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Changing the Observer and Moving Materials in Continuum Physics: Objectivity and Frame-Indifference

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Because one has to distinguish between changing the observer and changing the motion of a material with respect to a standard frame of reference, objectivity and material frame indifference are of different kind and have to be redefined. Besides objectivity and observer invariance of the balance and constitutive equations, the concept of standard frame dependence is introduced. The objectivity of time derivative operators is investigated with the result that the material time derivative is observer-independent. Although objective constitutive properties are standard frame dependent in general, representation theorems, and therefore the reduced forms of the constitutive equations remain still valid. A remark on non-objective force densities and inertial systems is made.

1 Introduction

There are a lot of publications on objectivity and frame-indifference (Noll, 1958; Truesdell and Noll, 1965; Truesdell, 1969; Bressan, 1972; Edelen, 1973; Muschik and Brunk, 1973; Bampi and Morro, 1980; Murdoch, 1982; Muschik, 1982; Müller and Ruggeri, 1993; Muschik, 1998; Svendsen and Bertram, 1999; Muschik, 1999; Bertram and Svendsen, 2001) [this is only a small selection of the huge flood of papers on this topic]. Although it is always assured that there are no remaining problems with respect to objectivity, the fact, that constitutive properties may depend on the motion of the material, is mostly out of scope. Reason for that is, that the conditions under which constitutive properties depend on motion are too "extreme", and that authors fear to lose the reduction of the constitutive equations.

In this paper the three different concepts of objectivity discussed by Bertram and Svendson (2001) are considered in detail. The first one is, that special quantities transform as an objective tensor under change of the observer. This means, that special observers "see" special tensor components of the same tensor representing the considered quantity. The second meaning of objectivity is equivalent to material frame-indifference which implies two statements: Constitutive mappings are observer-invariant, but depend on the motion of the material with respect to an arbitrary chosen frame of reference. The second statement of material frame-indifference is, that different uniform motions of the material with respect to the frame of reference do not influence material properties.

Balance equations representing physical laws are observer-invariant, that means, they are of the same shape for all observers, because no observer is distinguished. This generates tranformation properties for position and time derivatives. Especially non-objective force densities are discussed with respect to inertial frames. If the state variables and the constitutive properties are objective, the constitutive mapping is an isotropic function of the state variables, despite of its dependence on the material's motion.

2 Objectivity

2.1 Observers

An *observer* $\boldsymbol{B}$ at a fixed reference point $\boldsymbol{x}$, or synonymously a *frame* (of reference), is represented by a basis $\{\boldsymbol{e}^k\}$ (vector triad, or a tetrade in relativistic theories) or $\{\boldsymbol{e}_j\}$

$$\boldsymbol{e}_j \cdot \boldsymbol{e}^k = \delta_j^k, \qquad \boldsymbol{e}^k \boldsymbol{e}_k = \mathbf{1}, \tag{1}$$

and by a set of measuring devices for other physical quantities. The measuring devices generate tensor components of tensors of different order (e.g. of order 2)

$$A_{lk} \quad \longrightarrow \quad \mathcal{A} \ := \ \boldsymbol{e}^l A_{lk} \boldsymbol{e}^k. \tag{2}$$

Thus we can state, that the measuring devices of an observer are generating tensors of different order which belong to the physical properties of the objects at his point of reference.

2.2 Changing Observers

We denote an arbitrary but fixed observer $\boldsymbol{B^*}$ as *standard frame of reference* and another with respect to $\boldsymbol{B^*}$ arbitrarily moving frame $\boldsymbol{B}$ as the chosen observer. A change of frame in classical mechanics is described by a proper orthogonal time dependent rotation $(Q^j{}_l)(t) \equiv \mathbf{Q}(t)$ and by a time dependent translation $(c^j)(t) \equiv \boldsymbol{c}(t)$. Consequently the basis transforms

$$\boldsymbol{e}^{*j} \quad = \quad Q^j{}_l \boldsymbol{e}^l, \tag{3}$$

$$Q_i{}^l Q^m{}_l = \delta_i^m = Q^l{}_i Q_l{}^m \quad \longleftrightarrow \quad \mathbf{Q}\cdot\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}\cdot\mathbf{Q} = \mathbf{1}, \qquad \det\mathbf{Q} = 1. \tag{4}$$

The transformation of the position coordinates is given by

$$\boldsymbol{x}^* = \mathbf{Q}(t)\cdot\big(\boldsymbol{x} - \boldsymbol{c}(t)\big). \tag{5}$$

The components of the material velocity transform under changing the observer in the following way:

$$\boldsymbol{v}^* = \mathbf{Q}\cdot\boldsymbol{V} = \mathbf{Q}\cdot(\boldsymbol{v} + \boldsymbol{v}^{rel}) =: \mathbf{Q}\cdot\big(\boldsymbol{v} - \dot{\boldsymbol{c}} + \boldsymbol{\Omega}\cdot(\boldsymbol{x} - \boldsymbol{c})\big), \qquad \boldsymbol{\Omega} := \tilde{\mathbf{Q}}\cdot\dot{\mathbf{Q}}. \tag{6}$$

Here $\boldsymbol{v}^{rel}$ is the relative velocity between the two frames $\boldsymbol{B}^*$ and $\boldsymbol{B}$, decomposed into the translational part $-\dot{\boldsymbol{c}}$ and the rotational one $\boldsymbol{\Omega}\cdot(\boldsymbol{x}-\boldsymbol{c})$. $\boldsymbol{\Omega}$ is the skewsymmetric spin tensor of the relative motion between the two observers. We now start out with the

Axiom I: *Balance equations are observer-invariant.*

This axiom states, that balance equations are of the following same shape for arbitrary observers

$$\frac{\partial}{\partial t}(\rho\Psi) + \nabla\cdot(\rho\boldsymbol{v}\Psi + \boldsymbol{\Phi}) + \Sigma = 0. \tag{7}$$

If equation (7) is written down in a chosen frame $\boldsymbol{B}$, then we obtain in an arbitrary standard frame $\boldsymbol{B}^*$

$$\partial_t^*(\rho^*\Psi^*) + \nabla^*\cdot(\rho^*\boldsymbol{v}^*\Psi^* + \boldsymbol{\Phi^*}) + \Sigma^* = 0. \tag{8}$$

This axiom has a very clear interpretation: Because no observer is distinguished, they all "see" their balance equations in the same form. If two equations are of the same shape for two different observers, we call them *observer-invariant* (or covariant in relativistic theories). The axiom, that balance equations are observer-invariant is not a new one (Noll, 1959; Bertram, 1989). The advantage of this axiom is, that it uses only quantities which are measurable for the corresponding observers independently of their mutual motion to each other. An other advantage is, that the axiom can be generalized relativistically, the only possibility to formulate a covariant theory.

We now want to introduce the notion of observer dependence and observer independence.

Definition: If a quantity is unchanged $\varphi = \varphi^*$ under changing the observer, it is called *observer-independent*.

In non-relativistic physics we have the following examples of observer-independent quantities: time $t = t^*$, mass density $\rho = \rho^*$, temperature $T = T^*$.

Definition: A quantity is called *observer-dependent*, if it changes its value under change of the observer: $\varphi \neq \varphi^*$

Examples of observer-dependent quantities are: material velocity $\boldsymbol{v} \neq \boldsymbol{v}^*$, angular velocity $\boldsymbol{\omega} =$ $= \frac{1}{2}\left[\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^{\sim}\right] \neq \boldsymbol{\omega}^*$, force density $\boldsymbol{k} \neq \boldsymbol{k}^*$. In (7) and (8) the homologous quantities Ψ and Ψ^*, Φ

and Φ^*, Σ and Σ^* are measurable quantities, determined by measurements in different frames $\boldsymbol{B}$ and $\boldsymbol{B}^*$. Now the question arises, how these homologous quantities of different observers are related to each other. In general the quantities of different tensor order appearing in (7) and (8) transform to each other as follows:

Order	Transformation
0th	$\Psi^* = \Psi + \Psi^{rel}$
1st	$\boldsymbol{\Phi}^* = \mathbf{Q} \cdot (\boldsymbol{\Phi} + \boldsymbol{\Phi}^{rel})$
2nd	$\boldsymbol{A}^* = \mathbf{Q} \cdot (\boldsymbol{A} + \boldsymbol{A}^{rel}) \cdot \tilde{\boldsymbol{Q}}$
$\vdots$	$\vdots$

This kind of transformation corresponds to (6) for tensors of first order. In all the other cases it is analogous.

In special cases not all of the "relative" quantities $(\bullet)^{rel}$ are non-zero. Therefore we introduce the

Definition: Quantities are called *objective*, if their relative parts vanish for all observers

$$\Psi^{rel} \equiv 0, \quad \boldsymbol{\Phi}^{rel} \equiv \mathbf{0}, \quad \boldsymbol{A}^{rel} \equiv \mathbf{0}, \quad \ldots . \tag{9}$$

In this sense objective tensors are those, whose relative parts of their components vanish for all observers. If a quantity is not objective (like the material velocity (6)), its components transform differently, and we speak of non-objective components. In a standard frame of reference the relative parts of all quantities are identically defined to zero according to the table above.

2.3 Derivatives

In non-relativistic theory, we consider a function of arbitrary range

$$f(\boldsymbol{x}^*, t) = f(\mathbf{Q} \cdot (\boldsymbol{x} - \boldsymbol{c}), t), \tag{10}$$

taking (5) into account. Its gradient is according to (5)

$$\nabla f(\boldsymbol{x}^*, t) = \nabla^* f(\boldsymbol{x}^*, t) \cdot \mathbf{Q}. \tag{11}$$

Now we prove the following

Proposition: Under changing the observer the following transformation equation for the partial position derivative is true

$$\nabla = \tilde{\mathbf{Q}} \cdot \nabla^* = \nabla^* \cdot \mathbf{Q}, \tag{12}$$

and we obtain

$$\nabla^* \cdot \boldsymbol{v}^* = \nabla \cdot \boldsymbol{v}. \tag{13}$$

Proof: Because of

$$\nabla \cdot \boldsymbol{v}^{rel} = \nabla \cdot [-\dot{\boldsymbol{c}} + \boldsymbol{\Omega} \cdot (\boldsymbol{x} - \boldsymbol{c})] = \boldsymbol{\Omega} : \mathbf{1} = 0, \tag{14}$$

we have according to (6) and (14)

$$\nabla^* \cdot \boldsymbol{v}^* = \nabla^* \cdot \mathbf{Q} \cdot \boldsymbol{V} = \nabla \cdot \boldsymbol{V} = \nabla \cdot \left(\boldsymbol{v} + \boldsymbol{v}^{rel}\right) = \nabla \cdot \boldsymbol{v}. \tag{15}$$

Definition: There are two time derivatives belonging to the different frames $\boldsymbol{B}^*$ and $\boldsymbol{B}$

$$\text{in } \boldsymbol{B}^* \quad : \qquad \frac{\partial}{\partial t}\Big|_{\boldsymbol{x}^*} \equiv \frac{\partial}{\partial t}^{*} =: \partial_t^*, \tag{16}$$

$$\text{in } \boldsymbol{B} \quad : \qquad \frac{\partial}{\partial t}\Big|_{\boldsymbol{x}} \equiv \frac{\partial}{\partial t} =: \partial_t. \tag{17}$$

Now, we can prove the

Proposition: The transformation of the partial time derivative under change of the observer is

$$\partial_t^* = \partial_t - \boldsymbol{v}^{rel} \cdot \nabla =: \partial_t + \partial_t^{rel}, \tag{18}$$

for arbitrary quantities in its domain.

Proof: We obtain by the chain rule

$$\frac{\partial}{\partial t} f(\boldsymbol{x}^*, t) = \left[\frac{\partial^*}{\partial t} + \frac{\partial}{\partial t}\boldsymbol{x}^* \cdot \nabla^*\right] f(\boldsymbol{x}^*, t). \tag{19}$$

According to (5)

$$\frac{\partial}{\partial t}\boldsymbol{x}^* = \dot{\mathbf{Q}} \cdot (\boldsymbol{x} - \boldsymbol{c}) - \mathbf{Q} \cdot \dot{\boldsymbol{c}} = \mathbf{Q} \cdot \boldsymbol{\Omega} \cdot (\boldsymbol{x} - \boldsymbol{c}) - \mathbf{Q} \cdot \dot{\boldsymbol{c}} \tag{20}$$

is valid. By use of $(6)_2$ we obtain

$$\frac{\partial}{\partial t}\boldsymbol{x}^* \cdot \nabla^* = (\boldsymbol{x} - \boldsymbol{c}) \cdot \tilde{\boldsymbol{\Omega}} \cdot \tilde{\mathbf{Q}} \cdot \nabla^* - \dot{\boldsymbol{c}} \cdot \tilde{\mathbf{Q}} \cdot \nabla^* = \Big[(\boldsymbol{x} - \boldsymbol{c}) \cdot \tilde{\boldsymbol{\Omega}} - \dot{\boldsymbol{c}}\Big] \cdot \nabla = \boldsymbol{v}^{rel} \cdot \nabla \tag{21}$$

by which (19) results in the transformation equation (18) we looked for. According to (18) the partial time derivative is non-objective, because its relative part is not identical to zero

$$\frac{\partial}{\partial t}^{rel} \equiv -\boldsymbol{v}^{rel} \cdot \nabla. \tag{22}$$

Definition: The *material time derivative*

$$\frac{d}{dt} := \partial_t + \boldsymbol{v} \cdot \nabla \tag{23}$$

is observer invariantly defined

$$\frac{d^*}{dt} = \partial_t^* + \boldsymbol{v}^* \cdot \nabla^*. \tag{24}$$

Now we prove the

Proposition: The material time derivative is observer-independent

$$\frac{d^*}{dt} = \frac{d}{dt}. \tag{25}$$

Proof: By (18) we obtain from (24) by use of (18), (6), and (12)

$$\frac{d^*}{dt} = \partial_t - \boldsymbol{v}^{rel} \cdot \nabla + \boldsymbol{v}^* \cdot \nabla^* = \partial_t - \boldsymbol{v}^{rel} \cdot \nabla + (\boldsymbol{v} + \boldsymbol{v}^{rel}) \cdot \tilde{\mathbf{Q}} \cdot \mathbf{Q} \cdot \nabla = \frac{d}{dt}. \tag{26}$$

Example I: For components of a tensor $\boxplus$ of order one, there exists a special observer-invariantly defined time derivative of order two

$$\mathbf{D}_t^{(1)} \cdot \boxplus := [\mathbf{1}\frac{d}{dt} - \boldsymbol{\omega}] \cdot \boxplus. \tag{27}$$

Proposition: The operator $\mathbf{D}_t^{(1)}$ is objective

$$\tilde{\mathbf{Q}} \cdot \mathbf{D}_t^{(1)^*} \cdot \mathbf{Q} = \mathbf{D}_t^{(1)}. \tag{28}$$

Here (26) and the transformation law of the frame-invariantly defined material spin matrix

$$\omega^*(\boldsymbol{x}^*, t) = \mathbf{Q} \cdot [\omega(\boldsymbol{x}, t) + \boldsymbol{\Omega}] \cdot \tilde{\mathbf{Q}} \tag{29}$$

have been used.

Example II: For components of a tensor $\otimes$ of order two there exists a special observer-invariantly defined objective time derivative of order two

$$\mathbf{D}_t^{(2)} \cdot \otimes := \left[\mathbf{1} \frac{d}{dt} - \boldsymbol{\omega} \right] \cdot \otimes - \otimes \cdot \boldsymbol{\omega}. \tag{30}$$

Example III: For components of a tensor of arbitrary order there exists the following observer-invariantly defined objective time derivative of order two

$$\hat{\mathbf{D}}_t := \mathbf{1} \left[\frac{d}{dt} + \boldsymbol{a} \cdot \nabla \right] + \frac{1}{2} \boldsymbol{\Omega}, \qquad (\boldsymbol{a} \text{ objective}). \tag{31}$$

2.4 Non-objective Force Density

Non-objective quantities are not rare. According to (6) the material velocity $\boldsymbol{v}$ and the acceleration are non-objective, and therefore also the force density $\boldsymbol{f}$ in the balance of momentum

$$\frac{\partial}{\partial t} (\rho \boldsymbol{v}) + \nabla \cdot \left(\rho \boldsymbol{v} \boldsymbol{v} - \tilde{\boldsymbol{T}} \right) - \boldsymbol{f} = 0 \tag{32}$$

is not objective. This results from the fact, that balance equations are presupposed to be observer-invariant. Historically a special standard frame of reference was of great importance: the *inertial frame.* For defining this frame one needs a rule or an other possibility by which we can decompose $\boldsymbol{f}$ into the so-called imposed and accelerated (inertial) part

$$\boldsymbol{f} = \boldsymbol{f}^{imp} + \boldsymbol{f}^{acc}. \tag{33}$$

This decomposition cannot be achieved by measuring devices (therefore one needs a rule or a model by which this decomposition theoretically can be done). The transformation properties of $\boldsymbol{f}^{imp}$ and $\boldsymbol{f}^{acc}$ are different: $\boldsymbol{f}^{imp}$ is presupposed to be objective

$$\boldsymbol{f}^{*imp} = \mathbf{Q} \cdot \boldsymbol{f}^{imp}, \tag{34}$$

from which follows that

$$\boldsymbol{f}^{*acc} = \mathbf{Q} \cdot \left(\boldsymbol{f}^{acc} + \boldsymbol{f}^{rel} \right) \tag{35}$$

is not objective.

Accepting the split (33) achieved by a theoretical tool, we are able to introduce the

Definition: A frame $\boldsymbol{B}^*_{in}$ is denoted as an *inertial frame*, if

$$\boldsymbol{f}^{*acc}_{in} = 0. \tag{36}$$

From (35) and (36) we obtain in an arbitrary frame

$$\boldsymbol{f}^{rel} = -\boldsymbol{f}^{acc}. \tag{37}$$

Therefore (8) is often interpreted as a balance equation in an inertial frame, because in this case

$$\boldsymbol{f}^*_{in} = \boldsymbol{f}^{*imp}_{in} \tag{38}$$

is valid. The force density in an inertial frame is identical to its objective part according to (34).

Up to now we did not consider constitutive properties, because balance equations, gradients, and time derivatives are defined without respect to special materials. Constitutive properties are treated in the next section.

3 Material Frame Indifference

According to Bertram and Svendsen (2001) material frame indifference consists of three distinct concepts: *Euclidean frame indifference* (EFI) which states, if physical quantities transform objectively or not under change of the observer. Consequently all considerations in Sect. 2 deals with EFI. The second concept is called *form invariance* (FI) which states, that the form of constitutive equations does not depend on the observer. As we will see later, FI is more than the observer-invariance of the constitutive equations. The third concept *indifference with respect to superimposed rigid body motions* (IRBM) requires, that constitutive equations are independent of arbitrary rotations of the material body with respect to a single observer. There are examples, that IRBM is not generally valid, e.g. a material rotating with respect to an arbitrary standard frame of reference has an other heat and electrical conductivity than the same material resting with respect to the same standard frame (Müller, 1972). Consequently we will not use IRBM which is replaced by an other concept (SFD) later on. We now consider two frames: the arbitrary standard frame of reference $\boldsymbol{B}^*$ and an other observer $\boldsymbol{B}$ which is also arbitrary, but different from $\boldsymbol{B}^*$. We introduce in $\boldsymbol{B}$ an abstract state space spanned by a set of variables z (domain of the constitutive mapping $\mathcal{M}$) (Muschik, 1990). Additionally we have the constitutive equation $M(\boldsymbol{x},t)$ which describes in $\boldsymbol{B}$ the properties of the material under consideration and which is generated by the constitutive mapping $\mathcal{M}$ defined on z. Mathematically we have a double-fibre bundle consisting of z and M on $(\boldsymbol{x},t)$. As already mentioned IRBM is not generally valid, and therefore we need in the domain of $\mathcal{M}$ an other entrance (marked by $\square^*$) for the description of the motion of the material with respect to the chosen standard frame of reference $\boldsymbol{B}^*$

$$\text{in } \boldsymbol{B}: \quad M(\boldsymbol{x},t) = \mathcal{M}(z(\boldsymbol{x},t);\square^*). \tag{39}$$

The experimental fact, that materials in generally perceive their motion with respect to the chosen standard frame of reference $\boldsymbol{B^*}$, should be called *standard frame dependence* (SFD) of the constitutive mapping. This constitutive property is observer-independent, because it depends only on the motion of the material with respect to the standard frame of reference which is the same for all observers.

As discussed SFD is described in (39) by the set $\square^*$. This set can be specified as follows: We introduce a third frame $\boldsymbol{B^0}$, the so-called local co-rotational rest frame, which is fixed at a material point $\boldsymbol{X}$ of position $\boldsymbol{x}^0(t)$. Consequently we have

$$\text{in } \boldsymbol{B}^0: \quad \boldsymbol{v}^0\left(\boldsymbol{x}^0\left(t\right),t\right) = \mathbf{0}, \qquad \boldsymbol{\omega}^0\left(\boldsymbol{x}^0\left(t\right),t\right) = \mathbf{0}, \qquad \mathbf{Q}(t) = \mathbf{Q}^0(t). \tag{40}$$

Here $\mathbf{Q}^0(t)$ characterizes the orthogonal transformation from the co-rotational rest frame to the standard frame of reference $\boldsymbol{B}^*$. This rotation of the co-rotational rest frame with respect to the standard frame of reference is described by the material spin matrix $\mathbf{\Omega}^0$.

The local motion of the material at $\boldsymbol{x}^0(t)$ is now described by a functional of the material's motion described locally by $\mathbf{Q}^0(t)$ and $\boldsymbol{c}^0(t)$

$$\mathcal{F}[\mathbf{Q}^0(t),\boldsymbol{c}^0(t)] \equiv \square^*. \tag{41}$$

A *constitutive family* is defined by (39) and (41). If the functional $\mathcal{F}$ in (41) is of differential type, we obtain

$${}^d\square^* = F[\mathbf{Q}^0,\boldsymbol{c}^0,\mathbf{\Omega}^0,\dot{\boldsymbol{c}}^0,\dot{\mathbf{\Omega}}^0,\ddot{\boldsymbol{c}}^0,...]. \tag{42}$$

Now the question arises, what happens to the constitutive family, if the observer is changed. Because no special observer is distinguished, we formulate the

Axiom II: *Constitutive equations are observer-invariant.*

According to (39) and axiom II (which is less than FI) the constitutive equations in $\boldsymbol{B}^*$ are

$$\text{in } \boldsymbol{B}^* : \quad M^*(\boldsymbol{x}^*, t) = \mathcal{M}^*(z^*(\boldsymbol{x}^*, t); \Box^*), \tag{43}$$

and in an other frame

$$\text{in } \boldsymbol{B}^\diamond : \quad M^\diamond(\boldsymbol{x}^\diamond, t) = \mathcal{M}^\diamond(z^\diamond(\boldsymbol{x}^\diamond, t); \Box^*). \tag{44}$$

$\Box^*$ is not influenced by changing the observer, because according to SFD the motion of the material with respect to the frame of reference $\boldsymbol{B}^*$ is independent of changing the observer $\boldsymbol{B}^* \to \boldsymbol{B}^\diamond$.

Comparing (39), (43), and (44) it seems, as if to each observer an own constitutive family belongs (different constitutive mappings $\mathcal{M}$, $\mathcal{M}^*$, and $\mathcal{M}^\diamond$), although the different observers "see" the same material. If this would really be true, we cannot define what identical materials are. Therefore we formulate the

Axiom III: The *principle of material frame indifference* states that
i) *constitutive mappings are observer-independent* (that is FI)

$$\mathcal{M}(\bullet) = \mathcal{M}^*(\bullet) = \mathcal{M}^\diamond(\bullet), \tag{45}$$

and
ii) *uniform motions of the material with respect to the standard frame of reference do not influence constitutive properties*, so that (42) results in

$${}^d\Box^* = F[\boldsymbol{\Omega}^0, \dot{\boldsymbol{\Omega}}^0, \ddot{\boldsymbol{c}}^0, ...]. \tag{46}$$

After having taken into account axiom III the constitutive family is now called *class of materials*. A special class of materials are the acceleration-insensitive ones for which we have

$$\text{in } \boldsymbol{B} : \quad M(\boldsymbol{x}, t) = \mathcal{M}(z(\boldsymbol{x}, t)). \tag{47}$$

For this class IRBM is valid. We now prove the following

Proposition: If the state variables z and the constitutive properties M are objective quantities, the constitutive map $\mathcal{M}$ is isotropic in the state variables.

Proof: We introduce the abstract mapping $\mathcal{T}$ describing the change of observer $\boldsymbol{B} \to \boldsymbol{B}^*$. The constitutive equations M and the state space variables z transform according to the table as follows

$$M^* = \mathcal{T}(M + M^{rel}) = \mathcal{M}(\mathcal{T}(z + z^{rel}); \Box^*). \tag{48}$$

Here $\mathcal{T}(z + z^{rel}) = z^*$ is the transformation of the set of state space variables. From (48) and (39) we obtain

$$(M + M^{rel}) = \mathcal{T}^{-1}\mathcal{M}(\mathcal{T}(z + z^{rel}); \Box^*) = \mathcal{M}(z; \Box^*) + M^{rel}. \tag{49}$$

If especially the constitutive equations and state variables are objective

$$M^{rel} \equiv 0, \qquad z^{rel} \equiv 0, \tag{50}$$

we obtain from (49)

$$M = \mathcal{T}^{-1}\mathcal{M}(\mathcal{T}(z); \Box^*) = \mathcal{M}(z; \Box^*). \tag{51}$$

This results in

$$\mathcal{M}(\bullet; \Box^*) = \mathcal{T}^{-1}\mathcal{M}(\mathcal{T}(\bullet); \Box^*). \tag{52}$$

Consequently $\mathcal{M}$ is an isotropic function of the state space variables, but depending on the motion of the material according to SFD. All representation theorems of isotropic functions remain valid for a class of materials. According to (52) a constitutive equation M (e.g. the heat flux density) may be objective, but standard frame-dependent.

4 Summary

Starting out with physical interpretations for three axioms

I: Balance equations are observer-invariant,

II: Constitutive equations are observer-invariant,

IIIa: Constitutive mappings are observer-independent,

IIIb: Uniform motions of the material with respect to the standard frame of reference do not influence constitutive properties,

we demonstrate, that in general constitutive equations depend on the motion of the material with respect to a chosen frame of reference. Despite this dependence of motion, the constitutive equations are isotropic functions on the state space. Consequently the usual reduction of the constitutive equations to normal forms is also possible in this more general case.

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