

# A Lagrangian Approach to Electromagnetic Bodies <sup>1</sup>

C. Trimarco

*A Lagrangian formulation for a deformable and moving electromagnetic body is here proposed in the framework of the Galilean approximation. As the proper choice of the independent electromagnetic fields for such a Lagrangian seems to be of basic importance, the first concern is with this choice. In this respect, a preliminary transformation of the electromagnetic fields in the referential frame of the solid body is suitably introduced. The resulting fields, which combine with the motion and with the deformation, enter into the proposed Lagrangian. Eventually, one shows that the related Lagrange equations provide both, the generalised equations of mechanics and the Maxwell equations, in the material form. Interesting quantities, such as canonical momenta and stresses, stem naturally from the adopted procedure and are commented hereby. These quantities are very helpful in describing the behaviour of defective materials.*

## 1 Introduction

A Lagrangian description for electromagnetic fields in vacuum, such as found in several textbooks, is based on a specific expression, from which the Maxwell equations stem as Lagrange equations (Becker, 1964; Jackson, 1962). In the presence of matter, one can invoke Lorentz and introduce two additional quantities: the polarisation  $\mathbf{P}$  and the magnetisation  $\mathbf{M}$ . These two fields typically pertain to the material body and are specified either by macroscopic constitutive equations or through a microscopic atomic model. Here, they are viewed as primary fields that identically vanish outside the region occupied by the body (Becker, 1964; Jackson, 1962; Toupin, 1956; Trimarco, 1994). Based on these quantities, one can assemble a Lagrangian density for a deformable body at rest as a result of different contributions. Specifically, the resulting Lagrangian can be written as the superposition of essentially two terms: the Lagrangian of the ‘pure’ fields  $\mathbf{E}$  and  $\mathbf{B}$ , such as in a vacuum, with a term that is due to the presence of the material. The contribution of the Lagrangian due to the presence of the material splits, in turn, in two additional terms. A first one accounts for the interaction of the polarisation  $\mathbf{P}$  with the electric field  $\mathbf{E}$  at the material point and of the magnetic induction  $\mathbf{B}$  with the magnetisation  $\mathbf{M}$ ; a second one, which only includes fields that are identically vanishing outside the body, accounts for the material response (Eringen and Maugin, 1990; Maugin, 1993; Mindlin, 1972; Toupin, 1956; Trimarco, 1994; Trimarco and Maugin, 2001). It is worth remarking that such a Lagrangian reduces to the electric enthalpy in a dielectric material (apart from an unimportant minus sign), if magnetic effects are disregarded (Mindlin, 1972; Toupin, 1956; Trimarco, 1994).

The case of a deformable and moving body needs to be treated with due care. In fact, the Maxwell equations do not preserve their form in moving bodies. As is known, Maxwell equations are invariant in form with respect to the Lorentz transformations. Therefore, the natural framework for these equations seems to be the 4-dimensional Minkowski space, in which the Lorentz transformations are naturally defined (Becker, 1964; Jackson, 1962; Penfield and Haus, 1967; Sommerfeld, 1952). However, there are cases in which the behaviour of an electromagnetic body is better understood in the three-dimensional physical space. This is the case for univocally defined macroscopic quantities such as stresses.

Having this in mind, we first note that in finite deformations a reference configuration has to be introduced. A corresponding energy density for the material can also be introduced. Thus, it may be convenient to express the possible Lagrangian as a density per unit volume of the reference configuration. The additional proposal is that all spatial fields should be transformed in the material referential frame in such a way that the Maxwell equations are preserved in form in this frame. By assuming that the Lagrangian depends on these transformed fields, its density per unit volume of the reference configuration is consistently introduced. This being assumed, one derives the complete set of the Lagrange equations, which include the Maxwell-Lorentz equations in material form and the constitutive relationships as well. Of course, all equations are coupled to one another as the electromagnetic fields also depend on the deformation and on the motion. In this context, one of the Lagrange equations can be interpreted as the balance of the physical momentum, as it stands as a generalisation of the Cauchy equation for the mechanical momentum.

The Lagrangian approach also drives the attention to the canonical momenta that naturally stem from the

---

<sup>1</sup> Contribution to the sixth conference “Current Ideas in Mechanics, Thermodynamics, and Related Fields (CIMRF 2001)”, Sept. 3-6, 2001, TU Berlin

proposed treatment. These momenta are usually disregarded in the classical description, but can be of interest for introducing additional quantities, which are physically meaningful: the material (or configurational) forces and the material momentum (Maugin and Trimarco, 2001; Trimarco and Maugin, 2001). The latter turns out to correspond to the crystal momentum in solids (Nelson, 1979; Peierls, 1985). These quantities play a fundamental role in the description of defective materials (Maugin, 1993; Maugin and Trimarco, 2001; Trimarco and Maugin, 2001).

## 2 Preliminary Recalls

The Maxwell-Lorentz equations consist of the Maxwell-Faraday equations, which are supplemented by additional relationships. In a frame at rest and in S.I. units they are (Trimarco and Maugin, 2001)

$$\begin{aligned}
\operatorname{div} \mathbf{D} &= \rho_e \\
\operatorname{div} \mathbf{B} &= 0 \\
\operatorname{curl} \mathbf{E} &= -\partial \mathbf{B} / \partial t \\
\operatorname{curl} \mathbf{H} &= \mathbf{j}_e + \partial \mathbf{D} / \partial t
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\mathbf{D} &= \epsilon_0 \mathbf{E} && \text{in } E_3 - \mathcal{V} \\
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} && \text{in } \mathcal{V} \\
\mathbf{H} &= (\mu_0)^{-1} \mathbf{B} && \text{in } E_3 - \mathcal{V} \\
\mathbf{H} &= (\mu_0)^{-1} \mathbf{B} - \mathbf{M} && \text{in } \mathcal{V}
\end{aligned}$$

$\rho_e$  and  $\mathbf{j}_e$  are the free charge density and the free current density, respectively.  $E_3$  represents the Euclidean physical space,  $\mathcal{V}$  the region occupied by the body. All fields and physical constants are such as currently introduced in textbooks (Becker, 1964; Jackson, 1962; Sommerfeld, 1952). Boundary conditions and initial conditions depend on the specific problem.

Within this framework, the fundamental fields are  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{P}$  and  $\mathbf{M}$ , whereas the fields  $\mathbf{D}$  and  $\mathbf{H}$  are conceived as a combination of the former ones.

A further step could be that of reducing the number of the fundamental fields. By taking into account the equations (2.1)<sub>2</sub> and (2.1)<sub>3</sub>, one can introduce the fields  $\Phi$  and  $\mathbf{A}$  instead of  $\mathbf{E}$  and  $\mathbf{B}$

$$\begin{aligned}
\operatorname{curl} \mathbf{A} &= \mathbf{B}, \\
(\mathbf{E} + \partial \mathbf{A} / \partial t) &= -\nabla \Phi.
\end{aligned} \tag{2.2}$$

As the two equations (2.2) do not specify univocally the fields  $\mathbf{A}$  and  $\Phi$ , additional conditions are needed. The gauge conditions solve this indeterminacy (Jackson, 1962; Stratton, 1941; Trimarco and Maugin, 2001).

A possible Lagrangian approach can be based on the following expression

$$L_c = \frac{1}{2} [\epsilon_0 E^2 - (\mu_0)^{-1} B^2] + \mathbf{E} \cdot \mathbf{P} + \mathbf{B} \cdot \mathbf{M} - w, \tag{2.3}$$

in which the dependence of  $L_c$  on  $\mathbf{E}$  and  $\mathbf{B}$  is to be understood through  $\Phi$  and  $\mathbf{A}$ .

The name of Lagrangian for  $L_c$  is justified *a posteriori*, in the sense that the two remaining Maxwell equations (2.1)<sub>1</sub> and (2.1)<sub>4</sub> can be identified with the Lagrange equations such as derived from (2.3), in the absence of charges and currents. The term  $w$  represents the response of the material. As mentioned, should the magnetic phenomena be disregarded  $L_c$  would reduce to the *electric enthalpy* (Mindlin, 1972; Toupin, 1956).

## 3 In Moving Frames

For moving bodies, one could be tempted to re-adapt the expression (2.3) by judiciously introducing the following co-moving fields (Becker, 1964; Eringen and Maugin, 1990; Jackson, 1962; Maugin, 1993; Nelson, 1979; Penfield and Haus, 1967; Schoeller and Thellung, 1992; Sommerfeld, 1952; Stratton, 1941)

$$\mathcal{E} = \mathbf{E} + \mathbf{v} \wedge \mathbf{B} \quad \text{and} \quad \mathcal{M} = \mathbf{M} + \mathbf{v} \wedge \mathbf{P}, \tag{3.1}$$

which are the fields that interact with the material. Eventually one can write

$$L_c = \frac{1}{2} [\epsilon_0 E^2 - (\mu_0)^{-1} B^2] + \mathcal{E} \cdot \mathbf{P} + \mathcal{M} \cdot \mathbf{B} + \frac{1}{2} \rho \mathbf{v}^2 - w. \tag{3.2}$$

$\rho$  and  $\mathbf{v}$  are the material density and the physical velocity, respectively. The term  $w$  represents the energy due to the material response.

Unfortunately, this extension is not enough for a consistent description of the deformable and moving body such as presented in the previous section. One of the difficulties is that the Maxwell equations in moving frames differ from the classical Maxwell equation. In the Galilean approximation, they are

$$\begin{aligned} \operatorname{div} \mathbf{D} &= \rho_e \\ \operatorname{div} \mathbf{B} &= 0 \\ \operatorname{curl} \mathcal{E} + (\partial \mathbf{B} / \partial t)_{\text{conv}} &= 0 \\ \operatorname{curl} \mathcal{H} - (\partial \mathbf{D} / \partial t)_{\text{conv}} &= \mathbf{j}_e + \rho_e \mathbf{v}, \end{aligned} \quad (3.3)$$

where the time convected derivative  $(\partial \mathbf{a} / \partial t)_{\text{conv}}$  of a field  $\mathbf{a}$ , accounts for the time-derivative of its flux across a moving surface (Becker, 1964; Trimarco and Maugin, 2001)

$$(\partial \mathbf{a} / \partial t)_{\text{conv}} = (\partial \mathbf{a} / \partial t) + \operatorname{curl} (\mathbf{a} \wedge \mathbf{v}) + \mathbf{v} \operatorname{div} \mathbf{a}. \quad (3.4)$$

Although the set of equations (3.3) slightly differs from the set of equations (2.1), they are consistent with the classical Maxwell equations written in the global form.

However, care is needed in the constitutive relationships, even in the trivial case of a vacuum. In fact, the set of relationships (2.2) is not consistent with the set of equations (3.3). The consistency can be recovered by introducing the following additional fields and relationships

$$\mathcal{D} = \epsilon_0 \mathcal{E}, \quad \mathcal{H} = (\mu_0)^{-1} \mathcal{B}. \quad (3.5)$$

$\mathcal{D}$  and  $\mathcal{B}$  are defined as follows:

$$\mathcal{D} = \mathbf{D} + (1/c^2) \mathbf{v} \wedge \mathbf{H}; \quad \mathcal{B} = \mathbf{B} - (1/c^2) \mathbf{v} \wedge \mathbf{E}, \quad (3.6)$$

where  $(1/c^2) \equiv \epsilon_0 \mu_0$ .  $\mathcal{D}$  and  $\mathcal{B}$  turn out to represent the electric and the magnetic induction, respectively, in the moving frame. In addition, the relations (3.6), along with the relations (3.1), represent the full Lorentz transformations for the fields of interest, in the first order approximation for low velocities with respect to that of light.

This being remarked, difficulties may arise in developing the variational procedure, such as proposed in the previous section, basing on the expression (3.2). Nonetheless, the expression (3.2) will be useful for proposing a possible variational approach in the 3-dimensional space, as we shall see in the subsequent sections.

#### 4 The Material Transformations

The expression (3.2) can still be thought as the proper candidate to represent a Lagrangian. In this respect, one regards  $L_c$  as a density per unit volume of the current configuration of a body  $\mathcal{V}$ , as the electromagnetic fields are naturally defined in  $\mathcal{V}$ . For bodies that suffer finite deformations, a reference configuration  $V$  is also introduced, with respect to which the deformation gradient  $\mathbf{F} = \nabla_{\mathbf{R}} \chi$  is measured.  $\chi: (\mathbf{X}, t) \rightarrow \mathbf{x}; \mathbf{X} \in V, \mathbf{x} \in \mathcal{V}$ .  $V, \mathcal{V} \subset E_3 \equiv \{\text{Euclidean space}\}$ .  $\det \mathbf{F} \equiv J > 0$ . As the energy density  $w$  depends on  $\mathbf{F}$ , among the other possible fields, it seems a natural choice to introduce an energy density  $W$  per unit volume of the reference configuration. If the deformation is regular enough, the classical relationship  $W = J w$  holds true. These remarks suggest to transfer the spatial Lagrangian  $L_c$  in a referential frame and write

$$L = J L_c. \quad (4.1)$$

In addition,  $L$  is required to depend on fields, which have been properly transformed in the referential frame. The suggested transformation rules are the following

$$\begin{aligned} \mathcal{E} &= \mathbf{F}^T \mathbf{E}; & \mathcal{E}^* &= \mathbf{F}^T \mathbf{E} \equiv \mathcal{E} + \mathbf{V} \wedge \mathcal{B}; \\ \mathcal{D} &= \mathbf{J} \mathbf{F}^{-1} \mathbf{D}; & \mathcal{B} &= \mathbf{J} \mathbf{F}^{-1} \mathbf{B}; \\ \mathcal{H} &= \mathbf{J} \mathbf{F}^{-1} \mathbf{H}; & \mathcal{H} &= \mathbf{F}^T \mathcal{H} \equiv \mathbf{F}^T \mathbf{H} + \mathbf{V} \wedge \mathcal{D}; \\ \mathcal{M} &= \mathbf{F}^T \mathbf{M}; & \mathcal{M}^* &= \mathbf{F}^T \mathcal{M} \equiv \mathcal{M} - (\mathbf{V} \wedge \mathcal{B}), \end{aligned} \quad (4.2)$$

$\mathbf{V} = -\mathbf{F}^{-1} \mathbf{v}$ , where  $\mathbf{v}$  is the physical velocity. Basing on these transformed fields, the Maxwell equations turn out to preserve their form in the reference frame and read

$$\begin{aligned} \operatorname{div}_R \mathcal{D} &= J\rho_e \\ \operatorname{div}_R \mathcal{B} &= 0 \\ \operatorname{curl}_R \mathcal{E} + (\partial \mathcal{B} / \partial t)|_X &= 0 \\ \operatorname{curl}_R \mathcal{H} - (\partial \mathcal{D} / \partial t)|_X &= \mathbf{j}_e. \end{aligned} \quad (4.3)$$

The electromagnetic potentials  $\phi$  and  $\mathbf{A}$  can now be introduced as in the classical treatment, based on the equations (4.3)<sub>2</sub> and (4.3)<sub>3</sub>. These potentials satisfy the following equations

$$\operatorname{curl}_R \mathbf{A} = \mathcal{B} \quad \text{and} \quad \mathcal{E} = -\dot{\phi} - \nabla_R \mathbf{A} \quad (4.4)$$

where  $\dot{\phi} \equiv (\partial \phi / \partial t)|_X$ . Details are found in (Maugin and Trimarco, 2001; Nelson, 1979; Trimarco and Maugin, 2001).

## 5 The Material Lagrangian

In terms of the *material* fields  $\phi$ ,  $\mathbf{A}$ ,  $\mathcal{P}$ ,  $\mathcal{M}^*$  and  $\mathbf{x}$ , such as introduced in the previous section, the expression (3.2) transforms as follows (Trimarco and Maugin, 2001)

$$\begin{aligned} L = & \frac{1}{2} \{ \epsilon_0 J \mathbf{C}^i \mathcal{E}^* - (\mu_0 J)^{-1} \mathcal{B} \cdot \mathbf{C} \mathcal{B} \} + \mathcal{P} \cdot \mathcal{E} + \mathcal{M}^* \cdot \mathcal{B} + \\ & + \frac{1}{2} \rho_0 v^2 - W(\mathbf{F}, \mathbf{F} \mathcal{P}, \mathbf{J} \mathbf{F}^{-1T} \mathcal{M}^*, \mathbf{X}), \end{aligned} \quad (5.1)$$

where  $\mathbf{C} \equiv \mathbf{F}^T \mathbf{F}$ .

The quantities in the expression (5.1) are per unit volume of the reference configuration and represent a specific form of the following more general Lagrangian  $L$

$$L(\phi, \dot{\phi}, \nabla_R \phi, \mathbf{A}, \dot{\mathbf{A}}, \nabla_R \mathbf{A}, \mathcal{P}, \dot{\mathcal{P}}, \nabla_R \mathcal{P}, \mathcal{M}^*, \dot{\mathcal{M}}^*, \nabla_R \mathcal{M}^*, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \mathbf{X}). \quad (5.2)$$

Accordingly, the related Lagrange equations are

$$\begin{aligned} (\partial / \partial t)|_X [\partial L / \partial \dot{\phi}] - \partial L / \partial \phi + \operatorname{div}_R [\partial L / \partial \nabla_R \phi] &= 0 \\ (\partial / \partial t)|_X [\partial L / \partial \dot{\mathbf{A}}] - \partial L / \partial \mathbf{A} + \operatorname{div}_R [\partial L / \partial \nabla_R \mathbf{A}] &= 0 \\ (\partial / \partial t)|_X [\partial L / \partial \dot{\mathcal{P}}] - \partial L / \partial \mathcal{P} + \operatorname{div}_R [\partial L / \partial \nabla_R \mathcal{P}] &= 0 \\ (\partial / \partial t)|_X [\partial L / \partial \dot{\mathcal{M}}^*] - \partial L / \partial \mathcal{M}^* + \operatorname{div}_R [\partial L / \partial \nabla_R \mathcal{M}^*] &= 0 \\ (\partial / \partial t)|_X [\partial L / \partial \dot{\mathbf{x}}] - \partial L / \partial \mathbf{x} + \operatorname{div}_R [\partial L / \partial \mathbf{F}] &= 0. \end{aligned} \quad (5.3)$$

As the dependence on  $\dot{\mathcal{P}}, \nabla_R \mathcal{P}, \dot{\mathcal{M}}^*, \nabla_R \mathcal{M}^*$  is disregarded in the expression of interest (5.1), equations (5.3)<sub>3</sub> and (5.3)<sub>4</sub> turn out to play the role of the constitutive equations for a large class of classical materials. The equations (5.3)<sub>1</sub> and (5.3)<sub>2</sub> correspond to the equations (3.3)<sub>1</sub> and (3.3)<sub>4</sub>. The details are omitted here and can be found in references (Maugin and Trimarco, 2001; Trimarco, 1994). We only remark that the relationship (2.2)<sub>2</sub> slightly changes in the present description, as one of the results is the following:

$$\mathcal{D} = \epsilon_0 \mathbf{C}^i \mathcal{E}^* + \mathcal{P}. \quad (5.4)$$

Notice that the inverse deformation tensor  $\mathbf{C}^{-1}$  plays the role of a metric tensor in the formula (4.4). In the presence of a material, this tensor unexpectedly affects a relationship, such as (2.2)<sub>1</sub>, that typically pertain to a vacuum. Of course, it is tacitly understood that the expression (4.4) reduces to (2.2)<sub>1</sub> in the absence of a material, as a notion such as deformation of a vacuum is meaningless in this context. A similar remark is concerned with the field  $\mathcal{H}$  (Trimarco and Maugin, 2001).

## 6 Stresses and Momenta

The equation (5.3)<sub>5</sub> can be interpreted as a generalisation of the mechanical balance of momentum. Thus, the physical momentum is identified with the quantity  $(\partial L / \partial \mathbf{x}^*)$ , which is a density per unit volume  $V$ . Accordingly, the quantity  $[-(\partial L / \partial \mathbf{F})]$  is identified with a Piola-Kirchhoff (like) stress (Eringen and Maugin, 1990; Maugin, 1993; Nelson, 1979; Toupin, 1956; Trimarco and Maugin, 2001; Truesdell and Toupin, 1960). Hence, the actual electromagnetic stress is a Cauchy-like stress that is given by

$$\begin{aligned} \mathbf{t}_M^{\text{em}} = & -J^{-1} (\partial L / \partial \mathbf{F}) \mathbf{F}^T = -1/2 [\varepsilon_0 \mathbf{E}^2 + (\mu_0^{-1}) \mathbf{B}^2 + 2(\mathbf{M} + \mathbf{v} \wedge \mathbf{P}) \cdot \mathbf{B}] \mathbf{I} + \\ & + [\varepsilon_0 \mathbf{E} \otimes \mathbf{E} + (\mu_0^{-1}) \mathbf{B} \otimes \mathbf{B}] - \varepsilon_0 \mathbf{E} \wedge \mathbf{B} \otimes \mathbf{v} + J^{-1} (\partial W / \partial \mathbf{F}) \mathbf{F}^T + (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \otimes \mathbf{P} - (\mathbf{M} + \mathbf{v} \wedge \mathbf{P}) \otimes \mathbf{B}. \end{aligned} \quad (6.1)$$

The corresponding physical momentum is

$$\mathbf{p} = J^{-1} (\partial L / \partial \mathbf{x}^*) = J^{-1} (\partial L / \partial \mathbf{v}) = \rho \mathbf{v} + \varepsilon_0 \mathbf{E} \wedge \mathbf{B}. \quad (6.2)$$

It is worth noting that the quantity  $(\partial L / \partial \mathbf{x}^*)$  also represents one of the canonical momenta that are introduced by the Lagrangian (6.1) or (6.2). With specific reference to the expression (6.1), a second canonical momentum appears, namely the quantity  $(\partial L / \partial \mathbf{A}^*)$ . Along with these canonical momenta one can envisage other canonical quantities, which are the argument of the divergence terms in the set of equations (6.3). A proper combination of these quantities can be shown to play a role in the theory of material inhomogeneities. In fact, they provide the form for the *material* momentum and of the *material* or *configurational* stress. Here below we only report the final formulas for these two quantities, which are discussed in (Trimarco and Maugin, 2001).

The configurational stress, or material stress, or also Eshelby stress is governed by the following formula:

$$\mathbf{b} = -L \mathbf{I} + \mathbf{F}^T (\partial L / \partial \mathbf{F}) + (\nabla_{\mathbf{R}} \phi) \otimes (\partial L / \partial \nabla_{\mathbf{R}} \phi) + (\nabla_{\mathbf{R}} \mathbf{A})^T (\partial L / \partial \nabla_{\mathbf{R}} \mathbf{A}), \quad (6.3)$$

The corresponding material momentum is settled through a similar procedure and is expressed by the following formula:

$$\mathbf{p} = -\mathbf{F}^T (\partial L / \partial \mathbf{v}) - [(\nabla_{\mathbf{R}} \mathbf{A})]^T (\partial L / \partial \mathbf{A}^*). \quad (6.4)$$

In the present case, this formula leads to the following result:

$$\mathbf{p} = \rho \mathbf{C} \mathbf{V} + \mathbf{P} \wedge \mathbf{B}. \quad (6.5)$$

## 7 Final Comment

The Lagrangian approach not only provides in a natural way the stress and the momentum, whose form is otherwise controversial. It also naturally introduces canonical momenta and stresses. These, in turn, address to the notion of *material* stress and *material* momentum. These quantities account for the behaviour of inhomogeneous materials. In fact, they enter a material balance law, which extends the Eshelby balance law for *material forces* in defective elastic materials (Maugin, 1993; Maugin and Trimarco, 2001; Trimarco and Maugin, 2001). Isolated or distributed defects in a material can be regarded as inhomogeneities of the material. In this view, one can say that the inhomogeneous material response generates a *material force*. This *force* does not explicitly appear in the classical balance equations. It does appear explicitly if one appeals to a non-classical variational procedure. In this procedure  $\mathbf{x}$  is conceived as the 'domain variable', whereas  $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$  (i.e. the inverse motion) represents the varying field, along with the electromagnetic fields. Hence, the explicit dependence of the Lagrangian on  $\mathbf{X}$ , which expresses the presence of the inhomogeneity, affects the proposed non-classical variation, contrary to the case of the classical variational approach. The notion of *material* momentum and *material* stress, along with the related balance law, stem straightforwardly in this non-classical framework. Details on this novel variational procedure can be found in the references (Maugin, 1993; Maugin and Trimarco, 2001; Trimarco and Maugin, 2001). It is worth remarking that the *material* momentum is also known as *quasi-momentum*, *crystal momentum* or *pseudomomentum* in the physics literature. In this context, the term  $\mathbf{P} \wedge \mathbf{B}$ , in the formula (6.5), is candidate to represent the pseudomomentum of light in a dielectric material. The discussion of this topic, though very appealing, is beyond the aim of the present paper. The interested reader is referred to the proper literature (Nelson, 1979; Peierls, 1985; Schoeller and Thellung, 1992).

## Literature

1. Becker, R.: Electromagnetic Fields and Interactions. v.1, Blackie & Son, London, (1964).
2. Eringen, A.C.; Maugin, G.A.: Electrodynamics of Continua. v.1-2, Springer-Verlag, N.Y., (1990).
3. Jackson, J.D.: Classical Electrodynamics. J. Wiley & Sons, N.Y., (1962).
4. Maxwell, J.C.: A Treatise on Electricity and Magnetism. v.1-2, (1891), Oxford Classic Text in the physical Sciences, Clarendon Press, Oxford, (1998, reprint).
5. Maugin, G.A.: Material Inhomogeneities in Elasticity. Chapman and Hall, London, v.3, Series 'Applied Mathematics and Mathematical Computation', (1993).
6. Maugin, G.; Trimarco C.: Elements of Field Theory in Inhomogeneous and Defective Materials. CISM courses and lectures n. 427 on 'Configurational Mechanics of Materials', Springer-Verlag, Wien, N.Y., (2001), 55-128.
7. Mindlin, R.D.: Elasticity, Piezoelectricity and Crystal Lattice Dynamics. J. of Elasticity, 2, (1972), 17-282.
8. Nelson, D.F.: Electric, Optic and Acoustic Interactions in Dielectrics. John Wiley, N.Y., (1979).
9. Peierls, R.: Momentum and Pseudomomentum of Light and Sound. Highlights of Condensed-Matter Physics, Ed. M.Tosi, LXXXIX, S.I.F. Bologna, (1985), 237-255.
10. Penfield, P.; Haus, H.A.: Electrodynamics of Moving Media. M.I.T. Press, Cambridge Mass., (1967).
11. Schoeller, H.; Thellung, A.: Lagrangian Formalism and Conservation Law for Electrodynamics in Nonlinear Elastic Dielectrics. Annals of Physics, 220, (1992), 18-39.
12. Sommerfeld, A.: Electrodynamics, Lectures on Theoretical Physics. v.3, Academic Press, N.Y., (1952).
13. Stratton, J.A.: Electromagnetic Theory. McGraw-Hill, N.Y., (1941).
14. Toupin, R.A.: The Elastic Dielectrics. J. Rational Mech. and Anal., 5, (1956), 849-915.
15. Trimarco, C.: The Toupin-Mindlin Theory of Dielectrics in The Light of View of Mossotti's Idea. Bull. of Polish Academy of Sciences, 42-3, (1994), 429-438.
16. Trimarco, C.; Maugin, G.: Material Mechanics of Electromagnetic Bodies. CISM courses and lectures n. 427 on 'Configurational Mechanics of Materials', Springer-Verlag, Wien, N.Y., (2001), 129-172.
17. Truesdell, C.A.; Toupin, R.A.: The Classical Theory of Fields. Handbuch der Physik, Bd.III/1, ed. S. Flügge, Springer-Verlag, Berlin, (1960).

---

*Address:* Carmine Trimarco, University of Pisa, Dipartimento di Matematica Applicata U. Dini, Via Bonanno 25/B, I-56122 Pisa, Italy, e-mail: trimarco@dma.unipi.it