} \tag{20}$$

The first one is the well-known *pseudomomentum equation* and, the second is the *scalar moment of pseudomomentum equation.* Equation (20) holds for linear, non–homogeneous thermoelasticity and represents a balance law for scalar moment of pseudomomentum or canonical momentum. The corresponding balance equation in physical space is not often used, because it does not play any role in the description of the equilibrium or the motion of a body as does, for instance, the momentum or angular momentum equation. In the case of physical space, the factors that balance the rate of scalar moment of momentum are referred to as *scalar moments* or *virials.* So the right hand side term of equation (20) is a sort of *material scalar moment* or *material virial.*

The above–obtained results can be compared to previously published work of other researchers. We must especially refer to the work of Dascalu and Maugin (1995) for G–N thermoelasticity, and Maugin (1993) and Fletcher (1976) for elasticity. Let us return to equation (12), which represents the canonical momentum conservation law. It can be written in the form

$$Div((\rho_R \frac{\mathbf{v}^2}{2} - \Psi)\mathbf{I} + \mathbf{TF} - \beta \otimes \mathbf{S}) - \frac{D}{Dt}(\rho_R \mathbf{F}^T \mathbf{v} + \eta\beta) = 0. \tag{21}$$

Equation (21) coincides with the corresponding one deriving through a vectorial approach by Dascalu and Maugin (1995). Conservation laws (12), (13) and (18) restricted to the case of elasticity are in full agreement with the corresponding ones given by Maugin (1993) and Fletcher (1976). Introducing the definitions (Maugin, 1999)

$$\mathcal{P}_L^{th} := -(\rho_R \dot{\mathbf{x}}\mathbf{F} + \eta\beta_L) = -\mathbf{p}\cdot\mathbf{F} - \eta\beta = \mathbf{P}^{mech} - \eta\beta, \tag{22}$$

$$\mathbf{b}^{th} := b_{LK} = -(L\delta_{LK} + T_{Li}x_{i,K} - S_L\beta_K), \tag{23}$$

$$\mathbf{f}^{inh} := \frac{\partial L}{\partial \mathbf{X}} = \frac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\nabla_R \rho_R - \frac{\partial \Psi}{\partial \mathbf{X}}, \tag{24}$$

$$\mathcal{H} := \frac{1}{2}\rho_R \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + e, \quad \mathbf{Q} := \mathbf{T}\dot{\mathbf{x}} - \mathbf{S}\Theta, \tag{25}$$

for the *canonical momentum* of the present approach, the *canonical material stress* tensor (Eshelby stress tensor), the *material force* of true inhomogeneities, the *Hamiltonian density* (total energy density) and the *material energy flux*, respectively the equations (12), (13) and (20) can be written in vectorial form as follows

$$-Div\mathbf{b}^{th} + \frac{D\mathcal{P}^{th}}{Dt} = \mathbf{f}^{inh}, \qquad \frac{DH}{Dt} - \nabla_R \cdot \mathbf{Q} = 0, \tag{26}$$

$$\begin{aligned} Div(-\mathbf{b}^{th}\cdot\mathbf{X} \quad + \quad & \mathbf{Q}t + \mathbf{T}\cdot\mathbf{x} - \mathbf{S}\alpha) + \\ & \frac{D}{Dt}(-\mathcal{H}t + \mathcal{P}^{th}\cdot\mathbf{X} - \mathbf{p}\cdot\mathbf{x} - \eta\alpha) = \mathbf{f}^{inh}\cdot\mathbf{X}. \end{aligned} \tag{27}$$

We recall that the equation (27) holds for non-homogeneous but *linear* thermoelasticity. Under this restriction and in the framework of elasticity it can be compared with equation (4.89) of Maugin (1993). Assuming, as in standard continuum thermodynamics, that entropy and heat flux are related by the usual relation $\mathbf{S} = \mathbf{q}/\Theta$, equation (26b) will take the classical form of the energy conservation equation (Maugin and Berezovski, 2000)

$$\frac{DH}{Dt} - \nabla_R \cdot (\mathbf{T}\cdot\mathbf{v} - \mathbf{q}) = 0. \tag{28}$$

## 5 Conclusions

Summing up, we have deduced from the Lagrangian density (9) all field equations and balance laws for the theory of material inhomogeneous, finite deformable, thermoelastic conductors of heat. As a matter of fact, equations (11a) and (28) are the local balance equations of linear momentum and energy, respectively. These are all formally identical to those of the classical thermoelasticity of conductors (e.g., as recalled in (Maugin and Berezovski, 2000)). Only the equation of canonical momentum (26a) differs from the originally obtained by Epstein and Maugin (1995) in material thermoelasticity. But, abstractions being made of material inhomogeneities, it is the same as the one obtained by direct algebraic manipulations by Dascalu and Maugin (1995) in the dissipationless formulation of thermoelasticity. Indeed, the canonical momentum (22) is made of two parts, a strictly mechanical part – which is no other than the pull back, changed of sign, of the physical momentum – and a purely thermal part. In addition, the canonical stress tensor (23) contains a contribution of $\beta$ because, by its very definition, it captures material gradients of all fields. One should note that the source term in equation (26a) has no energetic contents. Furthermore, contrary to common use, even the entropy equation (11b) is source free. This means that in absence of material inhomogeneities, all equations obtained are strict *conservation laws*, hence the qualification of *dissipationless theory*. In this rather strange – we admit it – approach, the entropy flux and heat flux are derived from the free energy, on the same footing as entropy density, and stress.

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