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Buckling Analysis of Non-orthotropic Laminates by Means of B-Spline Functions

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Whereas the buckling of rectangular laminate plates with symmetrical layup and an effectively orthotropic behaviour is well understood, the situation gets more complicated when an arbitrary symmetrical layup is admitted. Then twist coupling has to be taken into account. This can be accomplished in a consistent way by an energetic variational approach. In doing so, the use of B-spline functions turns out to be very appropriate for the deflection representation in connection with various kinds of boundary conditions. Validating finite element analyses show the good efficiency of B-spline functions for the buckling analysis. The effect of twist coupling on the resultant buckling loads turns out to be considerable. It can be both conservative and non-conservative and should always be taken into account carefully.

1 Introduction

Within thin-walled light-weight laminate structures under inplane compressive and/or shear forces buckling is one of the undesired failure modes that should be considered already in early structural design phases.

For the case of a rectangular plate of isotropic material the corresponding buckling behaviour has been investigated in much detail and is well understood (Pflüger, 1975, Brush et al., 1975). The buckling analysis of plane laminates with symmetrical layup and effectively orthotropic behaviour is also available. In that case an analogous displacement representation can be applied for the solution of the buckling equation, at least for edges that are all simply supported.

The buckling problem of a rectangular laminate plate, however, gets more difficult when an arbitrary symmetrical layup is admitted. Then, in general, twist coupling has to be taken into account in the respective constitutive laminate behaviour and this makes the buckling equation more difficult to be satisfied together with the given boundary conditions (Rohwer, 1991).

In the following this case is considered in a basic and systematic way by means of an energetic variational approach. In doing so, for the buckling modes first a deflection representation is chosen in the form of a trigonometric double sine series, as it is known from the case of effectively orthotropic laminates. Furtheron, as an alternative a buckling mode representation is chosen by means of a double B-spline series. In each case buckling loads and buckling modes can be determined through the principle of minimum potential energy. In particular, this works also for arbitrary twist coupling and thus the impact of twist coupling on the resultant buckling behaviour can be clearly assessed.

The presented buckling analysis approach in particular is studied in regard of its convergence characteristics. For the validation of the obtained results comparative finite element buckling analyses have been performed by means of a commerically available finite element code and a good correspondence is obtained.

2 Basic Setting

A rectangular laminate plate is to be considered within the x-y-plane shown in Figure 1 with the dimensions a in x-direction and b in y-direction. The thickness of the laminate is denoted by h so that the bottom and top surfaces of the laminate correspond to $z = -h/2$ and $z = h/2$, respectively. The laminate is assumed to consist of K single plies with a symmetrical layup and the laminate midplane at $z = 0$. Beyond this the layup may be arbitrary. In real technical applications the single ply material

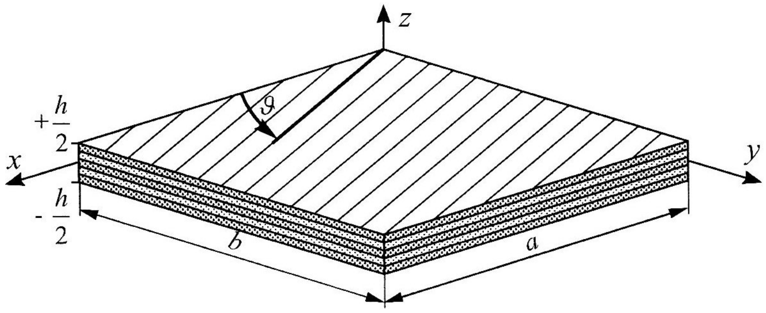

Figure 1: Considered Laminate Plate

quite often is a polymer matrix that is unidirectionally reinforced by carbon fibers (CFRP, carbon fiber reinforced plastics).

Assuming linear elasticity the behaviour of the laminate is to be described by classical laminate theory (Jones, 1975, Vinson et al., 1987, Leissa, 1995, Narita et al., 1990/1992). In doing so, the effective constitutive laminate behaviour is represented by the so- called laminate stiffness matrix through which the cross-sectional forces N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy} on the one hand and the laminate midplane strains ε_x^0, ε_y^0, γ_{xy}^0 and curvatures κ_x, κ_y, κ_{xy} on the other hand are interrelated as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{1}$$

The laminate stiffness matrix consists of the inplane stiffnesses A_{ij}, the coupling stiffnesses B_{ij} and the bending stiffnesses D_{ij}. For symmetrical layups all coupling stiffnesses B_{ij} vanish. Then the bending and buckling behaviour mainly is given by the bending stiffnesses D_{ij}. The calculation of the laminate stiffnesses from the basic single ply material properties is standard and can be found elsewhere (Jones, 1975). It is to be assumed that pure inplane forces N_x, N_y, N_{xy} are applied on the considered laminate in such a way that buckling occurs. Then the buckling deflection w in z-direction is dominant in comparison to the inplane displacements u and v. The consideration of equilibrium in z-direction in the deformed state (geometrically nonlinear) then leads to the following buckling differential equation (Narita, 1995, Baharlou et al., 1987, Dickinson et al., 1986):

$$\begin{aligned} D_{11}w_{,xxxx} + 4D_{16}w_{,xxxy} + 2(D_{12} + 2D_{66})w_{,xxyy} + 4D_{26}w_{,xyyy} + D_{22}w_{,yyyy} \\ = N_x w_{,xx} + 2N_{xy}w_{,xy} + N_y w_{,yy} \end{aligned} \tag{2}$$

Within this equation a comma denotes the partial derivative with respect to the given index arguments.

In some special cases when the twist coupling stiffnesses vanish, $D_{16} = D_{26} = 0$, this buckling equation can be solved in a closed-form manner. For a laminate with all simply supported edges, for example, under unidirectional compressive force N_x the buckling modes are given by

$$w(x, y) = w_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad , \tag{3}$$

where m and n are positive integers. The corresponding buckling loads are

$$N_{x,crit} = -\pi^2 \left(D_{11} \left(\frac{m}{a}\right)^2 + 2(D_{12} + 2D_{66}) \left(\frac{n}{b}\right)^2 + D_{22} \left(\frac{n}{b}\right)^4 \left(\frac{a}{m}\right)^2 \right) \tag{4}$$

3 RITZ Method for Buckling Analysis

An energetic approximate approach to the buckling problem is to choose a RITZ representation for the displacements in accordance with all given geometrical boundary conditions and then to minimize the second variation of the potential energy with respect to all introduced free constants. This is a very classical kind of buckling analysis that in principle has proven its usefulness in innumberable cases.

For the considered laminate problem it means that the second variation of the potential energy has to be minimized. In terms of the displacement variation w this second variation reads

$$\begin{aligned}\delta^2\Pi \quad = \quad & \int_A [(D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{16}w_{,xx}w_{,xy} + \\ & 4D_{26}w_{,yy}w_{,yx} + 4D_{66}w_{,yx}^2) - (N_x w_{,x}^2 + N_{xy}w_{,x}w_{,y} + N_y w_{,y}^2)]dA \end{aligned} \tag{5}$$

Generally speaking, according to the Ritz method, for the normal deflection w a series representation is chosen that satisfies the geometrical boundary conditions. As a representation a product of a function $X_m(x)$ of the single variable x and a function $Y_n(y)$ of the single variable y is chosen as follows:

$$w(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}X_m(x)Y_n(y) \tag{6}$$

The quantities a_{mn} are free constants to be determined.

When the buckling load case is represented as a scalar multiple λ of a given initial load state N_x^0, N_y^0, N_{xy}^0 such that

$$N_{x,crit} = \lambda N_x^0 \qquad N_{y,crit} = \lambda N_y^0 \qquad N_{xy,crit} = \lambda N_{xy}^0 \tag{7}$$

then the minimization of the second variation of the potential energy (5) with respect to the coefficients a_{mn} leads to an eigenvalue problem where the quantity λ plays the role of the eigenvalue and the coefficients a_{mn} are the components of the corresponding eigenvector. In detail, the eigenvalue problem can be stated as (Narita, 1995):

$$\begin{aligned} & (\boldsymbol{K} - \lambda\boldsymbol{L})\boldsymbol{a} = \mathbf{0} \\ \text{or} \quad & \left(\begin{bmatrix} k_{1111} & \cdots & k_{M1N1} \\ \vdots & \ddots & \vdots \\ k_{1M1N} & \cdots & k_{MNMN} \end{bmatrix} - \lambda \begin{bmatrix} l_{1111} & \cdots & l_{M1N1} \\ \vdots & \ddots & \vdots \\ l_{1M1N} & \cdots & l_{MNMN} \end{bmatrix} \right) \begin{bmatrix} a_{11} \\ \vdots \\ a_{MN} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \tag{8}$$

Herein the components of the matrices $\boldsymbol{K}$ and $\boldsymbol{L}$ are given as

$$\begin{aligned} k_{\overline{m}m\overline{n}n} \quad = \quad & \frac{1}{2}\sum_{\overline{m}=1}^{M}\sum_{\overline{n}=1}^{N} D_{11}I_{\overline{m}m\overline{n}n}^{2200} + D_{12}(I_{\overline{m}m\overline{n}n}^{2002} + I_{\overline{m}m\overline{n}n}^{0220}) + D_{22}I_{\overline{m}m\overline{n}n}^{0022} \\ & +2D_{16}(I_{\overline{m}m\overline{n}n}^{2101} + I_{\overline{m}m\overline{n}n}^{1210}) + 2D_{26}(I_{\overline{m}m\overline{n}n}^{1021} + I_{\overline{m}m\overline{n}n}^{1012}) + 4D_{66}I_{\overline{m}m\overline{n}n}^{1111} \end{aligned} \tag{9}$$

$$l_{\overline{m}m\overline{n}n} \quad = \quad \sum_{\overline{m}=1}^{M}\sum_{\overline{n}=1}^{N}(N_x I_{\overline{m}m\overline{n}n}^{1100} + N_y I_{\overline{m}m\overline{n}n}^{0011} + N_{xy}(I_{\overline{m}m\overline{n}n}^{0110} + I_{\overline{m}m\overline{n}n}^{1001})) \tag{10}$$

where the quantities $I_{\overline{m}m\overline{n}n}^{ijkl}$ mean the following integrals over the laminate area:

$$I_{\overline{m}m\overline{n}n}^{ijkl} = \int\int_A \frac{\partial^{(i)}X_{\overline{m}}}{\partial x^{(i)}}\frac{\partial^{(j)}X_m}{\partial x^{(j)}}\frac{\partial^{(k)}Y_{\overline{n}}}{\partial y^{(k)}}\frac{\partial^{(l)}Y_n}{\partial y^{(l)}}dA = \int_0^a \frac{\partial^{(i)}X_{\overline{m}}}{\partial x^{(i)}}\frac{\partial^{(j)}X_m}{\partial x^{(j)}}dx \int_0^b \frac{\partial^{(k)}Y_{\overline{n}}}{\partial y^{(k)}}\frac{\partial^{(l)}Y_n}{\partial y^{(l)}}dy \tag{11}$$

For simple functions $X_m(x)$, $Y_n(y)$ the integrals (11) can be calculated in a closed-form analytical manner. For the solution of the eigenvalue problem (8) standard numerical algorithms can be used.

For the case of all simply supported laminate edges an almost classical choice of the functions X_m and Y_n are sine functions:

$$X_m(x) = \sin\frac{m\pi x}{a} \qquad Y_n(y) = \sin\frac{n\pi y}{b} \tag{12}$$

leading to simple closed-form expressions for the integrals (11). For other kinds of boundary conditions the sine functions may be not appropriate. In the case of a clamped edge, for example, the geometrical condition of vanishing inclinations $\partial w/\partial x = 0$, $\partial w/\partial y = 0$ can not be satisfied by this representation.

4 B-Spline Functions

A rather general and flexible representation of various kinds of functional behaviour is possible by means of B-spline functions (De Boer, 1972). Within this notation B-spline stands for Basic-spline because all splines can be represented through B-spline functions. If a real interval $[x_0, x_M]$ is subdivided into M subintervals of equal length Δx with the partition $x_0 < x_1 < x_2 < ... < x_{M-1} < x_M$ a normalized equidistant B- spline $N_{i,k}$ of order k is a polynomial of degree $k-1$ which is non- zero within k adjacent subintervals and which is continuously differentiable up to the derivative $(k-2)$ at the segment interfaces x_i, x_{i+1}, ... , x_{i+k}. Within each interval $[x_j, x_{j+1}]$, $i \le j \le i+k-1$ the B-spline $N_{i,k}(x)$ allows the representation

$$N_{i,k}(x) = \sum_{m=0}^{k-1} a_m x^m \tag{13}$$

with appropriately defined constants a_m. Outside these intervals the spline vanishes:

$$N_{i,k} = 0 \quad \text{for} \quad x \le x_i \quad \text{and} \quad x \ge x_{i+k} \tag{14}$$

Furtheron, at an arbitrary real argument the sum of all B-splines in total gives unity:

$$\sum_{i=0}^{M} N_{i,k}(x) = 1 \tag{15}$$

which means a normalization. It can be shown that the B-spline functions allow the following recursive definition:

$$\begin{aligned} N_{i,1}(x) &= \begin{cases} 1 & \text{for} \quad x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \\ N_{i,k}(x) &= \frac{x - x_i}{(k-1)\Delta x} N_{i,k-1}(x) + \frac{x_{i+k} - x}{(k-1)\Delta x} N_{i+1,k-1}(x) \end{aligned} \tag{16}$$

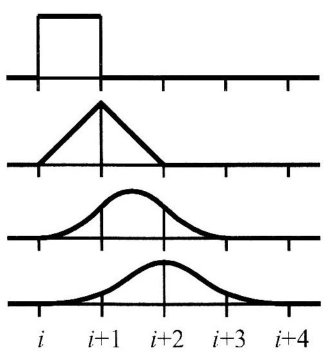

Figure 2: Normalized B-Splines of the Orders 1 to 4

If within each subinterval of length Δx the normalized variable ξ is used with

$$\xi = \xi_i(x) = \frac{x - x_i}{\Delta x} \tag{17}$$

the following direct segmentwise representation can be given for the B- splines of order 1 to 4:

$$\begin{aligned}
N_{i,1} &= \begin{cases} 1 & \text{for} \quad 0 \leq \xi < 1 \\ 0 & \text{otherwise} \end{cases} \\
N_{i,2} &= \xi N_{i,1} + (1-\xi) N_{i+1,1} \\
N_{i,3} &= \frac{\xi^2}{2} N_i, 1 + \left(-\xi^2 - \xi + \frac{1}{2}\right) N_{i+1,1} + \frac{(1-\xi)^2}{2} N_{i+2,1} \qquad (18) \\
N_{i,4} &= \frac{\xi^3}{6} N_{i,1} + \left(-\frac{1}{2}\xi^3 + \frac{1}{2}\xi^2 + \frac{1}{2}\xi + \frac{1}{6}\right) N_{i+1,1} \\
&\quad + \left(\frac{1}{2}\xi^3 - \xi^2 + \frac{2}{3}\right) N_{i+2,1} + \frac{(1-\xi)^3}{6} N_{i+3,1} \qquad (19)
\end{aligned}$$

The respective functional behaviour is illustrated by Figure 2.

5 Buckling Analysis by Means of B-Spline Functions

For the representation of the laminate buckling deformations equidistant B- splines of order 4 are to be employed leading to

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} X_m(x) Y_n(y) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} N_{m,4}(x) N_{n,4}(y) \tag{20}$$

The differentiation of the B-spline $N_{m,4}$ with respect to the normalized variable ξ simply gives:

$$\begin{aligned}
\frac{\partial N_{i,4}}{\partial \xi} &= \frac{1}{2}\xi^2 N_{i,1} + \left(-\frac{3}{2}\xi^2 + \xi + \frac{1}{2}\right) N_{i+1,1} + \left(\frac{3}{2}\xi^2 - 2\xi\right) N_{i+2,1} - \frac{1}{2}(1-\xi)^2 N_{i+3,1} \\
\frac{\partial^2 N_{i,4}}{\partial \xi^2} &= \xi N_{i+1,1} + (-3\xi + 1) N_{i+1,1} + (3\xi - 2) N_{i+2,1} + (1-\xi) N_{i+3,1} \qquad (21)
\end{aligned}$$

The derivatives with respect to the variable x can be calculated as

$$\frac{\partial^j N_{i,4}}{\partial x^j} = \frac{\partial^j N_{i,4}}{\partial \xi^j} \left(\frac{1}{\Delta x}\right)^j \tag{22}$$

The B-splines $N_{i,4}$ are very appropriate to approximate the laminate normal deflection away from the edges. Directly at the edge $x_0 = 0$ the spline $X_{0,4} = N_{0,4}$ ensures

$$w = w' = w'' = 0 \tag{23}$$

however it is not optimally adapted to boundary conditions where e.g. $w' \neq 0$ (simply supported edge) or $w'' \neq 0$ (clamped edge). It is advantageous to introduce an additional "boundary spline" $N_{L,4}$ at the "left" edge $x = x_0$ as a linear combination of the splines $N_{-3,4}$, $N_{-2,4}$, $N_{-1,4}$ defined on the "non-inside" intervals $[x_{-3}, x_1]$, $[x_{-2}, x_2]$, $[x_{-1}, x_3]$ in such a way that

$$N_{L,4} = a_{-3} N_{-3,4} + a_{-2} N_{-2,4} + a_{-1} N_{-1,4} \tag{24}$$

complies with the geometrically given boundary conditions. In the case of a simply supported edge with

$$w = w'' = 0 \qquad w' \neq 0 \tag{25}$$

the conditions $w = w'' = 0$ mean

$$\begin{aligned} \frac{1}{6}a_{-3} + \frac{2}{3}a_{-2} + \frac{1}{6}a_{-1} &= 0 \\ a_{-3} - 2a_{-2} + a_{-1} &= 0 \end{aligned} \tag{26}$$

which gives

$$a_{-2} = 0 \quad \text{and} \quad a_{-3} = -a_{-1} \tag{27}$$

With that the boundary spline $N_{L,4}^{(S)}$ can be represented as

$$N_{L,4}^{(S)} = \left(-\frac{1}{3}\xi^3 + \xi\right) N_{0,1} + \left(\frac{1}{2}\xi^3 - \xi^2 + \frac{2}{3}\right) N_{1,1} + \frac{1}{6}(1-\xi)^3 N_{2,1} \tag{28}$$

Herein the superscript (S) means "simply supported". In an analogous manner the boundary spline $N_{R,4}^{(S)}$ at the right edge $x = x_M$ can be given as

$$N_{R,4}^{(S)} = \frac{1}{6}\xi^3 N_{M-3,1} + \left(-\frac{1}{2}\xi^3 + \frac{1}{2}\xi^2 + \frac{1}{2}\xi + \frac{1}{6}\right) N_{M-2,1} + \left(\frac{1}{3}\xi^3 - \xi^2 + \frac{2}{3}\right) N_{M-1,1} \tag{29}$$

In the case of clamped edges with

$$w = w' = 0 \qquad w'' \neq 0 \tag{30}$$

the conditions $w = w' = 0$ lead to the following boundary splines:

$$\begin{aligned} N_{L,4}^{(C)} &= \left(-\frac{11}{12}\xi^3 + \frac{3}{2}\xi^2\right) N_{0,1} + \left(\frac{7}{12}\xi^3 - \frac{5}{4}\xi^2 + \frac{1}{4}\xi + \frac{7}{12}\right) N_{1,1} + \frac{1}{6}(1-\xi)^3 N_{2,1} \\ N_{R,4}^{(C)} &= \frac{1}{6}\xi^3 N_{M-3,1} + \left(-\frac{7}{12}\xi^3 + \frac{1}{2}\xi^2 + \frac{1}{2}\xi + \frac{1}{6}\right) N_{M-2,1} \\ &\quad + \left(\frac{11}{12}\xi^3 - \frac{5}{4}\xi^2 - \frac{1}{4}\xi + \frac{7}{12}\right) N_{M-1,1} \end{aligned} \tag{31}$$

The calculation of the integrals (11) is relatively easy because it is just the segmentwise integration of polynomials and thus can be done in a closed-form analytical manner.

6 Validation by Finite Element Analysis, Results and Discussion

For actual numerical examples first a quadratic laminate plate has been considered with the edge length dimensions $a = b = 1000$mm. The layup consists of 5 plies with alternating layup angles of $\pm 45°$, so that a $[45°/-45°/45°/-45°/45°]$-laminate is given. The single ply thickness is assumed to be 1mm. The engineering constants of the single ply material are given as

$$E_1 = 138000 MPa \qquad E_2 = 8960 MPa \qquad \nu_{12} = 0.3 \qquad G_{12} = 7100 MPa \tag{32}$$

Two different kinds of buckling loading have been considered (see Figure 3): a uniaxial compressive loading N_x in x-direction and a (positive) shear loading by N_{xy}. The edges of the laminate plate first

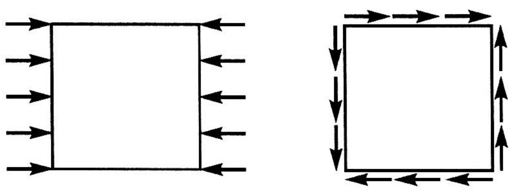

Figure 3: Considered Buckling Load Cases for Rectangular Laminate Plate

M × N	double sine series	double B-spline series
4×4	27.5354	27.7966
6×6	27.2446	27.4429
8×8	27.0849	27.2519
10×10	26.9821	27.1281
12×12	26.9099	27.0402
14×14	26.8561	26.9741
16×16	26.8143	26.9223

Table 1: Buckling Loads $N_{x,crit}$ (in N/mm) for Simply Supported $[45°/-45°/45°/-45°/45°]$ Plate under Unidirectional Compressive Loading According to Double Sine Series Representation and B-Spline Representation

layup angle ϑ	ABAQUS	double sine series	double B-spline series
0°	19.649	18.673	18.674
15°	21.513	20.563	21.608
30°	25.404	24.820	24.944
45°	26.667	27.085	27.252
60°	23.892	23.926	24.055
75°	14.499	14.388	14.431
90°	10.950	10.749	10.751

Table 2: Critical Buckling Loads $N_{x,crit}$ (in N/mm) for Unidirectional Compressive Loading of Simply Supported $[\vartheta/-\vartheta/\vartheta/-\vartheta/\vartheta]$ Plate

have been assumed to be all simply supported so that both the Ritz representation of the buckling deformation by a double sine series and by a double B-spline series are applicable. In order to study the convergence of the Ritz series approach the numbers M and N of included series terms have been varied from $M \times N = 4 \times 4$ to $M \times N = 16 \times 16$. In Table 1 the predicted buckling loads are given according to the double sine series and according to the double B-spline series with included boundary splines of the kind (28) and (29). Obviously the convergence in all cases is rather good. Relating to the 16×16 results the 8×8 predictions do not deviate more than about 1 percent and thus will be employed in the sequel. A comparison of the double sine series results with those of the B-spline series shows that the sine series representation is slightly more appropriate for the considered boundary conditions (i.e. gives lower buckling loads), but there is no serious difference of the corresponding results.

For validation comparative buckling analyses have been performed by means of the finite element code ABAQUS (ABAQUS, 1997). In doing so, a finite element discretization with 20×20 4-node shell elements with the respective laminate layup has been employed. First a simply supported laminate with a $[\vartheta/-\vartheta/\vartheta/-\vartheta/\vartheta]$ layup has been considered under unidirectional compressive loading N_x for a set of varying layup angles ϑ. The angle ϑ is given by rotation from the x-axis towards the y-axis as indicated in Figure 1. In Table 2 the buckling loads are given as they are predicted by finite element analysis and the corresponding Ritz series approaches (with 8×8 series terms). The agreement of the Ritz approach results with the finite element results is fairly good. It still can be somewhat improved with a finer finite element meshing.

Next, the same set of plates under unidirectional compressive loading N_x has been considered for the case of boundary conditions where in addition to a vanishing deflection w the plate inclinations $\partial w/\partial x$ and $\partial w/\partial y$ are suppressed ("clamped edges"). In this case the double sine series representation (12) can no longer be used for the normal deflection w because this representation does not satisfy the geometrical boundary conditions $\partial w/\partial x = \partial w/\partial y = 0$. Nevertheless, the B-spline representation (20) with respective boundary splines of the kind (31) can be employed in this case without difficulty. As can be seen from the results of Table 3 again a fairly good agreement can be stated between the finite element predictions and the B-spline series results.

The buckling analysis presented so far takes into account arbitrary twist coupling. The impact of twist coupling on buckling can be assessed by comparison of the predicted buckling loads for the cases with and without twist coupling stiffnesses D_{16}, D_{26}. Such an assessment first is done for a simply supported

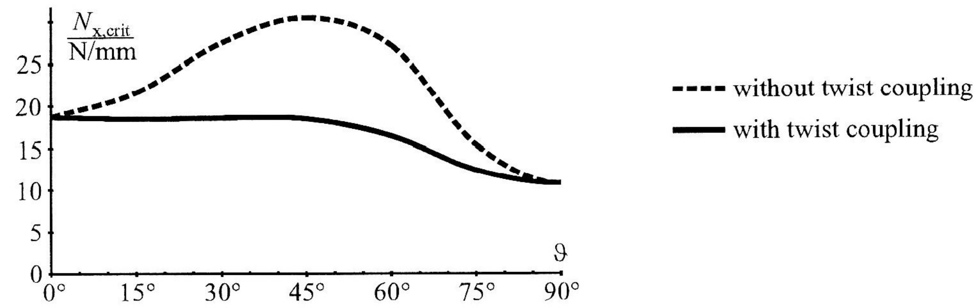

Figure 4: Critical Buckling Load $N_{x,crit}$ (in N/mm) for Simply Supported Single Layer Plate under Uniaxial Compressive Loading

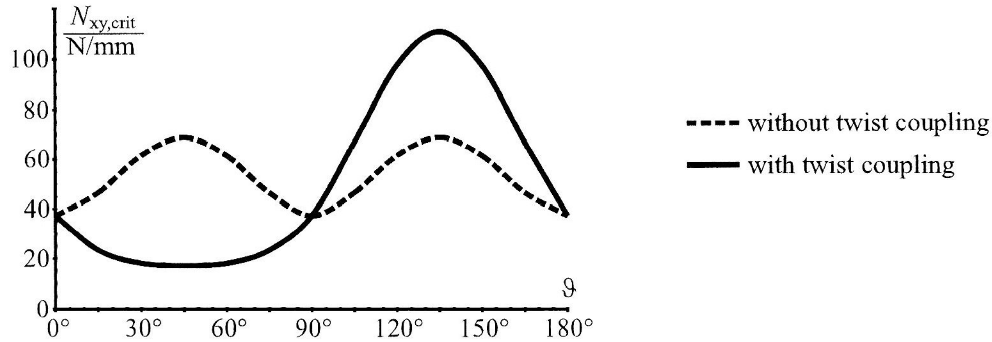

Figure 5: Critical Buckling Load $N_{xy,crit}$ (in N/mm) for Simply Supported Single Layer Plate under Pure Shear Loading

plate consisting of a single unidirectional ply with orientation ϑ. For varying ply angle ϑ Figure 4 shows the resultant unidirectional compressive buckling load $N_{x,crit}$ as it is predicted by the Ritz B-spline approach taking into account twist coupling (solid line) and neglecting twist coupling (dashed line). In the orthotropic cases $\vartheta = 0°$ and $\vartheta = 90°$ there is no twist coupling and, of course, there is no difference between the two corresponding predictions. For intermediate angles, however, a significant difference is possible. For the range of $\vartheta = 30°...60°$ with neglected twist coupling a (non-conservative) buckling load is obtained that is up to 65 percent too high. A respective comparison for a pure shear loading N_{xy} is shown in Figure 5 for a variation of the layup angle ϑ within the range $0°...180°$. In this case the error due to the neglect of twist coupling can become both conservative and non-conservative. The buckling load prediction can become both, about 40 percent too small or about 300 percent too large. This demonstrates well, that the twist coupling stiffnesses D_{16}, D_{26} of a laminate should always be taken into account carefully, at least when the twist coupling stiffnesses are sufficiently large. With the Ritz B-spline representation (20) this is possible for rather general combinations of boundary conditions. In doing so, the involved numerical effort is clearly lower than in the case of a finite element buckling analysis. The same kinds of buckling analyses can be performed for any specified symmetric laminate layup without additional difficulties. As an example of a multilayered laminate this has been done for a five ply layup of the kind $[\vartheta, -\vartheta, \vartheta, -\vartheta, \vartheta]$. For the case of all simply supported edges Figure 6 shows

layup angle ϑ	ABAQUS	double sine series	double B-spline series
0°	69.319	—	69.353
15°	66.122	—	66.032
30°	60.039	—	60.552
45°	53.750	—	56.953
60°	41.102	—	43.114
75°	29.309	—	30.367
90°	25.191	—	25.517

Table 3: Critical Buckling Loads $N_{x,crit}$ (in N/mm) for Unidirectional Compressive Loading of Clamped $[\vartheta/-\vartheta/\vartheta/-\vartheta/\vartheta]$ Plate

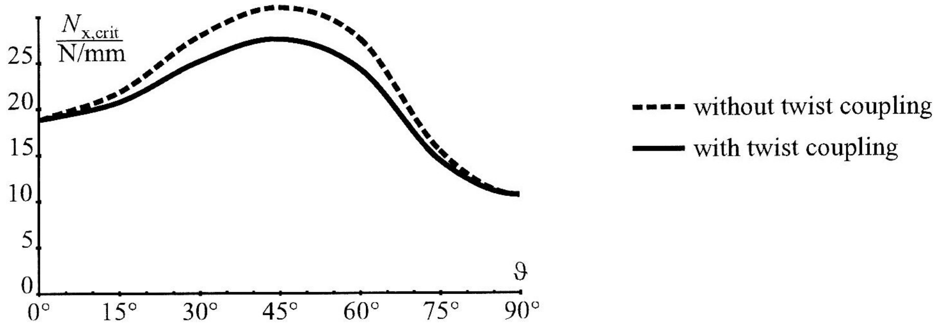

Figure 6: Critical Buckling Load $N_{x,crit}$ (in N/mm) for Simply Supported $[\vartheta/-\vartheta/\vartheta/-\vartheta/\vartheta]$ Laminate Plate under Uniaxial Compressive Loading

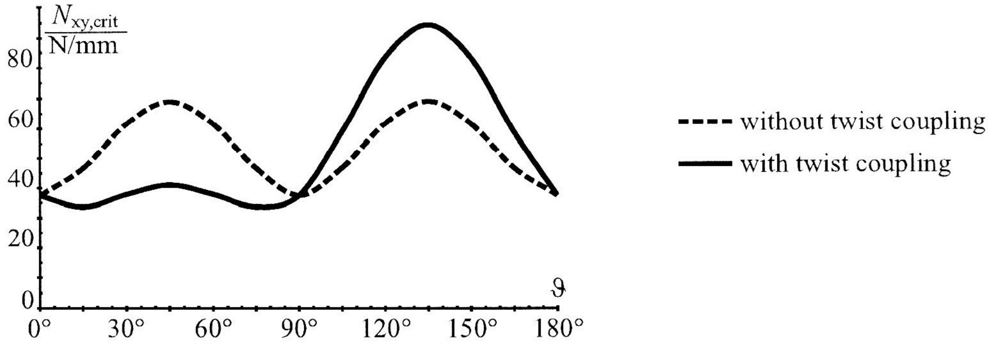

Figure 7: Critical Buckling Load $N_{xy,crit}$ (in N/mm) for Simply Supported $[\vartheta/-\vartheta/\vartheta/-\vartheta/\vartheta]$ Laminate Plate under Pure Shear Loading

the critical compressive buckling loads $N_{x,crit}$. In the same way Figure 7 shows the critical pure shear buckling loads $N_{xy,crit}$.

7 Conclusions

Depending on the given laminate layup with its resultant bending stiffnesses D_{ij}, the prescribed boundary conditions and the considered loads, in general, a laminate buckling problem can not be solved exactly through the buckling differential equation. As an energetic approximate approach the Ritz method works in very many cases. For all simply supported rectangular laminate plates a displacement representation in form of a double sine series is well approved. It is easy to implement, shows good convergence and gives sufficiently precise results. It can, however, not be applied for more general kinds of boundary conditions.

For more general boundary conditions a buckling deformation representation in form of a double B-spline series can be applied in a similar way as the double sine series. More general boundary conditions can be taken into account through appropriately defined boundary splines. A numerical implementation has been done successfully and shows also good convergence characteristics and gives sufficiently precise results. The obtained buckling predictions could be validated by comparative finite element analyses. In comparison with the finite element analyses, however, the implemented B-spline buckling analysis clearly shows more computational efficiency, at least when finite elements with displacement representations of low polynomial order are used. If a larger number of buckling analyses is required, as e.g. within an iterative optimization procedure, the computational efficiency of the B-spline buckling analysis is a decisive advantage.

By means of the B-spline buckling analysis the impact of twist coupling on the critical buckling load has been investigated. In contrast to the case of pure uniaxial compressive load N_x, where the neglect of twist coupling leads to unrealistic high buckling loads, in the case of pure shear loading N_{xy} it has been observed that the proper consideration of twist coupling may lead to both, higher and lower buckling loads, compared to the case where it is simply neglected.

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