TECHNISCHE MECHANIK, Band 22, Heft 4, (2002), 283-290
Manuskripteingang: 26. August 2002

Orbit Transfer by Means of a Ward Spiral

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The Ward spiral occurs as a result of the study of the effects of drag on the orbit of a satellite. The Ward spiral is also suitable as a climb path when transferring from a lower to a higher orbit, if both orbits are circular. A Ward transfer to a larger orbit is described in detail and compared to the well-known Hohmann transfer, and it is shown that a Ward transfer can have the advantage of a shorter transfer time.

1 Introduction

In a study of the effects of a constant drag on a point satellite in a circular orbit at high altitude Ward (2000) has shown that the altitude change rate can be integrated in closed form. The result is a spiral, which is valid for as many loops, as the basic simplifying assumptions are close to being realistic. A Ward spiral represents the actual orbit the better the more closely the shape of each loop resembles a circle, or in other words the smaller the drag. Strictly speaking, the satellite does after all no longer travel on a perfect circle when it is experiencing drag.

When a satellite is subjected to a forward thrust, a similar train of thought leads to an outward Ward spiral, which becomes the path along which the satellite climbs to a higher altitude. In principle, both inward and outward Ward spirals can serve as a transfer orbit to a lower or higher altitude, respectively.

In the present paper, the transfer from a circular base orbit to a higher circular target orbit is studied in detail.

2 The Inward Ward Spiral

The inward Ward spiral arises as the integral of the orbit radius change rate (Rimrott and Salustri, 2001) of a satellite subjected to drag. In parameter form, it is given by

$$r = \frac{r_0}{\left(1+\frac{D}{m}\sqrt{\frac{r_0}{\mu}}\,t\right)^2} \tag{1}$$

where

r_0 = initial orbit radius, km

D = drag force, kN

m = point satellite mass, kg

μ = 398 601 km^3/s^2 for the Earth as point master

t = time, s

Equation (1) can also be written in polar form

$$r = \frac{r_0}{(1+c\,\theta)^2} \tag{2}$$

See the plot of Figure 1. The coefficient

$$c = \frac{D}{m}\frac{r_0^2}{\mu} \tag{3}$$

The velocity of a point satellite on an inward Ward spiral increases at the rate of

$$\dot{v} = \frac{D}{m} \tag{4}$$

where $\dot{v}$ is in the direction of motion, and D is in the opposite direction (Figure 2), i.e., the drag causes a speeding up of the satellite as it descends to a lower altitude.

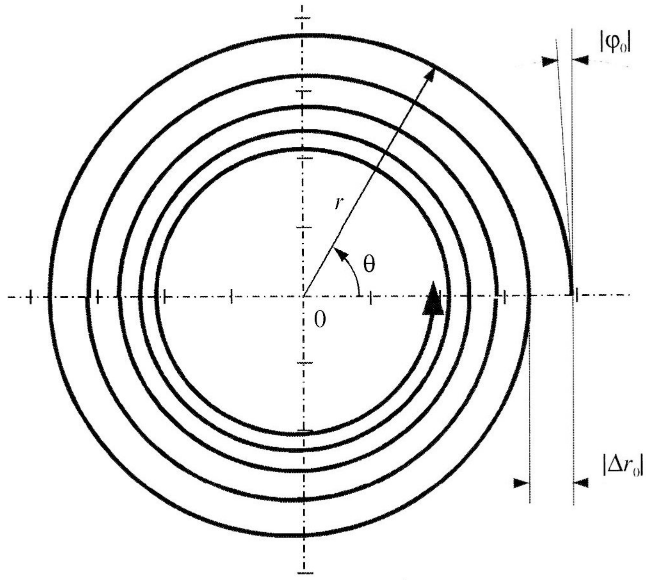

Figure 1. Five Loops of an Inward Ward Spiral with $c = 0.012$

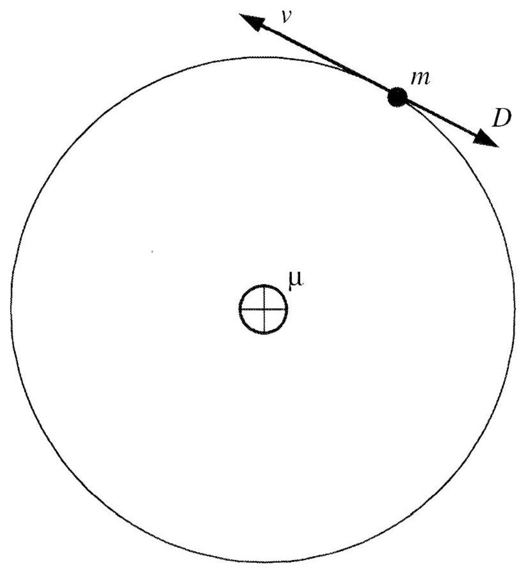

Figure 2. Satellite Velocity and Drag

3 Hohmann Transfer Revisited

The Hohmann transfer requires a powered velocity kick Δv_1 as the satellite moves from circular base orbit of radius r_1 and velocity

$$v_1 = \sqrt{\frac{\mu}{r_1}}$$

to a transfer ellipse of major axis $r_1 + r_2$ in the ellipse's perigee. The velocity kick required is

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) \tag{5}$$

Thereafter, the satellite coasts along the transfer ellipse until it reaches the apogee. The change of speed along the elliptical transfer orbit (Figure 3) is

$$\Delta v_{ell} = -\left(\sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}\right) \tag{6}$$

At the apogee, a second velocity kick is required to insert the satellite into the circular target orbit of radius r_2 and velocity

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$

The second velocity kick is

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right) \tag{7}$$

The sum of the velocity kicks amounts to

$$\Delta v_1 + \Delta v_{ell} + \Delta v_2 = -\left(\sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{r_2}}\right) = -(v_1 - v_2) \tag{8}$$

as is to be expected.

The sum of the two powered velocity kicks is known as the *characteristic velocity*

$$\Delta v_{characteristic} = \Delta v_1 + \Delta v_2 \tag{9}$$

The time elapsed between leaving the base orbit and arrival at the target orbit is

$$t_H = \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}} \tag{10}$$

As an example, let the task to be move a satellite (m = 100 000 kg) from a circular orbit with radius r_1 = 6600 km to a circular orbit with r_2 = 7000 km

$$\Delta v_1 = 7.771\left(\sqrt{\frac{2(7000)}{6600 + 7000}} - 1\right) = 0.113451\,\text{km/s}$$

$$\Delta v_{ell} = -\left(7.771\sqrt{\frac{2(7000)}{6600+7000}} - 7.546\sqrt{\frac{2(6600)}{6600+7000}}\right) = -0.450025 \text{ km/s}$$

$$\Delta v_2 = 7.546\left(1-\sqrt{\frac{2(6600)}{6600+7000}}\right) = 0.111799 \text{ km/s}$$

The sum of the three velocity kicks is

$$\Delta v_1 + \Delta v_{ell} + \Delta v_2 = -0.225000 \text{ km/s}$$

and the transfer time (10) is

$$t_H = \pi\sqrt{\frac{(6600+7000)^3}{8(398\,601)}} = 2.790.256 \text{ s} = 46.504 \text{ min}$$

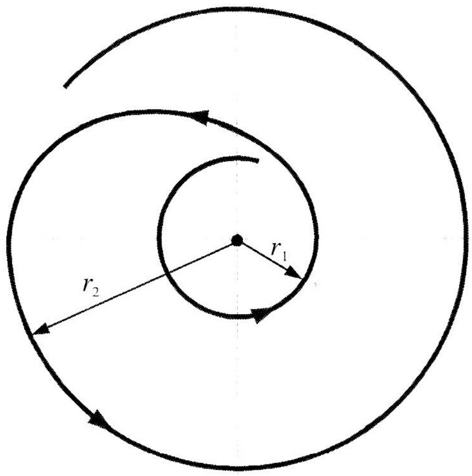

Figure 3. Hohmann Transfer

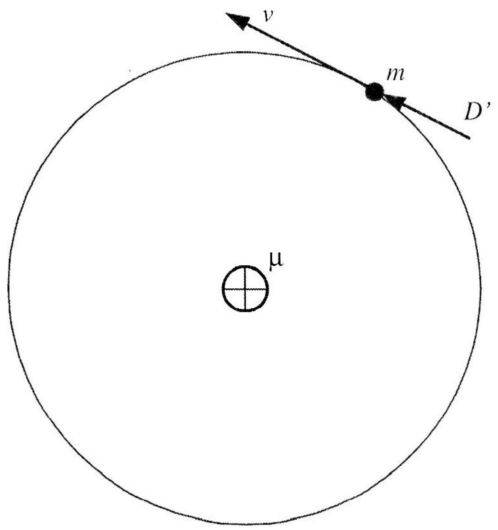

Figure 4. Satellite Velocity and Forward Thrust

4 The Outward Ward Spiral

Let us assume a circular orbit of radius r of a point satellite of mass m, traveling at a velocity

$$v = \sqrt{\frac{\mu}{r}}$$

and subject to a constant thrust D' in tangential direction (Figure 4). Then the power of the thrust is

$$P = D'v = D'\sqrt{\frac{\mu}{r}} \tag{11}$$

The orbital energy (Rimrott, 1989) of a point satellite in the gravitational field of a point master ($\mu = 398\,601$ km^3/s^2 for the Earth) is

$$E = -\frac{\mu m}{2a} \tag{12}$$

where a is the semi-major axis of the orbit ellipse. Differentiating with respect to time we obtain the power

$$\dot{E} = P = \frac{\mu m}{2a^2}\dot{a} \tag{13}$$

With $r = a$ for a near-circular orbit, equation (13) becomes

$$P = \frac{\mu m}{2r^2}\dot{r} \tag{14}$$

Equating equations (11) and (14) gives us the climb rate

$$\dot{r} = \frac{2D'}{\sqrt{\mu}m}r^{3/2} \tag{15}$$

representing a differential equation which can be integrated in closed form, with $r = r_1$ for $t = 0$ as initial conditions. The result is a parameter equation of the Ward spiral

$$r = \frac{r_1}{\left(1 - \frac{D'}{m}\sqrt{\frac{r_1}{\mu}}t\right)^2} \tag{16}$$

which we will refer to as an *outward* Ward spiral, in contradistinction to the *inward* Ward spiral of equation (1).

Equation (16) written in polar form reads

$$r = \frac{r_1}{(1 - c'\theta)^2} \tag{17}$$

An outward Ward spiral is depicted in Figure 5. Coefficient c' is defined as

$$c' = \frac{D'}{m}\frac{r_1^2}{\mu} \tag{18}$$

The velocity of a point satellite on an outward Ward spiral changes according to

$$\dot{v} = -\frac{D'}{m} \tag{19}$$

Since D' and m are both positive quantities, the acceleration $\dot{v}$ is negative. Thus, the satellite slows down as it climbs along an outward Ward spiral towards a higher altitude.

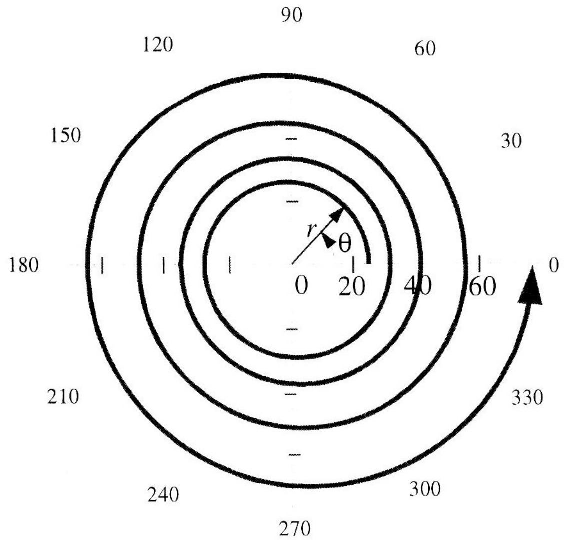

Figure 5. Four Loops of an Outward Ward Spiral with $c' = 0.0168$

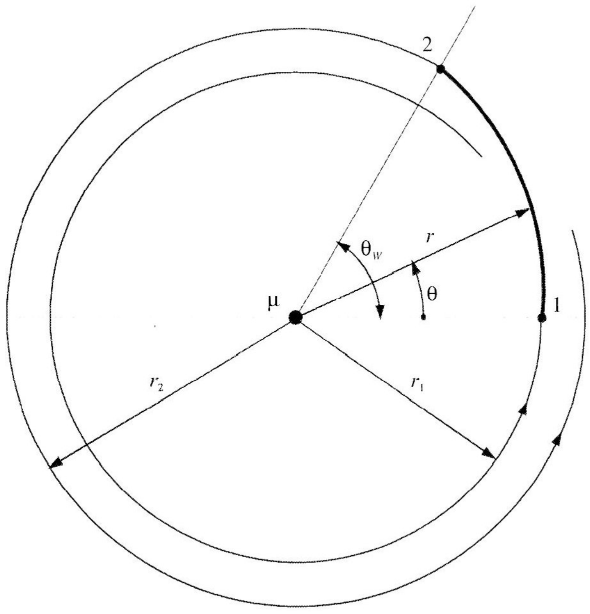

Figure 6. Ward Transfer between Points 1 and 2

5 Transfer along a Ward Spiral

Let us now investigate the transfer of a point satellite of constant mass m from a circular orbit of radius r_1 and velocity

$$v_1 = \sqrt{\frac{\mu}{r_1}}$$

to a higher circular orbit of radius r_2 and velocity

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$

Available is an onboard jet engine that delivers a constant thrust D' (Figure 4). Then the satellite will move along a Ward spiral (Figure 6)

$$r = \frac{r_1}{\left(1 - \frac{D'}{m}\sqrt{\frac{r_1}{\mu}}\,t\right)^2} \tag{20}$$

until it reaches the new orbit radius r_2 at time t_w

$$r_2 = \frac{r_1}{\left(1 - \frac{D'}{m}\sqrt{\frac{r_1}{\mu}}\,t_w\right)^2} \tag{21}$$

At that moment, the jet engine is turned off, causing $\dot{r} = 0$ from equation (15), and the satellite now moves on a new circular orbit.

From equation (21), the time to traverse the Ward spiral orbit is

$$t_w = \frac{m}{D'}\sqrt{\frac{\mu}{r_1}}\left(1 - \sqrt{\frac{r_1}{r_2}}\right) = \frac{m}{D'}(v_1 - v_2) \tag{22}$$

The angle swept out during a Ward transfer, from equations (16), (17), (18) and (22) is

$$\theta_w = \frac{m}{D'}\frac{\mu}{r_1^2}\left(1 - \sqrt{\frac{r_1}{r_2}}\right) \tag{23}$$

For our example of a Ward transfer, we shall aim for a short transfer time. To this end, we select a thrust of $D' = 25\ \text{kN}$. The transfer spiral is then (Figure 6)

$$r = \frac{6600}{\left(1 - \frac{25}{100\,000}\sqrt{\frac{6600}{398\,601}}\,t\right)^2} = \frac{6600}{(1 - 0.000032\,t)^2}$$

The target orbit is reached when $r = r_2 = 7000\ \text{km}$. The transfer time is

$$t_w = \frac{100\,000}{25}\sqrt{\frac{398\,601}{6600}}\left(1 - \sqrt{\frac{6000}{7000}}\right) = 903.167\ \text{s} = 15.053\ \text{min}$$

which is shorter than t_H. The velocity change rate is

$$\dot{v} = \frac{D'}{m} = -\frac{25}{100\,000} = 0.00025 \text{ km/s}^2$$

giving a velocity change during the transfer of

$$\Delta v = \dot{v} t_w = -0.225 \text{ km/s}$$

as required. It represents the difference between the two orbital velocities

$$v_1 = 7.771 \text{ km/s} \qquad \text{and} \qquad v_2 = 7.546 \text{ km/s}$$

The polar angle θ_w (Figure 6) swept out is

$$\theta_w = \frac{100\,000}{25} \frac{398\,601}{6600^2}\left(1 - \sqrt{\frac{6600}{7000}}\right) = 1.061 \text{ rad} = 60.8°$$

6 Conclusion

The transfer between circular orbits by means of the Ward spiral has been introduced and described for a transfer from a circular base orbit to a higher target orbit. The Ward transfer has been compared to the standard Hohmann transfer. Apart from differing transfer orbit shapes, a substantial difference between the two is that of transfer times, which for a Ward transfer is shown to be dependent on the thrust, and can be noticeably shorter.

Literature

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