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Axially Symmetric Deformations of Thin Flexible Multi-layered Shells of Revolution

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Professor Eliza M. Haseganu died in January 2002.

Large axisymmetric deflections of thin elastic multi-layered shells of revolution are studied. By using a variational principle for the three-dimensional non-linearly elastic body, the constitutive relations for a two-dimensional shell theory are derived. By using asymptotic expansions the problem of the shell deformations under axial force is solved. The dependence of the limiting point on the shell geometry and on the elastic parameters is obtained. As an example the two-layered shell is examined.

1 Introduction

The system of equations of the two-dimensional shell theory of Kirchhoff–Love type contains three kinds of relations: the geometrical relations, the equilibrium equations, and the constitutive relations between the reference shell surface deformations and the stress resultants and the stress couples. The relations of the first and of the second kind may be written in the exact form although often their approximate (for example linear) versions are used. Only the constitutive relations introduce an error in the shell theory.

The commonly used constitutive relations of the Kirchhoff–Love type for the axisymmetric deformations of shells of revolution are

$$T_i = K(\varepsilon_i + \nu\varepsilon_j) \qquad M_i = D(\kappa_i + \nu\kappa_j) \qquad \text{with} \qquad i,j = 1,2, \qquad i \neq j \tag{1}$$

where T_i and M_i are the stress resultants and the stress couples, ε_i and κ_i are the stretching and the bending deformations of the midsurface, K and D are the stiffness coefficients.

For most linear problems the relative error δ of relations (1) is of the order of the relative shell thickness h: $\delta = O(h)$ (see Novozhilov and Finkelshtein, 1943). The following investigations (Goldenweizer, 1976) support this estimation for the stress states if their variation is not very large. In nonlinear problems the error δ depends on the deformation level $\varepsilon = \max\{|\varepsilon_{ij}|\}$, where ε_{ij} are the components of the deformation Green tensor. Analysis shows that for $\varepsilon = O(h)$ the estimation $\delta = O(h)$ is valid. If the deformation level ε is larger, then the more general estimation $\delta \sim \max\{h, \varepsilon\}$ for the error of relations (1) is obtained (see Pietraszkiewicz, 1989; Axelrad, 2000). Tovstik (1997b) described the problem of the axial compression of a shell of revolution with the comparatively large deformation $\varepsilon \sim \sqrt{h}$. For this problem relations (1) lead to the error $\delta \sim \sqrt{h}$. In further papers Tovstik (1996), Tovstik (1997a) presented more general constitutive relations than (1) for which in this problem the estimation $\delta \sim h$ holds. In these papers the constitutive relations are found for an isotropic homogeneous non-linearly elastic material.

The main objective of this paper is to derive constitutive relations for a two-dimensional theory of thin multi-layered shells made of a non-linearly elastic material. By these relations the stress resultants and the stress couples are expressed through deformations of the shell reference surface. Here the material is again supposed to be isotropic, but its elastic properties depend on the normal to the reference surface coordinate. In partial cases we get multi-layered shells. By asymptotic simplification of the three-dimensional elastic potential energy, a two-dimensional expression is constructed, and the stress resultants and the stress couples are found as partial derivatives of this two-dimensional elastic energy.

The compression of a shell with non-negative Gaussian curvature under axial load applied to the shell edges is studied. If the shell edge is free in radial direction then it loses stability by axisymmetric deformation and the critical load may be found as a limit point of the curve "force – axial deflection" (see Tovstik, 1997a; Tovstik, 1999). The same type of the stability loss under axial compression takes place for shells of revolution, the generatrix of which has an angle (see Tovstik, 1999). In these cases, a large stretching and bending axisymmetric deformations are localized near the shell edge or near the angle. The method of asymptotic expansions of the solution of the singularly perturbed system of ordinary differential equations into powers of a small parameter μ connected with the relative shell thickness is used. For the homogeneous shell this problem was solved by Tovstik (1997a, 1999). Here the approximate expression of the critical force for the multi-layered shell is found, and its dependence on the shell geometry and on the elastic parameters is examined. As an example a two-layered shell of revolution consisting of two conic shells is studied.

2 The Axisymmetric Deformation of the Shell

For the non-homogeneous shell the neutral surface does not exist in all cases, and we introduce some surface of revolution as a reference surface. Let this surface be described by the following relations (see Figure 1)

$$r_0 = r_0(s_0) \qquad \theta_0 = \theta_0(s_0) \qquad r_0' = \cos\theta_0 \qquad (\,)' \equiv \frac{d}{ds_0}. \tag{2}$$

Here s_0 is the length of generatrix, r_0 is the distance between the current point of the reference surface and the axis of symmetry, θ_0 is the angle between the shell normal and the axis. The main radii of curvature R_{10} and R_{20} of the neutral surface before the deformation are obtained from

$$\frac{1}{R_{10}} = \theta_0' \qquad \text{and} \qquad \frac{1}{R_{20}} = \frac{\sin\theta_0}{r_0}. \tag{3}$$

We denote the same values after the deformation as s, r, θ, R_1, R_2. Formulas similar to (2) and (3) are valid.

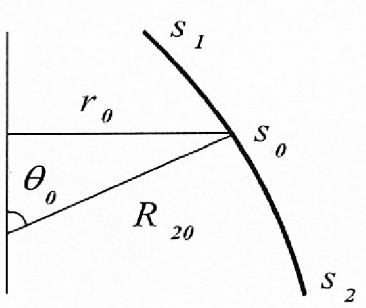

Figure 1. Shell of Revolution before Deformation

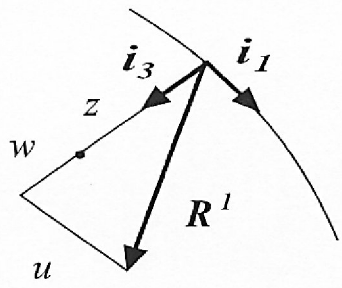

Figure 2. Displacements after Deformation

The stretching deformations of the reference surface ε_1 and ε_2 and the changes of its curvature κ_1 and κ_2 are

$$\begin{aligned} &\varepsilon_1 = s' - 1 \quad \text{and} \quad \varepsilon_2 = \frac{r}{r_0} - 1 \qquad \text{with} \qquad r' = (1+\varepsilon_1)\cos\theta \\ &\kappa_1 = \frac{1}{R_1} - \frac{1}{R_{10}} = \frac{\theta'}{1+\varepsilon_1} - \theta_0' \qquad \kappa_2 = \frac{1}{R_2} - \frac{1}{R_{20}} = \frac{\sin\theta}{r} - \frac{\sin\theta_0}{r_0} \end{aligned} \tag{4}$$

From relations (3) and (4) it follows that

$$(r_0\varepsilon_2)' = (1+\varepsilon_1)\cos\theta - \cos\theta_0 \tag{5}$$

In the three-dimensional body occupied by the shell before the deformation, we introduce the orthogonal system of curvilinear co-ordinates $q_1 = s_0$, $q_2 = \varphi$, $q_3 = z$, where φ is the angle in a circular direction, and z is the distance from the current point to the reference surface. Let $s_1 \le s_0 \le s_2$, $0 \le \varphi \le 2\pi$, $-h_1 \le z \le h_2$ $(h_1 + h_2 = h)$. The square of the distance between infinitely close points is

$$(d\mathbf{R}^0)^2 = H_i^2 dq_i^2 \equiv g_{ij}^0 dq_i dq_j \qquad H_1 = 1 + z\theta_0' \quad H_2 = r_0 + z\sin\theta_0 \quad H_3 = 1 \tag{6}$$

where H_i are Lamé's coefficients, and g_{ij}^0 are the covariant components of the metric tensor before the deformation. To describe the position of the point $(s_0,\ \varphi,\ z)$ after the deformation, we use mobile Cartesian co-ordinates with the unit vectors $\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3$, which are connected to the deformed reference surface. The point $(s_0,\ \varphi,\ z)$ position after the deformation is described by the vector (see Figure 2)

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{R}^1 \qquad \text{with} \qquad \mathbf{R}^1 = \mathbf{i}_1 u + \mathbf{i}_3(z + w) \tag{7}$$

The functions $u(s_0, z)$ and $w(s_0, z)$ describe the shear deformation and the stretching normal to the reference surface correspondingly. If the Kirchhoff–Love hypotheses are valid then $u(s_0, z) = w(s_0, z) \equiv 0$.

The covariant components of the metric tensor after the deformation, g_{ij}, are the following

$$g_{ij} = \mathbf{R_i R_j} \qquad \text{with} \qquad \mathbf{R_i} = \frac{\partial \mathbf{R}}{\partial q_i}$$

We find the components ε_{ij} of the Cauchy–Green deformation tensor $\mathbf{E}$ from

$$g_{ij} - g_{ij}^0 = 2H_i H_j \varepsilon_{ij}. \tag{8}$$

For an axisymmetric deformation $\varepsilon_{12} = \varepsilon_{23} = 0$.

For convenience in the asymptotic analysis we go to dimensionless variables (with the sign $\hat{}$) by

$$\{r,\ r_0,\ s,\ s_0,\ z,\ h\} \ = \ R_0\,\{\hat{r},\ \hat{r}_0,\ \hat{s},\ \hat{s}_0,\ \hat{z},\ h\} \tag{9}$$

where R_0 is the shell typical scale and h is the shell thickness. Then we omit the sign $\hat{}$. We study the state of strong shell bending, which is accompanied by comparatively large deformations, and accept the same assumptions about the stress-strain state as in (Tovstik, 1996)

$$\begin{aligned}
&z \sim \mu^2 \qquad \{\varepsilon_i,\ \varepsilon_{ii}\} \sim \mu \qquad \varepsilon_{13} \sim \mu^2 \qquad w \sim \mu^3 \qquad u \sim \mu^4 \\
&\{r_0,\ r,\ \theta_0,\ \theta,\ \kappa_2\} \sim 1 \qquad \{\theta',\ \kappa_1\} \sim \mu^{-1} \qquad \{\theta_0',\ r_0'\} = O(1) \\
&\frac{\partial y}{\partial s_0} \sim \frac{y}{\mu} \qquad \text{with} \qquad y = \{\varepsilon_i,\ \varepsilon_{ij},\ \theta,\ u,\ w\} \\
&\frac{\partial y}{\partial z} \sim \frac{y}{\mu^2} \qquad \text{with} \qquad y = \{\varepsilon_{ij},\ u,\ w\} \qquad \mu = c_0\sqrt{h}
\end{aligned} \tag{10}$$

where μ is a small parameter, and the constant $c_0 \sim 1$ will be chosen later. We put the sign $\sim$ between the values of the same asymptotic order, and the sign $O(\cdot)$ gives the upper estimation. We remind that the strain $\varepsilon \sim \mu$ is comparatively large, because for example strains $\varepsilon \sim \mu^2$ correspond to the critical stresses of the cylindrical shell buckling under axial compression (Tovstik, and Smirnov, 2001).

In the following expressions for strains ε_{ij} we keep the terms of the orders μ and μ^2 and omit the terms of the order $O(\mu^3)$

$$\varepsilon_{11} = \varepsilon_1 + (\varepsilon_1 + z\tau_1)^2/2 + (z+w)\tau_1 + O(\mu^3) \qquad \text{with} \quad \tau_1 = \theta' - \theta_0'$$
$$\varepsilon_{13} = (u_z + w')/2 + O(\mu^3)$$
$$\varepsilon_{22} = \varepsilon_2 + \varepsilon_2^2/2 + (z+w)\tau_2 + O(\mu^3) \qquad \text{with} \quad \tau_2 = \frac{\cos\theta - \cos\theta_0}{r_0} \tag{11}$$
$$\varepsilon_{33} = w_z + w_z^2/2 + O(\mu^3)$$

where τ_1 and τ_2 are the approximate expressions for changes of curvature.

3 The Asymptotic Simplification of the Potential Energy

The shell material is supposed to be elastic and isotropic. Let the potential energy density per unit volume before the deformation $\Phi(I_1, I_2, I_3)$ be given as a function of invariants of the deformation tensor $\mathbf{E}$. We use the following invariants

$$I_1 = \varepsilon_{ii} = I_1^0 + \varepsilon_{33} \qquad \text{with} \qquad I_1^0 = \varepsilon_{11} + \varepsilon_{22}$$
$$I_2 = \varepsilon_{ij}\varepsilon_{ji} = I_2^0 + \varepsilon_{33}^2 + O(\mu^4) \qquad \text{with} \qquad I_2^0 = \varepsilon_{11}^2 + \varepsilon_{22}^2 \tag{12}$$
$$I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = I_3^0 + \varepsilon_{33}^3 + O(\mu^4) \qquad \text{with} \qquad I_3^0 = \varepsilon_{11}^3 + \varepsilon_{22}^3$$

Here we take into account that $\varepsilon_{12} = \varepsilon_{23} = 0$ and $\varepsilon_{13} \sim \mu^2$.

The stresses σ_{ij} are equal to

$$\sigma_{ij} = \frac{\partial \Phi}{\partial \varepsilon_{ij}} = A_1\delta_{ij} + A_2\varepsilon_{ij} + A_3\varepsilon_{ik}\varepsilon_{kj} \qquad \text{with} \qquad A_k = k\frac{\partial\Phi}{\partial I_k} \qquad k = 1, 2, 3 \tag{13}$$

where δ_{ij} is the Kronecker's symbol.

For the 5-constants Mindlin elasticity theory

$$\Phi = \frac{1}{2}\lambda I_1^2 + GI_2 + \alpha_1 I_1^3 + \alpha_2 I_1 I_2 + \alpha_3 I_3 \qquad \text{with} \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)} \tag{14}$$

we get

$$A_1 = \lambda I_1 + 3\alpha_1 I_1^2 + \alpha_2 I_2 \qquad A_2 = 2G + 2\alpha_2 I_1 \qquad A_3 = 3\alpha_3 \tag{15}$$

Here E is Young's modulus and α_j are the additional elastic constants taking into account the non-linear dependence of stresses on strains. For the aims of the asymptotic analysis we suppose that the order of the constants α_j does not essentially exceed E. The potential (14) gives the general form of the square dependence of stresses on strains. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$ then relations (13) lead to Hooke's law.

In contradistinction to Tovstik (1996), we suppose now that all elastic constants $(\lambda, G, E, \nu, \alpha_1, \alpha_2, \alpha_3)$ depend piecewise continuously on z.

The potential energy Π of the elastic deformations is equal to

$$\Pi = \pi \int_{s_1}^{s_2} \int_{-h_1}^{h_2} \Phi \, r_0(s_0) \, dz \, ds_0 \tag{16}$$

To deliver the constitutive relations we suppose that body forces and external surface forces are absent. Then it is possible to find the equilibrium state by minimizing the potential energy. After minimizing the potential density Φ in ε_{33} we obtain

$$
\begin{aligned}
&\varepsilon_{33} = -\frac{\nu}{1-\nu}\, I_1^0 + O(\mu^2) \qquad \text{and} \qquad \min_{\varepsilon_{33}} \Phi = \Phi^0(\varepsilon_{11}, \varepsilon_{22}) + O(E\,\mu^4) \\
&\Phi^0 = \frac{E}{2(1-\nu^2)} \left((1-\nu) I_2^0 + \nu \left(I_1^0\right)^2 \right) + \beta_1 \left(I_1^0\right)^3 + \beta_2\, I_1^0 I_2^0 + \beta_3\, I_3^0 \\
&\beta_1 = \frac{(1-2\nu)^3\alpha_1 + \nu^2(1-2\nu)\alpha_2 - \nu^3\alpha_3}{(1-\nu)^3} \qquad \beta_2 = \frac{1-2\nu}{1-\nu}\alpha_2 \qquad \beta_3 = \alpha_3
\end{aligned}
\tag{17}
$$

After integrating the relation $\partial w/\partial z = \varepsilon_{33} + O(\mu^2)$ we find

$$
w = -\int_0^z \frac{\nu}{1-\nu} \left(\varepsilon_1 + \varepsilon_2 + z(\tau_1 + \tau_2)\right)\, dz + O(\mu^4) \tag{18}
$$

Now due to relations (10) and (18) the strains ε_{11} and ε_{22} are the functions of 5 main arguments $(\varepsilon_1,\, \varepsilon_2,\, \tau_1,\, \tau_2,\, z)$. We introduce the function

$$
\Psi(\varepsilon_1,\, \varepsilon_2,\, \tau_1,\, \tau_2) = \int_{-h_1}^{h_2} \Phi^0(\varepsilon_1,\, \varepsilon_2,\, \tau_1,\, \tau_2,\, z)\, dz \tag{19}
$$

Calculations of integral (19) in z due to relations (17), (10) and (18) lead to the following relation

$$
\begin{aligned}
\Psi = \frac{1}{2}\, &\big[K_0(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_1^3 + \varepsilon_2^3) + K_0^\nu(2\varepsilon_1\varepsilon_2 + \varepsilon_1^2\varepsilon_2 + \varepsilon_1\varepsilon_2^2) + \\
&K_1((2\varepsilon_1 + 3\varepsilon_1^2)\tau_1 + (2\varepsilon_2 + 3\varepsilon_2^2)\tau_2) + K_1^\nu((2\varepsilon_2 + \varepsilon_2^2 + 2\varepsilon_1\varepsilon_2)\tau_1 + (2\varepsilon_1 + \varepsilon_1^2 + 2\varepsilon_1\varepsilon_2)\tau_2) + \\
&K_2((1 + 3\varepsilon_1)\tau_1^2 + (1 + 3\varepsilon_2)\tau_2^2) + K_2^\nu(2\tau_1\tau_2(1 + \varepsilon_1 + \varepsilon_2) + \varepsilon_2\tau_1^2 + \varepsilon_1\tau_2^2) + \\
&K_3(\tau_1^3 + \tau_2^3) + K_3^\nu(\tau_1\tau_2^2 + \tau_1^2\tau_2)\big] - \\
&\big[L_{00}(\varepsilon_1\tau_1 + \varepsilon_2\tau_2) + L_{00}^\nu(\varepsilon_2\tau_1 + \varepsilon_1\tau_2) + L_{10}({\tau_1}^2 + \tau_2^2) + 2L_{10}^\nu\tau_1\tau_2\big]\,(\varepsilon_1 + \varepsilon_2) - \\
&\big[L_{01}(\varepsilon_1\tau_1 + \varepsilon_2\tau_2) + L_{01}^\nu(\varepsilon_2\tau_1 + \varepsilon_1\tau_2) + L_{11}(\tau_1^2 + \tau_2^2) + 2L_{10}^\nu\tau_1\tau_2\big]\,(\tau_1 + \tau_2) + \\
&N_{10}(\varepsilon_1 + \varepsilon_2)^3 + 3N_{11}(\varepsilon_1 + \varepsilon_2)^2(\tau_1 + \tau_2) + 3N_{12}(\varepsilon_1 + \varepsilon_2)(\tau_1 + \tau_2)^2 + N_{13}(\tau_1 + \tau_2)^3 + \\
&N_{20}(\varepsilon_1 + \varepsilon_2)(\varepsilon_1^2 + \varepsilon_2^2) + N_{21}((\tau_1 + \tau_2)(\varepsilon_1^2 + \varepsilon_2^2) + 2(\varepsilon_1\tau_1 + \varepsilon_2\tau_2)(\varepsilon_1 + \varepsilon_2)) + \\
&N_{22}((\varepsilon_1 + \varepsilon_2)(\tau_1^2 + \tau_2^2) + 2(\varepsilon_1\tau_1 + \varepsilon_2\tau_2)(\tau_1 + \tau_2)) + N_{23}(\tau_1 + \tau_2)(\tau_1^2 + \tau_2^2) + \\
&N_{30}(\varepsilon_1^3 + \varepsilon_2^3) + 3N_{31}(\varepsilon_1^2\tau_1 + \varepsilon_2^2\tau_2) + 3N_{32}(\varepsilon_1\tau_1^2 + \varepsilon_2\tau_2^2) + N_{33}(\tau_1^3 + \tau_2^3)
\end{aligned}
\tag{20}
$$

where the elastic coefficients are the following

$$
\begin{aligned}
&K_n = \int_{-h_1}^{h_2} \frac{E\, z^n}{1-\nu^2}\, dz && K_n^\nu = \int_{-h_1}^{h_2} \frac{E\,\nu\, z^n}{1-\nu^2}\, dz \\
&L_{mn} = \int_{-h_1}^{h_2} \frac{E\, z^m}{1-\nu^2} \left(\int_0^z \frac{\nu\, z^n}{1-\nu}\, dz \right) dz && L_{mn}^\nu = \int_{-h_1}^{h_2} \frac{E\,\nu\, z^m}{1-\nu^2} \left(\int_0^z \frac{\nu\, z^n}{1-\nu}\, dz \right) dz \\
&N_{mn} = \int_{-h_1}^{h_2} \beta_m\, z^n\, dz \qquad \text{with} && m, n = 0, 1, 2, 3
\end{aligned}
\tag{21}
$$

We remind that the elastic constants E, ν, β_m depend on z. If $h_1 = h_2 = h/2$ and all these constants are even then we call it a shell with symmetric cross-section. For such a shell some of the coefficients (21) vanish

$$K_1 = K_1^\nu = K_3 = K_3^\nu = L_{00} = L_{00}^\nu = L_{11} = L_{11}^\nu = N_{m1} = N_{m3} = 0 \qquad \text{with} \quad m = 1, 2, 3 \tag{22}$$

4 The Constitutive Relations

The stress resultants T_1 and T_2 and the stress couples may be found as partial derivatives of the function $\Psi(\varepsilon_1, \varepsilon_2, \tau_1,$ by its arguments

$$T_j = \frac{\partial \Psi}{\partial \varepsilon_j} \qquad M_j = \frac{\partial \Psi}{\partial \tau_j} \qquad \text{with} \qquad j = 1, 2 \tag{23}$$

where the stress resultants and the stress couples are related to the unit length before deformation. We get from equations (23) and (20) the following

$$\begin{aligned}
T_1 &= K_0\varepsilon_1 + K_0^\nu\varepsilon_2 + K_1\tau_1 + K_1^\nu\tau_2 + T_1^* + O(E_0 h\mu^3) \\
T_2 &= K_0\varepsilon_2 + K_0^\nu\varepsilon_1 + K_1\tau_2 + K_1^\nu\tau_1 + T_2^* + O(E_0 h\mu^3) \\
M_1 &= K_1\varepsilon_1 + K_1^\nu\varepsilon_2 + K_2\tau_1 + K_2^\nu\tau_2 + M_1^* + O(E_0 h^2\mu^3) \\
M_2 &= K_1\varepsilon_2 + K_1^\nu\varepsilon_1 + K_2\tau_2 + K_2^\nu\tau_1 + M_2^* + O(E_0 h^2\mu^3)
\end{aligned} \tag{24}$$

where $T_j^* = O(E_0 h\mu^2)$ and $M_j^* = O(E_0 h^3)$ $(j = 1, 2)$ depend non-linearly on the stretching and bending deformations (ε_j and τ_j) of the reference surface and are equal to

$$\begin{aligned}
T_1^* &= a_1\varepsilon_1^2 + a_2\varepsilon_1\varepsilon_2 + a_3\varepsilon_2^2 + a_4\varepsilon_1\tau_1 + a_5\varepsilon_1\tau_2 + a_6\varepsilon_2\tau_1 + a_7\varepsilon_2\tau_2 + a_8\tau_1^2 + a_9\tau_1\tau_2 + a_{10}\tau_2^2 \\
M_1^* &= b_1\varepsilon_1^2 + b_2\varepsilon_1\varepsilon_2 + b_3\varepsilon_2^2 + b_4\varepsilon_1\tau_1 + b_5\varepsilon_1\tau_2 + b_6\varepsilon_2\tau_1 + b_7\varepsilon_2\tau_2 + b_8\tau_1^2 + b_9\tau_1\tau_2 + b_{10}\tau_2^2
\end{aligned} \tag{25}$$

$$\begin{aligned}
&a_1 = 3(K_0/2 + N_{10} + N_{20} + N_{30}) && a_2 = 2a_3 = K_0^\nu + 6N_{10} + 2N_{20} \\
&a_4 = 2b_1 = 3K_1 - 2L_{00} + 6(N_{11} + N_{21} + N_{31}) && a_5 = 2b_3 = K_1^\nu - 2L_{00}^\nu + 6N_{11} + 2N_{21} \\
&a_6 = a_7 = b_2 = K_1^\nu - L_{00} - L_{00}^\nu + 6N_{11} + 2N_{21} && a_8 = b_4/2 = 3K_2/2 - L_{10} - L_{01} + 3(N_{12} + N_{22} + N_{32}) \\
&a_9 = b_5 = b_7 = K_2^\nu - 2L_{10}^\nu - L_{01} - L_{01}^\nu + 6N_{12} + 2N_{22} && a_{10} = b_6/2 = K_2^\nu/2 - L_{10} - L_{01}^\nu + 3N_{12} + N_{22} \\
&b_8 = 3(K_3/2 - L_{11} + N_{13} + N_{23} + N_{33}) && b_9 = 2b_{10} = K_3 - 2L_{11} - 4L_{11}^\nu + 6N_{13} + 2N_{23}
\end{aligned} \tag{26}$$

Function Ψ is symmetric with respect to indices 1 and 2. So in order to find T_2^* and M_2^* in relations (24) it is enough to replace in equations (25) ε_1 and τ_1 by ε_2 and τ_2 respectively and vice versa. In equations (24), E_0 is a scale of Young's modulus, and the relative error of the constitutive relations (24) is of the order of μ^2 or of the order of the relative shell thickness h/R. Non-linear terms in equations (24) are of the order of μ compared to the linear ones. Due to equations (21) the following estimations are valid

$$\{K_0,\, K_0^\nu,\, a_1,\, a_2,\, a_3\} \sim E_0 h \qquad \{K_2,\, K_2^\nu,\, a_8,\, a_9,\, a_{10},\, b_4,\, b_5,\, b_6,\, b_7\} \sim E_0 h^3 \tag{27}$$

$$\{K_1,\, K_1^\nu,\, a_4,\, a_5,\, a_6,\, a_7,\, b_1,\, b_2,\, b_3\} = O(E_0 h^2) \qquad \{K_3,\, K_3^\nu,\, b_8,\, b_9,\, b_{10}\} = O(E_0 h^4) \tag{28}$$

For shells with symmetric cross-section the coefficients in the left sides of estimations (28) are equal to zero.

We choose the reference surface so that $K_1 = 0$. If Poisson's ratio $\nu =$ const, then simultaneously $K_1^\nu = 0$. In this case as for the symmetric cross-section the reference surface is the neutral surface and according to (24) in the linear approximation, stretching and bending deformations of the reference surface act independently.

Relations (24) were obtained by assuming that external surface and body forces are absent. As in Tovstik (1997a) it can be verified that relations (24) are valid with the same residual terms also in the presence of external forces of large enough magnitude. As a rule in the case of compression, the shell buckling takes place earlier than relations (24) become incorrect.

If all elastic constants in the potential energy (14) do not depend on z then relations (24) coincide exactly with those obtained in Tovstik (1996).

5 The Case of Axial Compression

Two-dimensional equilibrium equations in projections on the tangent and on the normal to the reference surface after deformation have the form

$$
\begin{aligned}
&(r_0 T_1)' - T_2 \cos\theta + r_0 \theta' Q_1 + r_0 p_1 = 0 \\
&(r_0 Q_1)' - T_2 \sin\theta - r_0 \theta' T_1 + r_0 p_3 = 0 \\
&(r_0 M_1)' - M_2 \cos\theta - r_0 (1 + \varepsilon_1) Q_1 = 0
\end{aligned}
\tag{29}
$$

where Q_1 is the shear stress resultant, and p_1, p_3 are projections of the external body forces and surface loads per unit area of the reference surface before deformation. We introduce the projections U and V of the stress resultants on the radial and on the axial directions respectively

$$
T_1 = U\cos\theta + V\sin\theta \qquad Q_1 = U\sin\theta - V\cos\theta \tag{30}
$$

Then the first two equations (29) assume the form

$$
(r_0 V)' + r_0(p_1 \sin\theta - p_3\cos\theta) = 0 \qquad (r_0 U)' - T_2 + r_0(p_1\cos\theta - p_3 \sin\theta) = 0 \tag{31}
$$

In the case of compression the estimation $T_1 = O(E_0 h \mu^2)$ is valid because the critical value of T_1 (as a limit point) is of the same order. From this estimation it follows that

$$
\varepsilon_1 = -\nu_0 \varepsilon_2 + O(\mu^2) \qquad \text{with} \qquad \nu_0 = \frac{K_0^\nu}{K_0} \tag{32}
$$

Now taking into account relations (32), $K_1 = 0$ and $\tau_2 = O(1)$ we rewrite and simplify relations (24) with an exactness which is sufficient for further expansions

$$
\begin{aligned}
T_1 &= K_0(\varepsilon_1 + \nu_0\varepsilon_2) + \hat{T}_1 + O(E_0 h \mu^3) & \hat{T}_1 &= K_1^\nu \tau_2 + c_1\varepsilon_2^2 + c_2\varepsilon_2\tau_1 + c_3\tau_1^2 \\
T_2 &= K_0(1-\nu_0^2)\varepsilon_2 + K_1^\nu\tau_1 + \hat{T}_2 + O(E_0 h\mu^3) & \hat{T}_2 &= \nu_0(T_1 - K_1^\nu\tau_2) + c_4\varepsilon_2^2 + c_5\varepsilon_2\tau_1 + c_6\tau_1^2 \\
M_1 &= K_1^\nu\varepsilon_2 + K_2\tau_1 + \hat{M}_1 + O(E_0 h^2\mu^3) & \hat{M}_1 &= K_2^\nu\tau_2 + c_7\varepsilon_2^2 + c_8\varepsilon_2\tau_1 + c_9\tau_1^2 \\
M_2 &= -\nu_0 K_1^\nu \varepsilon_2 + K_2^\nu\tau_1 + \hat{M}_1 + O(E_0 h^2\mu^3) & \hat{M}_2 &= K_2\tau_2 + c_{10}\varepsilon_2^2 + c_{11}\varepsilon_2\tau_1 + c_{12}\tau_1^2
\end{aligned}
\tag{33}
$$

$$c_1 = \nu_0^2 a_1 - \nu_0 a_2 + a_3 \qquad c_2 = a_6 - \nu_0 a_4 \qquad c_3 = a_8$$
$$c_4 = (1-\nu_0)[a_1(1+\nu_0+\nu_0^2) - \nu_0(a_2+a_3)] \qquad c_5 = \nu_0^2 a_4 + a_5 - \nu_0(a_6+a_7) \qquad c_6 = a_{10} - \nu_0 a_8$$
$$c_7 = \nu_0^2 b_1 - \nu_0 b_2 + b_3 \qquad c_8 = b_6 - \nu_0 b_4 \qquad c_9 = b_8$$
$$c_{10} = b_1 - \nu_0 b_2 + \nu_0^2 b_3 \qquad c_{11} = b_5 - \nu_0 b_7 \qquad c_{12} = b_{10}$$

where $\hat{T}_j$ and $\hat{M}_j$ are the comparatively small terms containing geometrical as well as physical non-linearities.

6 The Asymptotic Expansions

Let the axial force P be applied to the shell edges and the external body forces and surface loads be absent ($p_1 = p_3 = 0$). We introduce dimensionless variables by means of the following formulas

$$\{\varepsilon_1, \varepsilon_2\} = \mu\{\varepsilon_1^o, \varepsilon_2^o\} \qquad \{T_1, Q_1, U, V\} = K_0^*\mu^2\{T_1^o, Q_1^o, U^o, V^o\}$$
$$T_2 = K_0^*\mu T_2^o \qquad \{M_1, M_2\} = K_0^*\mu^3\{M_1^o, M_2^o\} \tag{34}$$

with

$$\mu^4 = \frac{K_2}{K_0^*} = c_0^4 h^2 \qquad K_0^* = K_0(1-\nu_0^2) \tag{35}$$

In this way the constant c_0 in relation (10) for μ is chosen. By the symbol ^{o} the dimensionless variables are marked. Substitutions (34) are introduced in such a manner that these variables are of the order 1. Later we shall omit the symbol ^{o}. Then the system of equations (5), (29), (31) for variables

$$V,\ U,\ \varepsilon_2,\ M_1,\ \theta \tag{36}$$

assumes the form

$$(r_0 V)' = 0,$$
$$\mu(r_0 U)' = (1-\nu_1^2)\varepsilon_2 + \nu_1 M_1 + \mu f_1 + O(\mu^2) \qquad f_1 = \frac{\hat{T}_2}{K_0^*\mu^2}$$
$$\mu(r_0\varepsilon_2)' = (1-\mu\nu_0\varepsilon_2)\cos\theta - \cos\theta_0, \tag{37}$$
$$\mu(r_0 M_1)' = r_0(1-\mu\nu_0\varepsilon_2)(U\sin\theta - V\cos\theta) + \mu f_3 + O(\mu^2) \qquad f_3 = M_2\cos\theta,$$
$$\mu\theta' = M_1 - \nu_1\varepsilon_2 + \mu f_4 + O(\mu^2) \qquad f_4 = \theta_0' - \frac{\hat{M}_1}{K_2}$$

Here f_1, f_3, f_4 are functions of τ_1. After excluding τ_1 by the relation $\mu\tau_1 = M_1 - \nu_1\varepsilon_2$ we express these functions through variables (36) and obtain the following formulas

$$f_1 = \nu_0(U\cos\theta + V\sin\theta) + d_1\varepsilon_2^2 + d_2\varepsilon_2 M_1 + d_3 M_1^2$$
$$f_3 = (\nu_2 M_1 - \nu_1(\nu_0+\nu_2)\varepsilon_2)\cos\theta \tag{38}$$
$$f_4 = \theta_0' - \nu_2\tau_2 - d_7\varepsilon_2^2 - d_8\varepsilon_2 M_1 - d_9 M_1^2$$

$$d_1 = \frac{c_4}{K_0^*} - \frac{\nu_1 c_5}{K_0^* \mu^2} + \frac{\nu_1^2 c_6}{K_0^* \mu^4} \qquad d_2 = \frac{c_5}{K_0^* \mu^2} - \frac{2\nu_1 c_6}{K_0^* \mu^4} \qquad d_3 = \frac{c_6}{K_0^* \mu^4}$$
$$d_7 = \frac{\mu^2 c_7}{K_2} - \frac{\nu_1 c_8}{K_2} + \frac{\nu_1^2 c_9}{K_2 \mu^2} \qquad d_8 = \frac{c_8}{K_2} - \frac{2\nu_1 c_9}{K_2 \mu^2} \qquad d_9 = \frac{c_9}{K_2 \mu^2} \tag{39}$$

$$\nu_0 = \frac{K_0^\nu}{K_0} \qquad \nu_1 = \frac{K_1^\nu}{K_0^* \mu^2} = \frac{K_1^\nu}{\sqrt{K_0^* K_2}} \qquad \nu_2 = \frac{K_2^\nu}{K_2} \tag{40}$$

All functions f_j and constants d_j and ν_j are of the order of 1. For the shell with symmetric cross-section $\nu_1 = d_2 = d_7 = d_9 = 0$.

7 Formulation of the Boundary Value Problem

As an example we study a compound shell of revolution consisting of two equal conic shells (see Figure 3 where the shell generatrix is shown). For this shell $\theta_0 =$ const and

$$r_0 = 1 - |s_0| \cos\theta_0 \qquad \text{for} \qquad |s_0| \le s_1 \tag{41}$$

As the shell scale R in relations (9) we take the cone radius at $s_0 = 0$. The shell edges $s_0 = \pm s_1$ are supposed to be clamped. Due to symmetry of the problem we will study only the part $-s_1 \le s_0 \le 0$ of the shell and we will satisfy the boundary conditions

$$\varepsilon_2 = 0 \qquad \theta = \theta_0 \qquad \text{at} \qquad s_0 = -s_1$$
$$U = 0 \qquad \theta = \theta_0 \qquad \text{at} \qquad s_0 = 0 \tag{42}$$

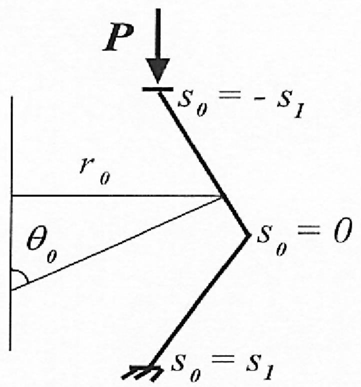

Figure 3. Generatrix of the Compound Shell

The first of equations (37) gives

$$V = \frac{C}{r_0} \qquad P = 2\pi R^2 K_0 \mu^2 C \tag{43}$$

where P is the axial force ($P < 0$ for compression) and we look for its limit value. At $\mu \to 0$ system (37) is a singularly perturbed one. Far from the edges $s_0 = 0$ and $s_0 = -s_1$ the membrane stress state has

$$U^m = \frac{C\cos\theta_0}{r_0 \sin\theta_0} \qquad \varepsilon_2^m = -\frac{\nu_0 \mu C}{r_0 \sin\theta_0} \qquad M_1^m = O(C\mu^3) \qquad \theta^m = \theta_0 + O(C\mu^2) \tag{44}$$

In the neighborhood of the generatrix angle $s_0 = 0$ estimations (10) are valid and system (37) may be used. It is connected with the fact that the shell is free in the radial direction ($U = 0$). Near the other edge $s_0 = -s_1$ the edge effect appears also but here due to the condition $\varepsilon_2 = 0$ the stresses are μ times smaller than near the edge $s_0 = 0$.

8 Asymptotic Solution of System (37)

We write system (37) in vector form

$$\mu \mathbf{x}' = \mathbf{F}(\mathbf{x},\, s_0,\, C,\, \mu) \qquad \mathbf{x} = \{U,\, \varepsilon_2,\, M_1,\, \theta\}, \tag{45}$$

perform the scale extension $\xi = s_0/\mu$ of the independent variable and seek its solution in the form

$$\mathbf{x} = \mathbf{x^0}(\xi) + \mu\mathbf{x^1}(\xi) + O(\mu^2) \qquad C = C^0 + \mu C^1 + O(\mu^2) \tag{46}$$

The accuracy of system (37) enables us to find explicitly written terms of the series (46). Taking account of the fact that $r_0(s_0) = 1 + \mu\xi\cos\theta_0$ and equating coefficients of μ^0 in system (37), we obtain in the zeroth approximation the following nonlinear system

$$\dot{\mathbf{x}}^\mathbf{0} = \mathbf{F^0} = \mathbf{F}(\mathbf{x^0},\, 0,\, C^0,\, 0) \tag{47}$$

or

$$\begin{aligned}
&\dot{U}^0 = (1 - \nu_1^2)\varepsilon_2^0 + \nu_1 M_1^0, \\
&\dot{\varepsilon}_2^0 = \cos\theta^0 - \cos\theta_0, \\
&\dot{M}_1^0 = U^0\sin\theta^0 - C^0\cos\theta^0, \\
&\dot{\theta}^0 = M_1^0 - \nu_1\varepsilon_2^0
\end{aligned} \tag{48}$$

The boundary conditions at $\xi \to -\infty$ confirm the fact that solution (46) approaches the membrane solution (44). For the zeroth approximation we get the following boundary conditions

$$\begin{aligned}
&U^0 \to \frac{C^0\cos\theta_0}{\sin\theta_0} \qquad \{\varepsilon_2,\, M_1\} \to 0 \qquad \theta^0 \to \theta_0 \qquad \text{at} \qquad \xi \to -\infty \\
&U^0 = 0 \qquad \theta^0 = \theta_0 \qquad \text{at} \qquad \xi = 0
\end{aligned} \tag{49}$$

Here a dot denotes a derivative with respect to ξ. If $\nu_1 = 0$ system (48) coincides with that studied in Kriegsmann and Lange (1980), Evkin and Korovaitsev (1992), and Tovstik (1997b).

In the first approximation we get the linear system

$$\dot{\mathbf{x}}^\mathbf{1} = \mathbf{L}(\xi)\mathbf{x}^1 + \mathbf{g} \qquad \text{with} \qquad \mathbf{L} = \frac{\partial\mathbf{F^0}}{\partial\mathbf{x^0}} \qquad \mathbf{g} = \frac{\partial\mathbf{F}}{\partial s_0}\xi + \frac{\partial\mathbf{F}}{\partial C}C^1 + \frac{\partial\mathbf{F}}{\partial\mu} \tag{50}$$

or

$$
\begin{aligned}
\dot{U}^1 &= (1-\nu_1^2)\varepsilon_2^1 + \nu_1 M_1^1 + g_1 \\
\dot{\varepsilon}_2^1 &= -\theta_1 \sin\theta^0 + g_2 \\
\dot{M}_1^1 &= U^1 \sin\theta^0 + (U^0\cos\theta^0 + C^0 \sin\theta^0)\theta^1 + g_3 - C^1\cos\theta^0 \\
\dot{\theta}^1 &= M_1^1 - \nu_1\varepsilon_2^1 + g_4
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
g_1 &= -(\xi\dot{U}^0 + U^0)\cos\theta_0 + \nu_0(U^0\cos\theta^0 + C^0\sin\theta^0) + d_1(\varepsilon_2^0)^2 + d_2\varepsilon_2^0 M_1^0 + d_3(M_1^0)^2 \\
g_2 &= -(\xi\dot{\varepsilon}_2^0 + \varepsilon_2^0)\cos\theta_0 - \nu_0\varepsilon_2^0\cos\theta^0 \\
g_3 &= M_1^0(\nu_2\cos\theta^0 - \cos\theta_0) - \nu_0\varepsilon_2^0(U_0\sin\theta^0 - C^0\cos\theta^0) + \xi C^0\cos\theta_0\cos\theta^0 - \nu_1(\nu_0+\nu_2)\varepsilon_2^0\cos\theta^0 \\
g_4 &= -\nu_2(\sin\theta^0 - \sin\theta_0) - d_7(\varepsilon_2^0)^2 - d_8\varepsilon_2^0 M_1^0 - d_9(M_1^0)^2
\end{aligned}
\tag{52}
$$

with the boundary conditions

$$
\begin{aligned}
&U^1 \to \frac{C^1\cos\theta_0}{\sin\theta_0} - \frac{C^0\cos^2\theta_0}{\sin^2\theta_0} \qquad \varepsilon_2^1 \to -\frac{\nu_0 C^0}{\sin\theta_0} \qquad M_1^1 \to -\frac{\nu_0\nu_1 C^0}{\sin\theta_0} \qquad \theta^1 \to 0 \qquad \text{at} \qquad \xi \to -\infty \\
&U^1 = 0 \qquad \theta^1 = 0 \qquad \text{at} \qquad \xi = 0
\end{aligned}
\tag{53}
$$

We introduce the shell shortening in the axial direction

$$
z = \int_{-s_1}^{0} \left((1+\mu\varepsilon_1)\sin\theta - \sin\theta_0\right)\, ds_0 \tag{54}
$$

For post-buckling axisymmetrical states $z = O(1)$ (see Tovstik, 1996). Here we study the pre-buckling state and the state near the limit point. Then $z = O(\mu)$ and we have the following expansion

$$
z = \mu z_1 + \mu^2 z_2 + O(\mu^3) \tag{55}
$$

with

$$
\begin{aligned}
z_1 &= \int_{-\infty}^{0} \left(\sin\theta^0 - \sin\theta_0\right)\, d\xi \\
z_2 &= \int_{-\infty}^{0} \left(\theta^1\cos\theta^0 - \nu_0\varepsilon_2^0\sin\theta^0\right)\, d\xi - \frac{C^0\log(1 - s_1\cos\theta_0)}{\sin^2\theta_0\cos\theta_0}
\end{aligned}
\tag{56}
$$

9 The Axial Force Calculation

We suppose that the value $z = \mu z_0$ is given and solve numerically the problem in the zeroth approximation (48), (49) with the additional condition $z_1 = z_0$. As a result we find the solution

$$
\mathbf{x}^0 = \mathbf{x}^0(s_0, z_0) \qquad \text{and} \qquad C^0 = C^0(z_0) \tag{57}
$$

The value C^0 for which $dC^0/dz_0 = 0$ corresponds according to relation (43) to the limit force in the zeroth approximation. To find the limit force more exactly we have two possibilities. Firstly we can numerically solve system (37) with the condition $z = \mu z_0$. Secondary we can study the first approximation (51), (53) of system (37) and find $C = C^0 + \mu C^1$. This way enables us to investigate the dependency of C on the shell elastic parameters.

The homogeneous problem (51), (53) has a non-zero solution, and the compatibility condition for the inhomogeneous problem has the following form

$$\int_{-\infty}^{0} \mathbf{g}\,\mathbf{x}^{\mathrm{c}}\,d\xi = C^1 \int_{-\infty}^{0} \theta^* \cos\theta^0\,d\xi \qquad \text{with} \qquad \dot{\mathbf{x}}^{\mathrm{c}} = -\mathbf{L}^{\mathrm{T}}\mathbf{x}^{\mathrm{c}} \tag{58}$$

or

$$\int_{-\infty}^{0} (g_1\varepsilon_2^* - g_2U^* + g_3\theta^* - g_4M_1^*)\,d\xi = C^1 \int_{-\infty}^{0} \theta^* \cos\theta^0\,d\xi \qquad \text{with} \qquad \dot{x}^* = \mathbf{L}\,\mathbf{x}^* \tag{59}$$

Here $^{\mathrm{T}}$ denotes transposition and $\mathbf{x}^* = \{U^*, \varepsilon_2^*, M_1^*, \theta^*\}$. From relation (59) the value C^1 may be found. As follows from the system (48), (49) the limit value C^0 depends only on two parameters, θ_0 and ν_1. In Figure 4 the dependency $C^0(\theta_0)$ for $\nu_1 = 0$ and $\nu_1 = \pm 0.3$ is presented. The relation

$$\eta = \frac{C^0}{C^*} \qquad \text{with} \qquad C^* = -2(1+\nu_1)\sin^2\theta_0 \tag{60}$$

is pictured. Here C^* due to relation (43) corresponds to the critical value of the force P. Indeed if we linearize system (37) near the membrane solution (44), $\mathbf{x}^{\mathbf{m}}$, and freeze the coefficients then we reduce the problem to the equation

$$\frac{d^4\hat{\theta}}{ds_0^4} - \left(\frac{C}{\sin\theta_0} + 2\nu_1\sin\theta_0\right)\frac{d^2\hat{\theta}}{ds_0^2} + \hat{\theta}\sin^2\theta_0 = 0 \qquad \text{with} \qquad \hat{\theta} = \theta - \theta_0 \tag{61}$$

Equation (61) has a solution $\hat{\theta} = a\sin(\lambda s_0)$ for $C = C^*$. It is interesting to note that the value C^* depends on the parameter ν_1 which takes into account the non-symmetry of the shell cross section. If we pull out the shell then parameter ν_1 changes its sign and the critical load may also essentially change. The fact that $\eta < 1$ (see Figure 4) marks the decreasing of the critical load due to the angle at $s_0 = 0$ (see Figure 3).

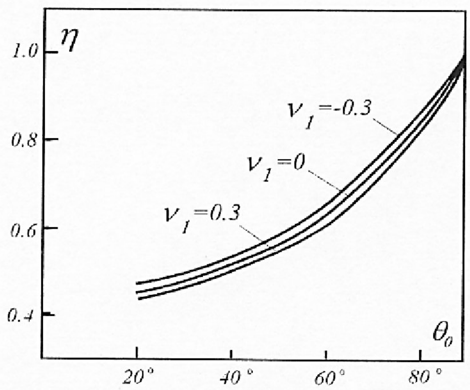

Figure 4. The Dependency $\eta = C^0/C^*$ on the Angle θ_0

The limit force parameter C_0 depends only on the distribution of the elastic moduli E and ν through the shell thickness and on the geometrical non-linearity of the problem. The correcting parameter C^1 may be calculated from equation (59) and it depends also on the nonlinear elastic moduli α_1, α_2, α_3 through the coefficients d_j.

10 The Two-layered Shell

Here we present some numerical results for the compound shell shown in Figure 3. The lower edge $s_0 = s_1$ is clamped. The upper edge $s_0 = -s_1$ is attached to the rigid body which is under action of the axial compression P. The other external forces are absent.

We study the two-layered shell. As scales we accept $R = r_0(0)$ and the Young modulus E_0 of the more stiffened material. The dimensionless parameters are: the relative shell thickness $h = 0.01$, the angle $\theta_0 = 45°$, the generatrix length $s_1 = 1/\sqrt{2}$ (see Figure 3), the layers thickness $h^{(1)} = h/6$, $h^{(2)} = 5h/6$, the Young moduli $E^{(1)} = E_0$, $E^{(2)} = E_0/5$, the Poisson ratios $\nu^{(1)} = 0.25$, $\nu^{(2)} = 0.5$. The material (1) is supposed to be linearly elastic, therefore the nonlinear elastic coefficients are equal to zero $\alpha_1^{(1)} = \alpha_2^{(1)} = \alpha_3^{(1)} = 0$. The material (2) is supposed to be incompressible and has the elastic potential

$$\Phi = G^{(2)} I_2 + E^{(2)} I_3 \alpha_3^{(2)} \qquad \text{with} \qquad G^{(2)} = E^{(2)}/3 \qquad \alpha_3^{(2)} = -8E^{(2)}/9 \tag{62}$$

the same as in the paper (Tovstik, 1999). As it was shown in this paper the coefficients β_j in relations (17) for this material are $\beta_1^{(2)} = -14E^{(2)}/9$, $\beta_2^{(2)} = -2E^{(2)}/3$, $\beta_3^{(2)} = -8E^{(2)}/9$. Now by using relations (17), (21), (26) and (33) we find the coefficients (39) and (40) in system (37). We find successively

$$K_0 = 0.400 E_0 h \quad K_2 = 0.038 E_0 h^3 \quad \mu = 0.0578 \quad \nu_0 = 0.389 \quad \nu_1 = 0.109 \quad \nu_2 = 0.407$$

$$d_1 = -0.776 \quad d_2 = -0.162 \quad d_3 = -1.449 \quad d_7 = -0.618 \quad d_8 = -1.539 \quad d_9 = -6.251$$

With these data system (37) was solved numerically. In Figure 5 in dimensionless variables the dependency of the axial force, P, on the shell shortening in axial direction, z, is presented. The limit point is marked by a dot. For the limit point $C = -0.562$, $z = -0.565$. From the approximate system (37) we get $C^0 = -0.581$ which is a good approximation for C.

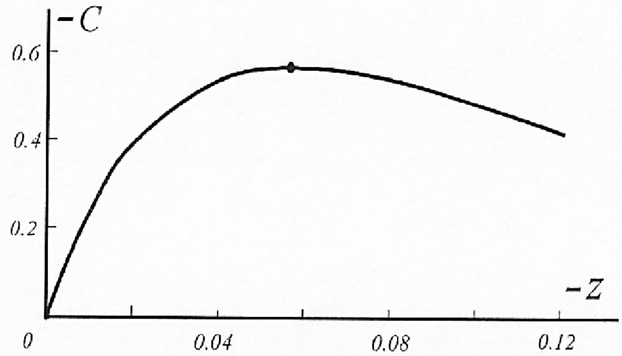

Figure 5. The Dependency $C(z)$

11 Conclusions

The problem of deriving two-dimensional shell theories from the three-dimensional theory of elasticity is constantly a point of attention of scientists. The first way which begins with Euler, Kirchhoff and Love consists in the acceptance of some hypotheses about the distribution of strains and stresses in the thickness direction. The second way consists of asymptotic expansions based on the small shell thickness. This way has two branches: the direct asymptotic solution of the three-dimensional (linear or non-linear) equations of the theory of elasticity (see Goldenveizer (1976), Goldenveizer et al. (1993), Goldenveizer (1994), Tovstik (1996), Tovstik (1997a), Agalovian (1997), Aslanyan et al. (2000) and others) and the variational approach based on the asymptotic simplification of the three-dimensional elastic potential energy (see Berdichevski (1983), Ciarlet (1998), Lods and Miara (1998), Miara (1998), this paper and others). Both these asymptotic approaches must give results of the same asymptotic exactness. It is verified for the problem studied in this paper. In the case when the elastic moduli do not depend on the normal coordinate these methods give the same expressions for the stress resultants and the stress couples.

Asymptotic expansions are based on the small shell thickness and on the following scaling of the independent variables. For linear problems the various scales lead to various two-dimensional shell models: membrane theory, flexible theory, moment theory. For non-linear problems an additional parameter, which marks the level of strains and stresses, appears. In this paper the space scaling is the same as in paper by Aslanyan et al. (2000) and the assumptions (10) about the level of strains ($\varepsilon \sin \sqrt{h}$) are accepted. For the large strains ($\varepsilon \sim 1$) the strict derivation of the constitutive relations from the three-dimensional theory of elasticity is apparently absent. There are the elasticity relations (see Chernykh, 1986) obtained by using the hypothesis similar to the Kirchhoff–Love one. For small (of the order of $\sqrt{h}$) deformations the comparison with formulas (24) gives the difference in the nonlinear summands depending on bending. The membrane part of the constitutive relations coincides with the relative error of the order h.

For multi-layered shells with non-symmetric cross-section the neutral surface does not exist in the general case and even in the linear approximation the constitutive relations (24) differ from relations (1). It is impossible to separate stretching and bending deformations. For multi-layered (especially for three-layered) shells often the stiffness of layers differs considerably. It is desirable in future to introduce in the asymptotic analysis an additional small parameter equal to the relation of these thicknesses. Also it is interesting to study for the multi-layered shells the large post-buckling deflections for which one part of the deformed shell is close to its mirror image reflected from the plane perpendicular to the shell axis (see Tovstik, 1996).

Acknowledgment

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Literature

1. Agalovian, L.A.: Asymptotic Theory of Anisotropic Plates and Shells. Moscow. Nauka, (1997), 414 p. (in Russian)

2. Aslanyan, A.G.; Movchan, A.B. and Selsil, 0.: Universal Asymptotic Algorithm for Elastic Thin Shells. Euro. J. Appl. Math., vol. 11, (2000), pp. 573–594.

3. Axelrad, E.L.: Shell Theory and its Specialized Branches. Int. J. Solids and Structures, vol. 37, (2000), pp. 1425–1451.

4. Berdichevski, V.L.: Variational Principles of Continuum Mechanics. Moscow, Nauka, (1983).

5. Chernykh, K.F.: Nonlinear Theory of Elasticity in Engineering Design. Leningrad, Mashinostroenie, (1986), 336 p. (in Russian)

6. Ciarlet, P.G.: Introduction to Linear Shell Theory. Ser. Appl. Math., Paris, (1998), 184 p.

7. Evkin, A.Ju. and Korovaitsev, A.V.: Asymptotic Analysis of the Post-buckling Stress-strain State of Shells of Revolution at Intense Flexion. Mech. Solids, vol. 27, no 1, (1992).

8. Goldenweizer, A.L.: Theory of Thin Elastic Shells. Moscow, Nauka, (1976), 512 p. (in Russian)

9. Goldenveizer, A.L.: Algorithms of the Asymptotic Construction of the Linear Two-dimensional Theory of Thin Shells and the Saint-Venant's Principle. J. Appl. Math. Mech., vol. 58, no 6, (1994).

10. Goldenveizer, A.L.; Kaplunov, Ju.D.; Nolde, E.V.: On the Timoshenko–Reissner Type Theories of Plates and Shells. Intern. J. Solids and Structures, vol. 30, no 5, (1993), pp. 675–694.

11. Kriegsmann, G.A. and Lange, C.G.: On Large Axisymmetrical Deflection States of Spherical Shells. J. of Elasticity, vol. 10, no 2, (1980), pp. 179–192.

12. Lods, V. and Miara, B.: Nonlinearly Elastic Shell Models: A Formal Asymptotic Approach. II. The Flexural Model. Arch. Rat. Mech. Anal., vol. 142, (1998), pp. 355–374.

13. Miara, B.: Nonlinearly Elastic Shell Models: A Formal Asymptotic Approach I. The Membrane Model. Arch. Rat. Mech. Anal., vol. 142, (1998), pp. 331–353.

14. Novozhilov, V.V. and Finkelshtein, R.M.: On the Error of the Kirchhoff–Love Hypothesis in the Shell Theory. Prikl. Mat. Mekh., vol. 7, no 5, (1943), pp. 331–340. (in Russian)

15. Pietraszkiewicz, W.: Geometrically Nonlinear Theories of Thin Elastic Shells. Adv. Mech., vol. 12, (1989), pp. 51–130.

16. Tovstik, P.E.: The Post-buckling Axisymmetrical Deflections of Thin Shells of Revolution under Axial Loading. Techn. Mech., Band 16, Heft 2, (1996), pp. 117–132.

17. Tovstik, P.E.: Axially Symmetric Deformation of Thin Shells of Revolution Made of a Non-linearly Elastic Material. J. Appl. Maths. Mechs, vol. 61, no. 4, (1997a), pp. 639–651.

18. Tovstik, P.E.: Asymptotic Methods on the Problem of Large Axisymmetric Deflections of the Thin Elastic Shell of Revolution. In: Integral Methods in Sci. and Eng., Longman, (ed. Constanda, Saranen, and Seikkala), vol.1, (1997b).

19. Tovstik, P.E.: Axisymmetrical Deformation of Shells of Revolution. In: Mechanics of Shells and Plates in XXI Sentury, Saratov Techn. Univ., (1999), pp. 152–174.

20. Tovstik, P.E. and Smirnov A.L.: Asymptotic Methods in the Buckling Theory of Elastic Shells. World Scientific. Series A, vol. 4. Singapore, New Jersey, London, Hong Kong, (2001), 348 p.

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