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Angular Momentum Collinearization of an Elastic Gyro with Hysteresis

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In the present paper it is shown in some detail how the energy dissipation in a deforming elastic gyro leads to a collinearization of the angular momenta, according to the collinearity principle.

1 Introduction

The collinearity principle (Rimrott, 1998) requires that independent dissipative gyro systems lose mechanical energy by adjusting their angular momenta such that they exhibit unidirectional collinearity while the resultant angular momentum remains constant.

Gyrodynamics is often referred to as Rigid Body Dynamics and that not without reason. Clear-cut results are – unfortunately – typically only obtainable by assuming rigidity of the bodies involved. The present paper attempts to include deformation of the gyro body in the analysis, in particular as it pertains to attitude drift.

Deformation of an elastic gyro is usually linearly proportional to the force acting. In an independent, i.e. torque-free, gyro there are only centrifugal and (reversed) Coriolis forces acting. Centrifugal forces lead typically to small increases in size, which have an only insignificant effect on the angular velocity components. Coriolis forces on the other hand can cause an attitude drift, if these forces lead to a deformation component perpendicular to the force, due to the material's hysteresis.

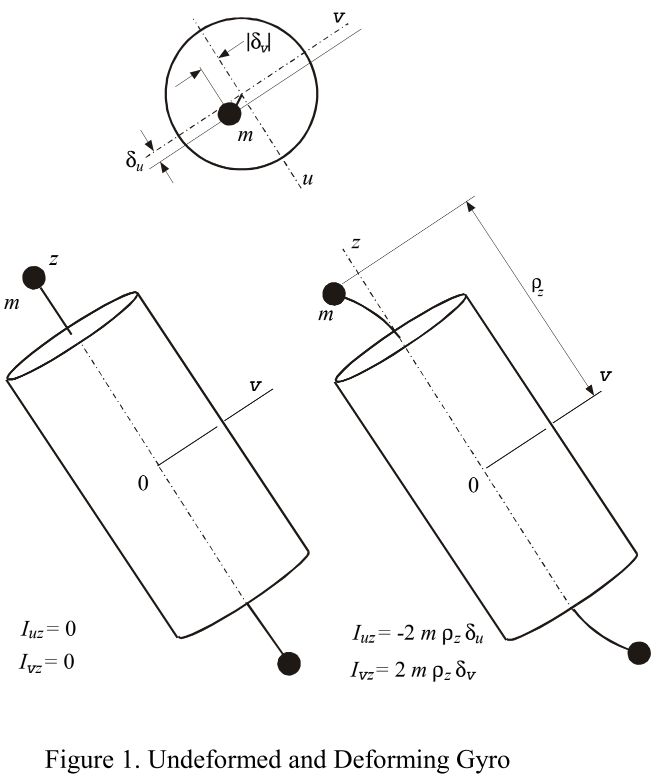

Figure 1. Undeformed and Deforming Gyro

In the subsequent analysis we use a rigid gyro as reference, and superimpose small size and shape changes due to the elasticity including hysteresis of the material. We specifically allow for small shape changes due to the material's hysteresis. Then we investigate the attitude behaviour that these small shape changes cause.

We realize that for a torque-free rigid gyro its kinetic energy is its sole mechanical energy. Its kinetic energy can readily be shown to be dependent on the gyro's attitude, expressed by the nutation angle, and to exhibit maxima and minima. A kinetic energy minimum defines a stable attitude. Any reduction in kinetic energy on the way to a stable attitude can only come about by internal energy losses. The sole possible energy sink is the gyro's hysteresis, since real gyros are made of elastic materials with hysteresis.

The assignment is then to find a deformation associated with the gyro structure's hysteresis, and to study its effect on the dynamics, i.e. the attitude of the gyro.

2 Inertia Tensor and Angular Velocity

We choose an axisymmetric gyro, with $A > C$, for our analysis. The rigid reference gyro has an inertia tensor of

$$[I]_{rigid} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \tag{1}$$

in $0uvz$ coordinates which rotate at

$$\mathbf{\Omega} = [\mathbf{e}_u \mathbf{e}_v \mathbf{e}_z] \begin{bmatrix} \omega_u \\ \omega_v \\ \dfrac{C}{A}\omega_z \end{bmatrix} \tag{2}$$

When deformed by a small amount the inertia tensor (1) changes to

$$[I] = \begin{bmatrix} A+\Delta A & I_{uv} & I_{uz} \\ I_{uv} & A+\Delta B & I_{vz} \\ I_{uz} & I_{vz} & C+\Delta C \end{bmatrix} \tag{3}$$

We recognize that $\Delta A << A,\ \Delta B << A,\ \Delta C << C$ such that $\Delta A,\ \Delta B,\ \Delta C$ can be neglected. Because of symmetry we set $I_{uv} = 0$. That leaves (Figure 1)

$$[I] = \begin{bmatrix} A & 0 & I_{uz} \\ 0 & A & I_{vz} \\ I_{uz} & I_{vz} & C \end{bmatrix} \tag{4}$$

If we were dealing with a rigid gyro (1) the angular velocity would simply be

$$\boldsymbol{\omega}_{rigid} = [\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z] \begin{bmatrix} 0 \\ \omega_v \\ \omega_z \end{bmatrix} \tag{5}$$

Because of the new inertia tensor (4) we expect that there are associated small changes in the angular velocity components of the undeformed gyro, such that

$$\boldsymbol{\omega} = [\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z]\begin{bmatrix} 0+\Delta\omega_u \\ \omega_v+\Delta\omega_v \\ \omega_z+\Delta\omega_z \end{bmatrix} \tag{6}$$

For simplicity we write $\Delta\omega_u = \omega_u$. After recognizing that $\Delta\omega_v << \omega_v$ and that $\Delta\omega_z << \omega_z$ we neglect small ratios and write

$$\boldsymbol{\omega} = [\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z]\begin{bmatrix} \omega_u \\ \omega_v \\ \omega_z \end{bmatrix} \tag{7}$$

3 The Global Angular Momentum

With the help of equations (4) and (7) we find for the constant global angular momentum of the deforming gyro

$$\mathbf{H} = [\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z]\begin{bmatrix} A\omega_u + I_{uz}\omega_z \\ A\omega_v + I_{vz}\omega_z \\ C\omega_z + I_{uz}\omega_u + I_{vz}\omega_v \end{bmatrix} \tag{8}$$

The v-component is

$$A\omega_v\left(1 + \frac{I_{vz}\omega_z}{A\omega_v}\right) \tag{9}$$

And since $I_{vz} << A$ we decide to neglect the second term in the brackets. A similar consideration applies to the z-component

$$C\omega_z\left(1+\frac{I_{uz}\omega_u}{C\omega_z}+\frac{I_{vz}\omega_v}{C\omega_z}\right) \tag{10}$$

Since $I_{uz} << C$ and $I_{vz} << C$ we can neglect the second and the third term in the brackets.

For the u-component

$$A\omega_u\left(1+\frac{I_{uz}\omega_z}{A\omega_u}\right) \tag{11}$$

on the other hand we note that the ratio may be is indeterminate and thus possibly finite, since $I_{uz} << A$ and $\omega_u << \omega_z$, and consequently we retain the second term in the brackets. As will be shown in the present paper it is the very term that describes the collinearization process of the angular momenta.

The constant global angular momentum ist consequently

$$\mathbf{H} = [\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z]\begin{bmatrix} A\omega_u + I_{uz}\omega_z \\ A\omega_v \\ C\omega_z \end{bmatrix} \tag{12}$$

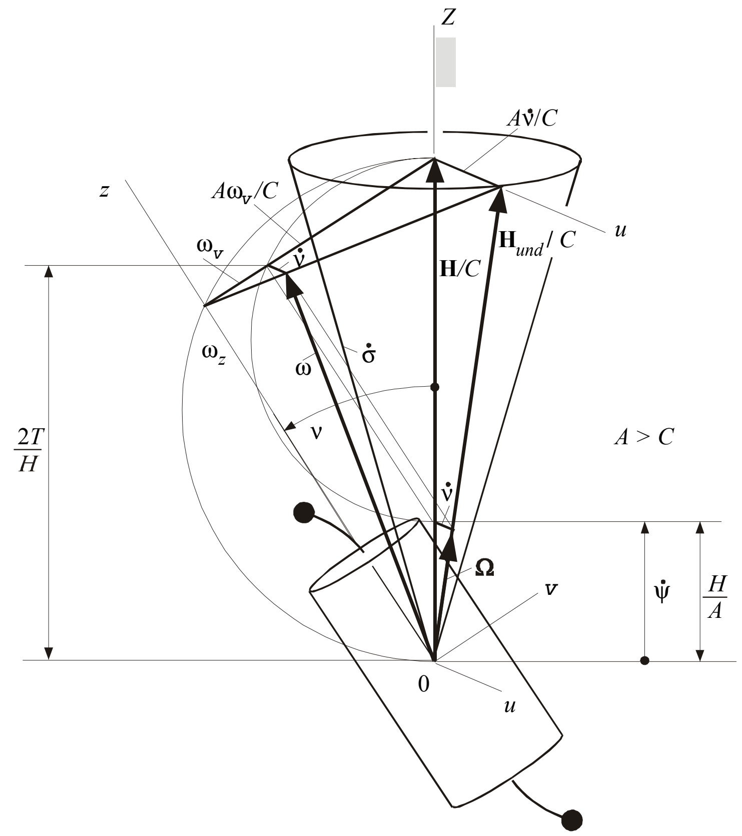

Figure 2. Deforming Gyro and Floating $0uvz$ Coordinate System

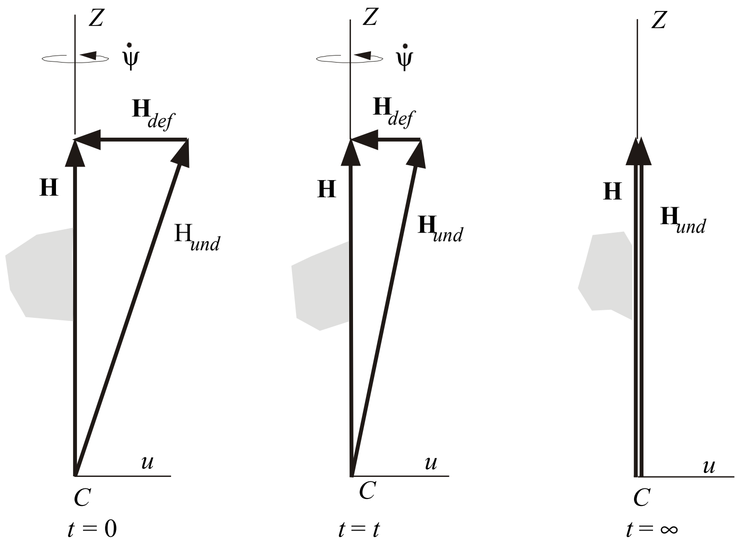

Figure 3. The Angular Momenta

The $0uvz$ coordinate system has been chosen such that

$$\mathbf{H} = [\mathbf{e}_u \; \mathbf{e}_v \; \mathbf{e}_z] \begin{bmatrix} 0 \\ A\omega_v \\ C\omega_z \end{bmatrix} \tag{13}$$

As a result we conclude that

$$A\omega_u + I_{uz}\omega_z = 0 \tag{14}$$

or

$$I_{uz} = -\frac{\omega_u}{\omega_z} A \tag{15}$$

And (Figure 2) we have

$$\omega_u = \dot{v} \tag{16}$$

We separate equation (12) into

$$\mathbf{H} = \mathbf{H}_{und} + \mathbf{H}_{def} \tag{17}$$

and write for the undeformed portion

$$\mathbf{H}_{und} = [\mathbf{e}_u \; \mathbf{e}_v \; \mathbf{e}_z] \begin{bmatrix} A\omega_u \\ A\omega_v \\ C\omega_z \end{bmatrix} \tag{18}$$

and for the angular momentum resulting from the deformation (Figure 3)

$$\mathbf{H}_{def} = [\mathbf{e}_u \; \mathbf{e}_v \; \mathbf{e}_z] \begin{bmatrix} I_{uz}\omega_z \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

4 The Deformation Angular Momentum

The influence of the deformation on the angular momentum, which affects the dynamics of the gyro profoundly, is measured by the inertia product I_{uz} which leads to an angular momentum (19).

The inertia product can be shown (Rimrott and Yu, 1989) to be

$$I_{uz} = -\beta\omega_v \, \omega_z \, \dot{\sigma} \tag{20}$$

provided we deal with a relatively stiff gyro whose lowest structural eigenfrequency is higher than any of the exciting frequencies ω_v, ω_z, or $\dot{\sigma}$, or in other words no resonance vibration occurs, i.e. there is no violent response of the gyro structure, during attitude drift.

The coefficient β is a measure of the internal energy dissipation with the dimension Ws[6]. The angular velocity components are

$$\omega_v = \frac{H}{A}\sin v \tag{21}$$

$$\omega_z = \frac{H}{C}\cos v \tag{22}$$

$$\dot{\sigma} = \frac{A-C}{A}\omega_z = \frac{A-C}{AC}H\cos v \tag{23}$$

such that

$$\mathbf{H}_{def} = [\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z]\begin{bmatrix} -\beta\frac{A-C}{A^2C^3}H^4\sin v\cos^3 v \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

5 The Collinearization Process

The internal energy dissipation causes a continuous reduction in the magnitude of the deformation angular momentum (24) as the nutation angle v approaches 90^0. From equations (15) and (16), and with the help of equations (20) to (23) we find that the nutation

$$\dot{v} = \beta\frac{A-C}{A^3C^3}H^4\sin v\cos^3 v \tag{25}$$

an equation that happens to be integrable in closed form if $\beta = \beta_0 =$ constant and results in

$$t = \frac{A^3C^3}{4\beta_0(A-C)H^4}\left(\tan^2 v - \tan^2 v_0 + \ln\frac{\tan^2 v}{\tan^2 v_0}\right) \tag{26}$$

where v_0 is the nutation angle at time $t = 0$.

6 The Kinetic Energy

We use as definition of the kinetic energy of the deforming gyro

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H} \tag{27}$$

and equations (7) and (12) to obtain

$$T = \frac{1}{2}\left(A\omega_u^2 + A\omega_v^2 + C\omega_z^2 + I_{uz}\omega_u\omega_z\right) \tag{28}$$

With the help of equation (14) we get

$$T = \frac{1}{2}A\omega_v^2 + \frac{1}{2}C\omega_z^2 \tag{29}$$

and write with the help of equations (21) and (22)

$$T = \frac{1}{2}\frac{H^2}{A}\left(1+\left(\frac{A}{C}-1\right)\cos^2 v\right) \tag{30}$$

At the beginning of the collinearization process $v = v_0$ and

$$T_i = \frac{1}{2}\frac{H^2}{A}\left(1+\left(\frac{A}{C}-1\right)\cos^2 v_0\right) \tag{31}$$

At the end we have $v = 90^\circ$ and

$$T_f = \frac{1}{2}\frac{H^2}{A} \tag{32}$$

The change in kinetic energy is

$$\Delta T = T_f - T_i = -\frac{1}{2}\frac{H^2}{A}\left(\frac{A}{C}-1\right)\cos^2 v_0 \tag{33}$$

It is negative, thus representing a loss of kinetic energy during the collinearization process. The rate of kinetic energy change is

$$\dot{T} = -\frac{A-C}{AC}H^2\dot{v}\sin v\cos v \tag{34}$$

7 The Energy Rates

We recall that the constant global angular momentum (13) of the deforming gyro is

$$\mathbf{H} = \left[\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z\right]\begin{bmatrix}0\\ A\omega_v\\ C\omega_z\end{bmatrix}$$

The undeformed gyro's angular velocity is, from equations (7) and (16),

$$\boldsymbol{\omega} = \left[\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z\right]\begin{bmatrix}\dot{v}\\ \omega_v\\ \omega_z\end{bmatrix} \tag{35}$$

The angular velocity of the rotating $0uvz$-coordinate system is, from equations (2) and (16),

$$\boldsymbol{\Omega} = \left[\mathbf{e}_u\ \mathbf{e}_v\ \mathbf{e}_z\right]\begin{bmatrix}\dot{v}\\ \omega_v\\ \dfrac{C}{A}\omega_z\end{bmatrix} \tag{36}$$

The energy dissipation rate (Rimrott, 1989) is given by

$$\dot{D} = \boldsymbol{\omega} \bullet (\boldsymbol{\Omega} \times \mathbf{H}) = \begin{vmatrix} \dot{v} & \omega_v & \omega_z \\ \dot{v} & \omega_v & \frac{C}{A}\omega_z \\ 0 & A\omega_v & C\omega_z \end{vmatrix} \tag{37}$$

and with ω_v and ω_z from equation (21) and (22) we eventually obtain for the energy dissipation rate

$$\dot{D} = \frac{A-C}{AC} H^2 \dot{v} \sin v \cos v \tag{38}$$

For each level of the nutation angle v we have then from equation (34) the kinetic energy rate, and for the potential (=elastic) energy rate we have

$$\dot{V} = 0 \tag{39}$$

because the elastic deformation does not change for the constant configuration of the rotating gyro in $0uvz$ configuration space.

Thus we conclude that

$$\dot{T} + \dot{V} + \dot{D} = 0 \tag{40}$$

8 The Energy Dissipation

The energy dissipation rate is given by equation (38). With the help of equation (25) we can write

$$\dot{D} = \beta \frac{(A-C)^2}{A^4 C^4} H^6 \sin^2 v \cos^4 v \tag{41}$$

By using equations (20) to (33) we can also write

$$\dot{D} = -\frac{A-C}{A^2 C^2} H^3 I_{uz} \sin v \cos^2 v \tag{42}$$

and obtain

$$I_{uz} = -\frac{A^2 C^2}{(A-C) H^3 \sin v \cos^2 v} \dot{D} \tag{43}$$

or

$$I_{uz} = -\frac{A}{(A-C)\omega_v \omega_z^2} \dot{D} \tag{44}$$

or

$$I_{uz} = -\frac{1}{\omega_v \omega_z \dot{\sigma}} \dot{D} \tag{45}$$

With the help of equation (20), we find eventually

$$\dot{D} = \beta \omega_v^2 \omega_z^2 \dot{\sigma}^2 \tag{46}$$

9 Internal Torques

The torque exerted on the deforming portion of the gyro is given by

$$\mathbf{M}_{def} = \dot{\mathbf{H}}_{def} = \overset{\circ}{\mathbf{H}}_{def} + \mathbf{\Omega} \times \mathbf{H}_{def} \tag{47}$$

We make use of equations (19) and (36) and find

$$\mathbf{M}_{def} = [\mathbf{e}_u\,\mathbf{e}_v\,\mathbf{e}_z]\begin{bmatrix} -A\ddot{v} \\ 0 \\ 0 \end{bmatrix} + \begin{vmatrix} \mathbf{e}_u & \mathbf{e}_v & \mathbf{e}_z \\ \dot{v} & \omega_v & \dfrac{C}{A}\omega_z \\ -A\dot{v} & 0 & 0 \end{vmatrix} \tag{48}$$

or

$$\mathbf{M}_{def} = [\mathbf{e}_u\,\mathbf{e}_v\,\mathbf{e}_z]\begin{bmatrix} -A\ddot{v} \\ -C\omega_z\dot{v} \\ A\omega_v\dot{v} \end{bmatrix} \tag{49}$$

The torque exerted on the phantom undeformed gyro is given by

$$\mathbf{M}_{und} = \dot{\mathbf{H}}_{und} = \overset{\circ}{\mathbf{H}} + \mathbf{\Omega} \times \mathbf{H}_{und} \tag{50}$$

From equations (18) and (36)

$$\mathbf{M}_{und} = [\mathbf{e}_u\,\mathbf{e}_v\,\mathbf{e}_z]\begin{bmatrix} A\ddot{v} \\ A\dot{\omega}_v \\ C\dot{\omega}_z \end{bmatrix} + \begin{vmatrix} \mathbf{e}_u & \mathbf{e}_v & \mathbf{e}_z \\ \dot{v} & \omega_v & \dfrac{C}{A}\omega_z \\ A\dot{v} & A\omega_v & C\omega_z \end{vmatrix} \tag{51}$$

The determinant vanishes and we can write

$$A\dot{\omega}_v = \dot{H}_v = (H\sin v)^{\cdot} = H\dot{v}\cos v = C\omega_z\dot{v} \tag{52}$$

and similarly

$$C\dot{\omega}_z = \dot{H}_z = (H\cos v)^{\cdot} = -H\dot{v}\sin v = -A\omega_v\dot{v} \tag{53}$$

such that

$$\mathbf{M}_{und} = [\mathbf{e}_u\,\mathbf{e}_v\,\mathbf{e}_z]\begin{bmatrix} A\ddot{v} \\ C\omega_z\dot{v} \\ -A\omega_v\dot{v} \end{bmatrix} \tag{54}$$

Comparing the two torques we see that

$$\mathbf{M}_{def} = -\mathbf{M}_{und} \tag{55}$$

or

$$\mathbf{M}_{def} + \mathbf{M}_{und} = 0 \tag{56}$$

i.e. the torques are internal, or in other words there is no external torque acting on the gyro during the collinearization process.

10 A Numerical Example

To convey a feeling for the relative magnitudes of the various quantities involved in the preceding calculations, let us look at a typical numerical example.

Let a torque-free gyro satellite be given, with

$$A = 3000 \text{ Ws}^3$$
$$C = 1000 \text{ Ws}^3$$
$$\nu = 30^\circ, \text{at } t = 0$$
$$\omega_\nu = 1 \text{ rad/s, at } t = 0$$
$$\beta = 0.002 \text{ Ws}^6$$

Then we find, for $t = 0$

$$\omega_z = \frac{A}{C \tan \nu} = \frac{3000}{1000 \tan 30^\circ} = 5.196 \text{ rad/s}$$

$$\dot{\sigma} = \frac{A-C}{A}\omega_z = \frac{3000-1000}{3000}5.196 = 3.464 \text{ rad/s}$$

$$\dot{\nu} = \beta\frac{\omega_\nu\omega_2^2\dot{\sigma}}{A} = 0.002\frac{\left(5.196^2\right)(3.464)}{3000} = 0.06235\left((10^{-3}\right)\text{rad/s} = 12.86 \text{ deg/h}$$

The angular momentum components of the undeformed gyro are

$$H_u = A\dot{\nu} = 3000\left(0.06235\right)10^{-3} = 0.187 \text{ Ws}^2$$

$$H_\nu = A\omega_\nu = 3000(1) = 3000 \text{ Ws}^2$$

$$H_z = C\omega_z = 1000\left(5.196\right) = 5196 \text{ Ws}^2$$

The magnitude of the angular momentum of the undeformed gyro is (Figure 3)

$$H_{und} = \sqrt{0.187^2 + 3000^2 + 5196^2} = 6000.000\,002\,914 \text{ Ws}^2$$

It is only slightly larger than the magnitude of the global angular momentum of the deforming gyro

$$H = \sqrt{H_\nu^2 + H_z^2} = \sqrt{3000^2 + 5196^2} = 6000 \text{ Ws}^2$$

The magnitude of the angular momentum of the deformation is

$$H_{def} = \left|I_{uz}\omega_z\right| = A\dot{\nu} = 3000(0.06235)10^{-3} = 0.187 \text{ Ws}^2$$

For the inertia product we find

$$I_{uz} = -\beta\omega_\nu\omega_z\dot{\sigma} = -0.002(1)\ 5.196\,(3.464) = -0.036 \text{ Ws}^3$$

which is obviously of much smaller magnitude than either inertia moment, A or C.

The kinetic energy is, at $t = 0$,

$$T_i = \frac{H^2}{2A}\left(1+\left(\frac{A}{C}-1\right)\cos^2 v_0\right) = \frac{6000^2}{2(3000)}\left(1+\left(\frac{3000}{1000}-1\right)\cos^2 30^\circ\right) = 15000 \text{ Ws}$$

At the end of the collinearization process, when $v = 90^\circ$, the kinetic energy is

$$T_f = \frac{H^2}{2A} = 6000 \text{ Ws}$$

The kinetic energy change due to collinearization is

$$\Delta T = T_f - T_i = -9000 \text{ Ws}$$

The kinetic energy rate, at $t = 0$, is

$$\dot{T} = -\frac{H^2}{A}\left(\frac{A}{C}-1\right)\dot{v}\sin v\cos v = -\frac{6000^2}{3000}\left(\frac{3000}{1000}-1\right)0.06135(10^{-3})\sin 30^\circ\cos 30^\circ$$

$$= -0.648 \text{ W}$$

The energy dissipation rate at $t = 0$, is

$$\dot{D} = \beta\,\omega_v^2\omega_z^2\dot{\sigma}^2 = 0.002\left(1^2\right)5.196^2\left(3.464^2\right) = 0.648 \text{ W}$$

The torque transmitted from the deformation to the undeformed gyro, at $t = 0$, has the components

$$M_u = A\ddot{v} = \text{negligible}$$

$$M_v = C\omega_z\dot{v} = 1000\,(5.196)\,0.062\,(10^{-3}) = 0.324 \text{ Ws}$$

$$M_z = -A\omega_v\dot{v} = -3000(1)\,0.062\,(10^{-3}) = -0.187 \text{ Ws}$$

and the magnitude

$$M_{und} = \sqrt{0^2+0.324^2+0.187^2} = 0.374 \text{ Ws}$$

It causes the tip of the $\mathbf{H}_{und}$-vector and the tail of the $\mathbf{H}_{def}$-vector to process at

$$\dot{\psi} = \frac{H}{A} = \frac{6000}{3000} = 2 \text{ rad/s}$$

about the **H**-vector (Figure 3).

11 Conclusion

In the preceding paper it has been shown, how a deformation due to an elastic gyro's hysteresis can be taken into account in establishing the collinearization process of the angular momenta. A constant global angular momentum is defined, consisting of the vector sum of an angular momentum of the basic undeformed gyro and the angular momentum of the deformation. It is then shown how the latter becomes smaller and smaller in the course of time until the angular momentum of the undeformed gyro becomes unidirectionally collinear with the global angular momentum.

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