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Short Communication
The "Rail-Unloading" Problem

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The "Rail-Unloading" Problem deals with a heavy beam of infinite or semi-infinite length, which is placed on both sides of a supporting point on planes with different levels. Analytical expressions for the contact lengths on both sides of the supporting point as well as for the maximum bending moment are presented.

1 Introduction

More than ten years ago the group of authors published a paper, Fischer et al. (1989), dealing with the uplift of a heavy, flexible and extensionable strip from the ground related to the problem of the uplift of a bottom plate of a container exerted by an earthquake. Recently another fine example appeared which allows for the application of the concept from Fischer et al. (1989), namely the unloading of a very long rail from a wagon, represented by a beam with stiffness EJ, E Young's modulus, J moment of inertia, and the weight q per unit length. Let us consider the following configuration, see Fig. 1.

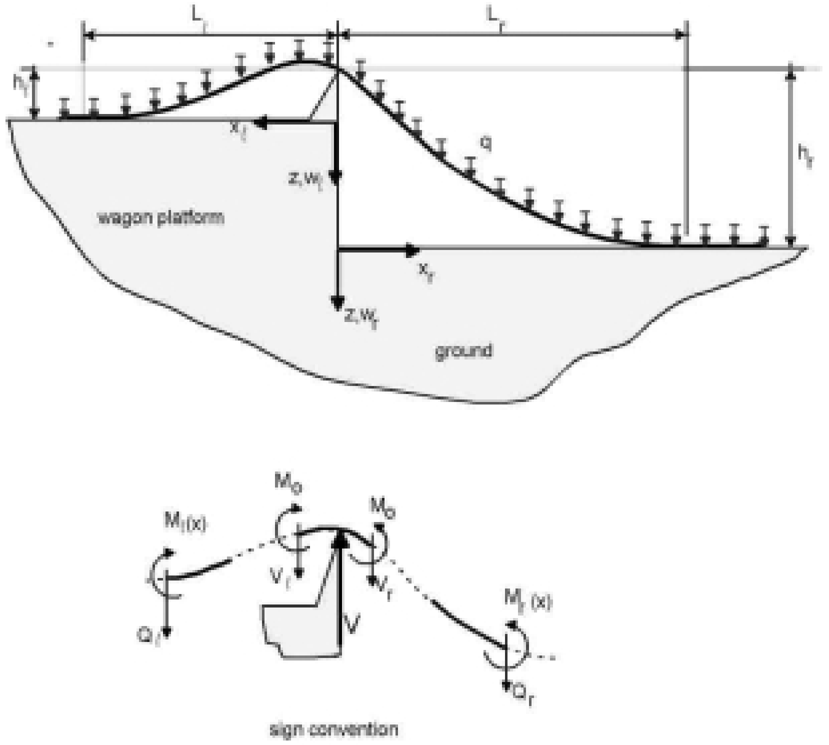

Figure 1: Schematic Representation of the "Mounting" Condition and the Definitions.

The rail is lifted from a wagon platform by an amount of h_ℓ on the left side. On the right side the rail is placed on the ground for later mounting by lowering it by an amount of h_r. The main goal is to be safe from plastification of the rail, which makes it necessary to know the maximum bending moment in the rail. The rail

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itself can be assumed as an infinitely long, heavy beam. Since finally an analytical expression can be presented for the maximum bending moment and its location, this solution is of interest for other mechanical applications, too. The authors are not aware of any similar solution in the open literature.

Defining the system as a two span bending beam with the left span length L_ℓ and the right span L_r, the integration of the differential equations for beam bending

$$-EJ\frac{d^2w}{dx^2} = M(x), \quad \frac{d^2M}{dx^2} = -q \tag{1}$$

makes it necessary to determine two sets of four integration constants, one for the left span and one for the right span. The following geometrical boundary conditions are available

$$x_\ell = L_\ell : w_\ell = 0\,, \qquad x_r = L_r : w_r = 0\,, \tag{2.1}$$

$$x_\ell = L_\ell : \frac{dw_\ell}{dx_\ell} = 0\,, \qquad x_r = L_r : \frac{dw_r}{dx_r} = 0\,, \tag{2.2}$$

$$x_\ell = 0 : w_\ell = -h_\ell\,, \qquad x_r = 0 : w_r = -h_r\,, \tag{2.3}$$

$$x_\ell = x_r = 0 : -\frac{dw_\ell}{dx_\ell} = \frac{dw_r}{dx_r}\,. \tag{2.4}$$

Since no jump in the moment $M(x)$ must occur at $x_\ell = x_r = 0$, a dynamic boundary condition follows as

$$x_\ell = x_r = 0 : M = M_o\,. \tag{2.5}$$

These eight boundary conditions allow for the determination of the eigth integration constants. However, two unknown quantities exist, the span lengths L_ℓ, L_r. Therefore, two further boundary conditions, the contact conditions, must be employed, being

$$x_\ell = L_\ell : \ \frac{d^2w_\ell}{dx_\ell^2} = 0\,, \quad x_r = L_r : \ \frac{d^2w_r}{dx_r^2} = 0\,. \tag{2.6}$$

Relation (2.6) means that the rail is placed on the platform to the left and on the ground to the rigth without any jump in the curvature. Therefore, no bending moment does exist in the contact points, $M_\ell(L_\ell) = M_r(L_r) = 0$. Applying the equilibrium equations leads to

$$V = q\frac{L}{2} - \frac{M_o}{L}\,, \tag{3.1}$$

$$M(x) = Vx + M_o - qx^2/2\,. \tag{3.2}$$

Relations (3.1), (3.2) are valid for the left span (label "ℓ") and the right span (label "r").

2 Problem Solution

Integration of the differential equation (1) with boundary conditions (2.1), (2.2) yields, for the right side,

$$-EJw_r = \frac{1}{6}V_r\left(x_r^3 - 3L_r^2x_r + 2L_r^3\right) + \frac{1}{2}M_o\left(x_r - L_r\right)^2 - \frac{1}{24}q\left(x_r^4 - 4L_r^3x_r + 3L_r^4\right). \tag{4}$$

Equation (4) is valid also for the left part substituting the label "r" by "ℓ". Insertion of (4) together with (3) into the boundary condition (2.4) yields

$$M_o = -\frac{1}{6} q \left(L_r^2 - L_r L_\ell + L_\ell^2 \right). \tag{5}$$

Now the remaining boundary condition (2.3) can be verified from (4) by inserting (3) and (5) yielding finally

$$24 EJ\, h_\ell / q = L_\ell^4 - \frac{2}{3} L_\ell^2 \left(L_r^2 - L_r L_\ell + L_\ell^2 \right), \tag{6.1}$$

$$24 EJ\, h_r / q = L_r^4 - \frac{2}{3} L_r^2 \left(L_r^2 - L_r L_\ell + L_\ell^2 \right). \tag{6.2}$$

By introducing $\chi = h_\ell / h_r$, $0 \le \chi \le 1$, as a known quantity, and $\lambda = L_\ell / L_r$ as an unknown quantity, equating (6.1) and (6.2) delivers due to the homogeneity in h and L

$$\lambda^4 + 2\lambda^3 - 2(1-\chi)\lambda^2 - 2\chi\lambda - \chi = 0 . \tag{7.1}$$

Some analysis allows to solve this nonlinear polynomial relation as

$$\lambda = \left(\sqrt{3} - 1 \right) + 0.47833\, \chi - 0.21038\, \chi^2 , 0 \le \chi \le 1 . \tag{7.2}$$

Note that $\lambda = 1$ for $\chi = 1$.

Reformulating (6.1) with λ allows to express L_r as

$$L_r = (EJ\, h_r / q)^{1/4} [72/(1 + 2\lambda(1-\lambda))]^{1/4} , \tag{8.1}$$

$$L_\ell = \lambda L_r . \tag{8.2}$$

The condition for the local maximum bending moment $M_{r,max}$, $\frac{dM_r}{dx_r} = 0$, delivers with (3.1), (3.2)

$$x_{r,max} = V_r / q , M_{r,max} = M_r \left(x_{r,max} \right) \tag{9.2}$$

and with (5), (8.1)

$$M_{r,max}^m = \left(EJ\, h_r q \right)^{1/2} \cdot \frac{\left(4 + 4\lambda - 3\lambda^2 - 2\lambda^3 + \lambda^4 \right)}{[72 \cdot (1 + 2\lambda(1-\lambda))]^{1/2}} , \tag{10.1}$$

$$x_{r,max} = \left(EJ_y\, h_r / q \right)^{1/4} \cdot \frac{(2 - \lambda(1-\lambda))}{[18 \cdot (1 + 2\lambda(1-\lambda))]^{1/4}} . \tag{10.2}$$

The suffix "m" in $M_{r,max}^m$ refers to the mounting condition.

Comparing $M_{r,max}^m$ with M_o^m , which follows from (5) and (8.1) as

$$M_o^m = -(EJ h_r q)^{1/2} \cdot \left[2 \left(1 - \lambda + \lambda^2 \right)^2 / (1 + 2\lambda(1-\lambda)) \right]^{1/2} , \tag{10.3}$$

yields that the maximum absolute value of the bending moment M_o^m at the support position is always larger than the field moment $M_{r,max}^m$.

A further bending problem may occur at the beginning of the unloading process. The left part of the rail remains as depicted in Figure 1. However, the right part of the rail bends down just before touching the ground. This situation is denoted as unloading condition with a suffix "u". At $x_r = L_r$ the contact condition (2.6) remains due to no bending moment at the right end. However, the boundary condition $\left.\frac{dw_r}{dx_r}\right|_{x_r=L_r}=0$ is no longer valid and must be replaced by the condition for no supporting force, $\left.\frac{d^3w_r}{dx^3}\right|_{x_r=L_r}=0$. Equilibrium yields

$$V_r^u = qL_r\,, \quad M_o^u = -q\,L_r^2/2\,. \tag{11}$$

The right part of the rail is a beam, clamped at $x_r = 0$ with the rotation $\left.\frac{dw_r}{dx_r}\right|_{x_r=0}$ at $x_r = 0$ and free at $x_r = L_r$, yielding

$$EJh_r = EJL_r \left.\frac{dw_r}{dx_r}\right|_{x_r=0} + q\,L_r^4/8\,. \tag{12.1}$$

Considering the left half of the rail, differentiation of (4) and inserting of (3.1) yield

$$-EJ\left.\frac{dw_\ell}{dx_\ell}\right|_{x_{\ell=0}} = -q\,L_\ell^3/12 - M_o^u\,L_\ell/2\,. \tag{12.2}$$

Engaging the transition condition (2.4) after inserting of (11) and (12.2) we obtain again a relation for $\lambda_u = L_\ell/L_r$ in the unloading case similar to (7.1), namely

$$\lambda_u^4 + 2\chi\lambda_u^3 - 2\lambda_u^2 - 6\chi\,\lambda_u - 3\chi = 0\,. \tag{13.1}$$

Some analysis allows again to solve this nonlinear polynomial relation as

$$\lambda_u = \sqrt{2} + 0.90785\,\chi - 1.04919\,\chi^2 + 0.45921\,\chi^3, \quad 0 \le \chi \le 1\,. \tag{13.2}$$

M_o^u follows as

$$M_o^u = -(EJh_r q)^{1/2} \cdot \left[6/\left(3 + 6\lambda_u - 2\lambda_u^3\right)\right]^{1/2}\,.$$

The dimensionless quantities $\widetilde{M}_o^m = M_o^m/(EJh_r q)^{1/2}$ and $\widetilde{M}_o^u = \left|M_o^u\right|/(EJh_r q)^{1/2}$ are depicted in Fig. 2 as a function of $\chi = h_\ell/h_r$.

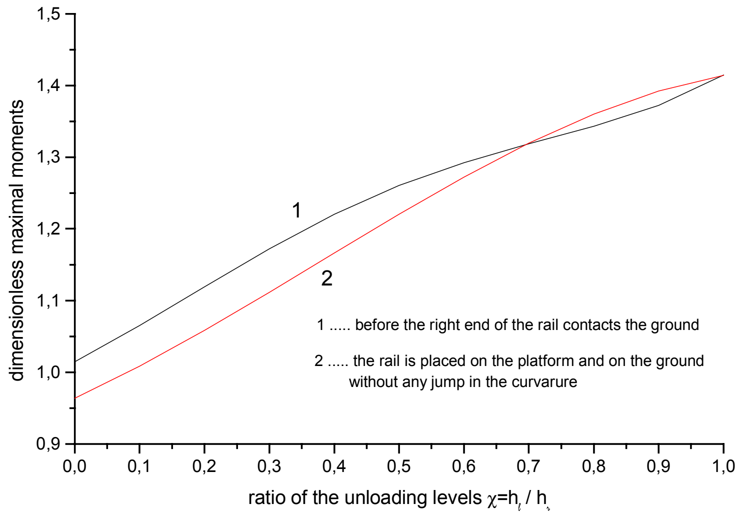

Figure 2: Dimensionless Moments $\widetilde{M}_o^u, \widetilde{M}_o^m$ as a Function of the Ratio of the Unloading Levels $\chi = h_\ell / h_r$.

3 Numerical Check

One may ask the question wether the application of the technical bending theory of beams, assuming small strains and small curvatures, can be justified. Since we look only for an elastic solution, small strains are assured. Therefore, a numerical check was run by applying COSMOS/M contact facilities (www.cosmosm.com), and using finite beam elements allowing for large displacements and rotations.

We check the case for $\lambda = 1$ yielding $L_r^2 = (24EJ\, h_r/q)^{1/2}$. By introducing the reference length $(EJ/q)^{1/3}$, we get with $\widetilde{L}_r = L_r/(EJ/q)^{1/3}$ and $\widetilde{h}_r = h_r/(EJ/q)^{1/3}$ the relation

$$\widetilde{L}_r = \sqrt[4]{72} \cdot \sqrt[4]{\widetilde{h}_r} \, .$$

This solution free $\widetilde{L}_r$ was verified by finite element studies. Accurate results for the concept at hand can be found for values of $\widetilde{h}_r$ up to 0.1. For a UIC 60 rail $(EJ/q)^{1/3}$ amounts to 21.791 mm. In conclusion, the concept at hand can be used for h_r up to 2,200 mm, which means for any practical configuration.

Acknowledgement

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Literature

1. Fischer, F.D.; Rammerstorfer, F.G.; Schreiner, W.: The extensible uplifted strip, Acta Mech., 80, (1989), 227-257.

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