TECHNISCHE MECHANIK, Band 21, Heft 1, (2001), 75-86
Manuskripteingang: 17. Oktober 2000

Nonlocal Analysis of Differential Equations of Induction Motors

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For a system of differential equations describing dynamics of the induction motor in cases of the external load in form of constant torque and of torque linearly dependent on angular rate with the help of qualitative theory of differential equations and the direct method of Lyapunov the conditions of dyhotonomous and global asymptotic stability are obtained.

1 Introduction

In this paper the stability of the induction motor on the basis of its full mathematical model is investigated. The model is a system of differential equations of the sixth order which describes the dynamics of the induction motor with generally accepted idealizing assumptions, the latter are presented in detail in Anderson (1977), Gorev (1985), Rodyukov (1997), White (1959). The main assumptions here are the following ones. In the first place the electromagnetic field in any cross-section of the idealized physical model of the induction motor with the face effects neglected is identical (the conjection of the flat model). Secondly we suppose that it is possible to describe the interaction of the electromagnetic processes in the stator winding and the rotor of the machine with the help of two symmetric linear electric circuits. Note that when investigating the static stability of the induction motor in practice often only its static mechanical performance is used. The latter gives the widely known Kloss formula in case the resistance of stator windings is neglected. Such approach is well agreed with the practical operation of the induction motor. But it is not rigorous enough from the point of view of mathematics.

To carry out the nonlocal analysis of the complete mathematical model we exploited the special choice of mobile coordinates (the generalized Park transformation) and special choice of hybrid variables (the quasicurrents and quasiflux linkages of the stator windings). All this gave us the opportunity to reduce the system of differential equations to an autonomous system of the fifth order. For the latter the Lyapunov functions were used. As a result, for various cases conditions for dissipativity, dichotomy and global stability were obtained and the domains of attraction for stable equilibria (dynamic stability) were evaluated. Similar functions were constructed in Leonov (1983) for the well known Lorenz system.

2 Mathematical Model

Let us consider the system of differential equations

$$
\begin{aligned}
&\left(L_s \frac{d}{dt} + R_s\right) i_\alpha^s + \kappa M \frac{d}{dt}\left(i_\alpha^r \cos\gamma - i_\beta^r \sin\gamma\right) = -u_m \sin\omega_s t \\
&\left(L_s \frac{d}{dt} + R_s\right) i_\beta^s + \kappa M \frac{d}{dt}\left(i_\alpha^r \sin\gamma + i_\beta^r \cos\gamma\right) = u_m \cos\omega_s t \\
&M \frac{d}{dt}\left(i_\alpha^s \cos\gamma + i_\beta^s \sin\gamma\right) + \left(L_r \frac{d}{dt} + R_r\right) i_\alpha^r = 0 \\
&M \frac{d}{dt}\left(-i_\alpha^s \sin\gamma + i_\beta^s \cos\gamma\right) + \left(L_r \frac{d}{dt} + R_r\right) i_\beta^r = 0 \\
&J\ddot{\gamma} = M\left[\left(i_\alpha^r i_\beta^s - i_\beta^r i_\alpha^s\right)\cos\gamma - \left(i_\alpha^r i_\alpha^s + i_\beta^r i_\beta^s\right)\sin\gamma\right] - M_H
\end{aligned}
\tag{2.1}
$$

Here variables $i_\alpha^s, i_\beta^s, i_\alpha^r, i_\beta^r$ are the currents of stator and rotor windings, γ is the angle of turn of the rotor; parameters R_s, L_s, R_r, L_r are resistances and inductances of appropriate windings, M is the peak significance of the mutual inductance between them, J is the moment of inertia of the rotor of the induction motor; $\omega_s = 2\pi f$, f, u_m are the frequency and the amplitude of the voltage brought to the stator windings, M_H is the moment of the load on the shaft of the induction motor, t is the current time, and κ is the parameter, characterizing a degree of influence of electromagnetic processes in a rotor on processes in the stator windings. The equations (2.1) to which the notation coincide with the equations (8-1b), and (8-2g)in White (1959). Neglecting the influence of electromagnetic processes in a rotor on processes in the stator windings, we assume in system (2.1) $\kappa = 0$.

For further transformations of the equations (2.1) we need the expressions for the flux linkages of the windings. For a case of the non-pole-salient electrical machine, which the induction motor is, we can receive these flux linkages from the formulas (3-3a) - (3-3e)in White (1959), with regard to the expressions for the inductances (3-40) - (3-49)in White (1959). In the notations accepted above they have the form

$$\begin{aligned} \psi_\alpha^s &= L_s i_\alpha^s \\ \psi_\beta^s &= L_s i_\beta^s \\ \psi_\alpha^r &= M(i_\alpha^s \cos\gamma + i_\beta^s \sin\gamma) + L_r i_\alpha^r \\ \psi_\beta^r &= M(-i_\alpha^s \sin\gamma + i_\beta^s \cos\gamma) + L_r i_\beta^r \end{aligned} \tag{2.2}$$

The equations (2.1) and the flux linkages (2.2) are written in so-called phase coordinates $\alpha - \beta$. For a mathematical research they are inconvenient. But they allow for the simplification of the writing to use a nonholonomic transformation of coordinates Park (Gorev, 1985). In case of the induction motor it can be written with the help of auxiliary orthogonal axes $u - v$, which rotate with arbitrary angular velocity and are at the angle γ_k to a magnetic axis of the phase α of the stator of the induction motor. Such transformation is given in White (1959). For our notations it takes the form

$$\begin{aligned} i_u^s &= i_\alpha^s \cos\gamma_k + i_\beta^s \sin\gamma_k \\ i_v^s &= -i_\alpha^s \sin\gamma_k + i_\beta^s \cos\gamma_k, \\ i_u^r &= i_\alpha^r \cos(\gamma_k - \gamma) + i_\beta^r \sin(\gamma_k - \gamma) \\ i_v^r &= -i_\alpha^r \sin(\gamma_k - \gamma) + i_\beta^r \cos(\gamma_k - \gamma) \end{aligned}$$

$$\begin{aligned} \psi_u^s &= \psi_\alpha^s \cos\gamma_k + \psi_\beta^s \sin\gamma_k \\ \psi_v^s &= -\psi_\alpha^s \sin\gamma_k + \psi_\beta^s \cos\gamma_k \\ \psi_u^r &= \psi_\alpha^r \cos(\gamma_k - \gamma) + \psi_\beta^r \sin(\gamma_k - \gamma) \\ \psi_v^r &= -\psi_\alpha^r \sin(\gamma_k - \gamma) + \psi_\beta^r \cos(\gamma_k - \gamma) \end{aligned} \tag{2.3}$$

$$\begin{aligned} u_u &= u_m(-\sin\omega_s t \cos\gamma_k + \cos\omega_s t \sin\gamma_k) = -u_m \sin(\omega_s t - \gamma_k) \\ u_v &= u_m(\sin\omega_s t \sin\gamma_k + \cos\omega_s t \cos\gamma_k) = u_m \cos(\omega_s t - \gamma_k) \end{aligned}$$

These transformations mean, that we go from real (phase) variables, written in two orthogonal coordinate systems, one rigidly connected with the stator of the induction motor and the other connected with the rotor, over to projections of these variable (to the quasivariables) on the same orthogonal axes $u - v$. In these axes the equations (2.1) and the expressions (2.2) linkages take the form:

$$\begin{aligned} L_s(\dot{i}_u^s - \dot{\gamma}_k i_v^s) &= -u_m \sin(\omega_s t - \gamma_k) \\ L_s(\dot{i}_v^s + \dot{\gamma}_k i_u^s) &= u_m \cos(\omega_s t - \gamma_k) \\ M[\dot{i}_u^s - (\dot{\gamma}_k - \dot{\gamma}) i_v^s] + R_r i_u^r + L_r[\dot{i}_u^r - (\dot{\gamma}_k - \dot{\gamma}) i_v^r] &= 0 \\ M[\dot{i}_v^s + (\dot{\gamma}_k - \dot{\gamma}) i_u^s] + R_r i_v^r + L_r[\dot{i}_v^r + (\dot{\gamma}_k - \dot{\gamma}) i_u^r] &= 0 \end{aligned} \tag{2.4}$$

$$J\ddot{\gamma} = M(i_u^r i_v^s - i_v^r i_u^s) - M_H$$

$$\begin{aligned} \psi_u^s &= L_s i_u^s, \qquad \psi_v^s = L_s i_v^s \\ \psi_u^r &= M i_u^s + L_r i_u^r, \qquad \psi_v^r = M i_v^s + L_r i_v^r \end{aligned} \tag{2.5}$$

Let us pass in the equations (2.4) and formulas (2.5) to the dimensionless form. For purpose, we introduce a dimensionless synchronous time τ, slip s, dimensionless variables, parameters, and the moment of load the formulas

$$
\begin{array}{llll}
\tau = \omega_s t & \dfrac{d\gamma}{d\tau} = 1 - s & & \\
\psi^s = \dfrac{u_m}{\omega_s}\,\bar{\psi}_s & \psi^r = \dfrac{u_m}{\omega_s}\dfrac{L_r}{M}\,\bar{\psi}^r & i^s = \dfrac{u_m}{\omega_s L_s}\,\bar{i}^s & i^r = \dfrac{u_m}{\omega_s L_s}\dfrac{L_r}{M}\,\bar{i}^r \\
\varepsilon_s = \dfrac{R_s}{\omega_s L_s} & \varepsilon_r = \dfrac{R_r}{\omega_s L_r} & \delta = \dfrac{u_m^2}{\omega_s^4 J L_s} & \\
\overline{M}_H = \dfrac{M_H}{\omega_s^2 J \delta} & \mu = 1 - \dfrac{M^2}{L_s L_r} & &
\end{array}
$$

where μ is the total leakage factor in air clearance of an induction motor.

Then, omitting the bar over the dimensionless magnitudes, we can write the equations (2.4) and the relations (2.5) in the form

$$
\begin{array}{l}
\dot{i}_u^s - \dot{\gamma}_k i_v^s + \varepsilon_s i_u^s = -\sin(\tau - \gamma_k) \\
-\dot{i}_v^s + \dot{\gamma}_k i_u^s + \varepsilon_s i_v^s = \cos(\tau - \gamma_k) \\
(1-\mu)[\dot{i}_u^s - (\dot{\gamma}_k - 1 + s) i_v^s] + \varepsilon_v i_u^r + \dot{i}_u^r - (\dot{\gamma}_k - 1 + s) i_v^r = 0 \\
(1-\mu)[\dot{i}_v^s + (\dot{\gamma}_k - 1 + s) i_u^s] + \varepsilon_v i_v^r + \dot{i}_v^r - (\dot{\gamma}_k - 1 + s) i_u^r = 0 \\
\dot{s} = -\delta[(i_u^r i_v^s - i_v^r i_u^s) - \overline{M}_H]
\end{array}
\tag{2.6}
$$

$$
\begin{array}{ll}
\psi_u^s = i_u^s & \psi_v^s = i_v^s, \\
\psi_u^r = (1-\mu) i_u^s + i_u^r & \psi_v^s = (1-\mu) i_v^s + i_v^r
\end{array}
\tag{2.7}
$$

where a dot over a variable means its differentiation with respect to synchronous time τ.

Let us remind, that in (2.6) and (2.7) the variables $\psi_u = \psi_u(\tau)$, $\psi_v = \psi_v(\tau)$, $i_u = i_u(\tau)$, $i_v = i_v(\tau)$ are the quasiflux linkages and the quasicurrents of appropriate windings, $s = s(\tau)$ is the slip, i.e. the relative difference of angular velocities of the rotor and the magnetic field of the stator, τ is the dimensionless time (angle of a turn of a magnetic field of a stator); the parameters ε_s, ε_r are the resistances of the stator and the rotor windings, μ is the total leakage factor in an air clearance, δ is an electromechanical constant, inversely proportional to the moment of inertia of the rotor; γ_k is the angle of a rotation of the axes $u - v$, $\overline{M}_H$ is the moment of the external load on the shaft of the rotor.

Let us assume in (2.6) $\gamma_k = \tau$, i.e. we pass to so-called synchronous axes of coordinates $x - y$ (which rotate synchronoasly with the magnetic field of the stator). Then we have

$$
\begin{array}{l}
\dot{i}_x^s - i_y^s + \varepsilon_s i_x^s = 0, \\
\dot{i}_y^s + i_x^s + \varepsilon_s i_y^s = 1 \\
(1-\mu)(\dot{i}_x^s - s i_y^s) + \varepsilon_v i_x^r + \dot{i}_x^r - s i_y^r = 0 \\
(1-\mu)(\dot{i}_y^s + s i_x^s) + \varepsilon_v i_y^r + \dot{i}_y^r + s i_x^r = 0 \\
\dot{s} = -\delta[(i_x^r i_y^s - i_y^r i_x^s) - \overline{M}_H]
\end{array}
\tag{2.8}
$$

$$
\begin{array}{ll}
\psi_x^s = i_x^s & \psi_y^s = i_y^s, \\
\psi_x^r = (1-\mu) i_x^s + i_x^r & \psi_y^s = (1-\mu) i_y^s + i_y^r
\end{array}
\tag{2.9}
$$

The equations (2.8) take the form

$$
\begin{array}{l}
\dot{i}_x^s = -\varepsilon_s i_x^s + i_y^s \\
\dot{i}_y^s = -\varepsilon_s i_y^s - i_x^s + 1
\end{array}
\tag{2.10}
$$

$$
\begin{aligned}
\dot{\psi}_x^r &= -\varepsilon_r \psi_x^r + s\psi_y^r + \varepsilon_r(1-\mu)x \\
\dot{\psi}_y^r &= -\varepsilon_r \psi_y^r - s\psi_x^r + \varepsilon_r(1-\mu)y \\
\dot{s} &= -\delta[(\psi_x^r y - \psi_y^r x) - \overline{M}_H]
\end{aligned}
\tag{2.11}
$$

3 Statement of the Results

Let us consider the autonomous system

$$
\dot{x} = f(x) \quad x \in \mathbf{R}^n \tag{3.1}
$$

Let us introduce several definitions.

Definition 3.1 Point $p \in \mathbf{R}^n$ is called the stationary point (the equilibrium state) of the system (3.1), if $x(t) \equiv p$ is a solution of this systems.

The set of all stationary points is called the stationary set.

Definition 3.2 We say that the system (3.1) is dychotomous, if any bounded for $t > 0$ the solution tends to stationary set as $t \to +\infty$.

Definition 3.3 We call that the system (3.1) dissipative, if all its solutions are infinitely extendable to the right and there exists such a number $R > 0$, that for any solution $x(t, x_0)$ of this system with the initial data $x(0, x_0) = x_0$ the following relations is true

$$
\overline{\lim_{t \to +\infty}} |x(t, x_0)| < R
$$

It follows from definition 3.3, that for each solution $x(t, x_0)$ of (3.1) there exists the moment T_{x_0} such, that

$$
|x(t, x_0)| < R \quad \text{for f} \quad t \geq T_{x_0}
$$

i.e.beginning with a moment T_{x_0}, the trajectory $x(t, x_0)$ is located in a full-sphere of a radius R.

Definition 3.4 System (3.1) is called globally asymptotically stable, if any solution of this system tends to a certain equilibrium as $t \to +\infty$.

With the help of the changes of variables

$$
\begin{aligned}
x &= \frac{\psi_x^r + \varepsilon_s \psi_y^r}{\varepsilon_r(1-\mu)} & y &= \frac{\psi_y^r - \varepsilon_s \psi_x^r}{\varepsilon_r(1-\mu)} \\
\psi_x &= \frac{i_x^s + \varepsilon_s i_y^s - 1}{\varepsilon_r(1-\mu)} & \psi_y &= \frac{i_y^s - \varepsilon_s i_x^s}{\varepsilon_r(1-\mu)}
\end{aligned}
\tag{3.2}
$$

we can reduce the system (2.10), (2.11) to the form more convenient for research:

$$
\begin{aligned}
\dot{\psi}_x &= -\alpha_s \psi_x + \psi_y \\
\dot{\psi}_y &= -\alpha_s \psi_y - \psi_x
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\dot{x} &= -\alpha_r x + sy + \alpha_r(1-\mu)\psi_x + 1 \\
\dot{y} &= -\alpha_r y - sx + \alpha_r(1-\mu)\psi_y \\
\dot{s} &= \delta[\alpha_r by + \alpha_r^2 b(1-\mu)(\psi_x y - \psi_y x) + M_H]
\end{aligned}
\tag{3.4}
$$

Here we put

$$
\alpha_s = \varepsilon_s \qquad \alpha_r = \varepsilon_r \qquad M_H = \overline{M}_H \qquad b = \frac{1-\mu}{1+\alpha_s{}^2}
$$

In this article the stability of the system (3.3), (3.4) is investigated for two cases - for the case of a constant moment M_H of the external load and for the case, when the moment M_H linearly depends on the slip of the rotor.

Theorem 3.1 *Let M_H=const. Then the following statements hold:*

1) If

$$0 < M_H < \min\left\{\frac{b}{2}, \frac{2\alpha_r^2}{\delta}\right\} \tag{3.5}$$

that the system (3.3), (3.4) is dychotomous;

2) If

$$\frac{b}{2} < M_H < \frac{2\alpha_r^2}{\delta} \tag{3.6}$$

all those solutions of (3.3), (3.4) are unbounded.

Theorem 3.2 *Let $M_H = \bar{\kappa}(1-s)$, $\bar{\kappa} = const.$ Then, if*

$$0 < \bar{\kappa}\delta \leq 3\alpha_r\sqrt{(\alpha_r + \bar{\kappa}\delta)(\alpha_r - 2\bar{\kappa}\delta)} \tag{3.7}$$

the system (3.3), (3.4) is globally asymptotically stable.

4 Investigation of a Model of the Third Order

The research of the system (3.3), (3.4) is largely based on the research of the system of the third order

$$\begin{aligned} \dot{x} &= -\alpha_r x + sy + 1 \\ \dot{y} &= -\alpha_r y - sx \\ \dot{s} &= \delta[\alpha_r by + M_H] \end{aligned} \tag{4.1}$$

obtained from equations (3.3), (3.4) by substitution the stationary solutions $\psi_x = 0$, $\psi_y = 0$ of the asymptotically stable linear system (3.3) in a system (3.4). Note that the system (4.1) is of independent interest for the theory of induction motors. It can be obtained from a complete mathematical model of induction motor with the help of the so-called quasistationary approach.

Note that a structure of the system (4.1) in the case $M_H = k(1-s)$ is similar to a structure of the well known Lorentz system (Sinaj, 1981). This circumstance allows to carry out the unlocal analysis of this system with the help of Lyapunov functions, considered in Leonov (1983).

Theorem 4.1 *Let M_H=const.*

1) If the condition (3.5) is true then the system (4.1)is dychotomous;

2) If $M_H = 0$, the system (4.1) is globally stable;

3) If the condition (3.6) is executed, all solutions of a system (4.1) are unbounded.

Proof. The change of variables

$$\sigma = s \qquad \eta = \nu\alpha_r y + \gamma \qquad z = -\alpha_r x - as + 1 \tag{4.2}$$

where

$$\gamma = \delta\, M_H \qquad \nu = \delta\, b \qquad a = \frac{\gamma}{\alpha_r \nu} \tag{4.3}$$

reduces the system (4.1) to the form

$$\begin{aligned} \dot{\sigma} &= \eta \qquad \dot{\eta} = -\alpha_r\eta + \nu\sigma z - \varphi(\sigma) \\ \dot{z} &= -\alpha_r z - \frac{1}{\nu}\sigma\eta - a\eta \end{aligned} \tag{4.4}$$

where $\varphi(\sigma) = -a\nu\sigma^2 + \nu\sigma - \alpha_r\gamma$.

Under condition (3.5) the stationary set Λ of the system (4.4) consists of two points, the stable point O_1 and the saddle O_2:

$$O_1\left(\frac{\alpha_r(\nu - \sqrt{\nu^2 - 4\gamma^2}\,)}{2\gamma},\ 0,0\right) \qquad O_2\left(\frac{\alpha_r(\nu + \sqrt{\nu^2 - 4\gamma^2}\,)}{2\gamma},\ 0,0\right)$$

Let us introduce the function

$$V_1 = \frac{\nu^2}{2}z^2 + \frac{1}{2}\eta^2 + \int\limits_{\sigma_1}^{\sigma}\varphi(\sigma)\,d\sigma \tag{4.5}$$

For solutions $x(t) = (\sigma(t), \eta(t), z(t))$ of the system (4.4) the relation

$$\begin{aligned} &\dot{V}_1(x(t)) = -\alpha_r\nu^2 z^2(t) - \nu\sigma(t)\eta(t)z(t) - \\ &-a\nu^2\eta(t)z(t) - \alpha_r\eta^2(t) + \nu\sigma(t)\eta(t)z(t) - \\ &-\eta(t)\varphi(\sigma(t)) + \varphi(\sigma(t))\eta(t) = -\alpha_r\eta^2(t) - \\ &-a\nu^2\eta(t)z(t) - \alpha_r\nu^2 z^2(t) \end{aligned} \tag{4.6}$$

holds. It is not difficult to verify that under the condition (3.5) the square forms in the right-hand side of the relation (4.6) are definitely negative by virtue of the Sylvester criterion (Gantmakher,1988). Therefore it follows from (4.6), that the function V_1 does not increase on some $(\tau, +\infty)$ along any trajectory of the system (4.4). Hence for any bounded trajectory $x(t, x_0)$ we receive by virtue of boundedness of the function $V_1(x(t, x_0))$ the existence of finite $\lim\limits_{t\to+\infty} V_1(x(t, x_0)) = L$.

From boundedness of a trajectory $x(t, x_0)$ it follows that the set Ω of ω limit points of x_0 is not empty. Let $y \in \Omega$. It is known (Nemytskij, 1949) that the trajectory $x(t, y) \in \Omega$ for all $t \in \mathbf{R}^1$. Therefore $V_1(x(t, y)) = L$ for all $t \in \mathbf{R}^1$. Using the relation (4.6), we receive the identities $\eta(t, y) \equiv 0$ and $z(t, y) \equiv 0$. From the system(4.4) and $\eta(t, y) \equiv 0$ we receive, that $\sigma(t, y) \equiv$ const. Therefore $\Omega \subset \Lambda$. Since the elements of Λ are isolated this inclusion implies the statement 1).

In case $M_H = 0$ $(\gamma = 0)$ the stationary set of the system (4.4) consists of a single point $(0,0,0)$, $\varphi(\sigma) = \nu\sigma$. In this case $V_1 = \frac{\nu^2}{2}z^2 + \frac{\eta^2}{2} + \frac{\nu}{2}\sigma^2$, $\dot{V}_1 \le 0$, and $\dot{V}_1 = 0$ on a set, not containing whole trajectories, except for a stationary point $(0,0,0)$. So all conditions of the Barbashin- Krasovsky theorem are valid and the system (4.4) is globally stable.

Assume that the condition (3.5) is satisfied and $x(t)$ is a bounded solution of the system (4.4). The condition (3.6) is a special case of condition (3.5). Therefore, as it was shown in item 1), $x(t)$ tends to a certain equilibrium. However under the condition (3.5) the stationary set of (4.4) is empty. We have received a contradiction, which proves the statement 3). The theorem 4.1 is proved.

For further investigation of stability of the system (4.1) we shall prove an analogy of the Barbashin - Krasovsky theorem (Barbashin, 1967).

Let us consider the system

$$\dot{x} = f(x) \qquad x \in \mathbf{R}^n \qquad f \in \mathbf{C} \tag{4.7}$$

Lemma 4.1 *Suppose that an unbounded positively invariant set G of the system (4.7) contains only one stationary point p. Suppose also that in a phase space of the system (4.7) a certain function v is determined which satisfies the following conditions: 1) $v(x) > 0$ for any $x \in G$, $x \neq p$; 2) $v\ (p) = 0$; 3) $\dot{v}(x)) \leq 0$ for all $x \in G$, 4) The set $\{x:\ \dot{v}(x) = 0\}$ does not contain the whole trajectories, except for a point p.; 5) $\lim v(x_n) = +\infty$ for any sequence $\{x_n\}$ such, that $|x_n| \to \infty$ for $n \to \infty$ and $x_n \in G$ for all n. Then any trajectory $x(t, x_0)$ of the system (4.7) such, that $x_0 \in G$ tends for $t \to +\infty$ to a point p.*

Proof. Let us consider a trajectory $x(t, x_0)$ of the point $x_0 \in G$ for $t > 0$. Because of the properties of $\dot{v}$ and the positive invariancy of G it is true that $x(t, x_0) \in G_1 = \{x:\ v(x)) \leq v(x_0)\} \cap G$ for all $t > 0$.

Let us prove the boundedness of the set G_1. Let us assume the opposite. Let $\{x_n\}$ be a sequence of x_n such, that for any n $\quad x_n \in G_1$ and $\lim |x_n| = \infty$ for $n \to \infty$. By virtue of a condition 3) $\lim\limits_{n\to\infty} v(x_n) \leq v(x_0)$, which contradicts to the condition 5).

Thus, G_1 is a bounded positively invariant set, and $p \in G_1$ and G_1 does not contain other stationary points.

Let us consider a trajectory $x(t, x_0)$ of $x_0 \in G_1$ for $t > 0$. By virtue of condition 5) and properties of G_1, function $v(x(t, x_0))$ does not increase and is bounded on $(0, +\infty)$. Therefore, for $t \to +\infty$ there is a finite $\lim v(x(t, x_0)) = L$.

The trajectory $x(t, x_0)$ is bounded on $(0, +\infty)$, therefore, the set Ω of its ω - limit points is not empty. Let $q \in \Omega$. It is known [14], that trajectory $x(t, q) \in \Omega$ for all $t \in \mathbf{R}^1$. Therefore, $v((x(t, q)) \equiv L$ for all $t \in R^1$ and therefore, $\dot{v}(x(t, q)) \equiv 0$ for all $t \in R^1$. By virtue of condition 4) of the Lemma $x(t, q) \equiv p$. Then $q = p$. The lemma 4.1 is proved.

Using (4.2), (4.3) and (4.5) we can present function V_1 in coordinates and parameters of the system (4.1)

$$V_1 = \frac{\delta^2 \alpha_r^2 b^2}{2}\left[\left(x - \frac{\alpha_r b - M_H s}{\alpha_r^2 b}\right)^2 + \left(y - \frac{M_H}{\alpha_r b}\right)^2\right] + \int_{s_1}^{s} \varphi(s)\, ds$$

where $\varphi(s) = \delta(\, - M_H s^2/\alpha_r \ + bs - \alpha_r M_H)$

$$s_{1,2} = \frac{\alpha_r}{2M_H}\left(b \mp \sqrt{b^2 - 4M_H^2}\right)$$

Let us introduce the function

$$W_1 = \frac{\alpha_r{}^2}{2} x^2 + \frac{\alpha_r{}^2}{2} y^2$$

and the sets

$$D_0 = \{X:\ W_1(X) \leq \Gamma_1\}$$
$$D_1 = \{X:\ 2V_1(X) < p_0^2,\ \ s < s_2\}$$
$$D_2 = \{X:\ W_1(X) \leq \Gamma_1,\ \ s \leq s_1\}$$

where

$$p_0 = \sqrt{2\int_{s_1}^{s_2} \varphi(s)\, ds}$$

$$\Gamma_1 = \frac{\alpha_r{}^2}{8\lambda(\alpha_r - \lambda)} \qquad \lambda \in [0, \alpha_r]$$

Theorem 4.2 *Let $M_H = const$, the condition (3.5) and the condition*

$$\Gamma_0 = \frac{p_0}{\delta b} - \frac{M_H\sqrt{\alpha_r^2 + s_2^2}}{\alpha_r b} \geq 1 \tag{4.8}$$

are fulfilled. If $\Gamma_1 = \Gamma_0^2/2$, then the set $G = D_1 \bigcup D_2$ is contained in the domain of attraction of a stable equilibrium of the system (4.1).

Proof. Let us prove, that the set D_0 is positively invariant for solutions of the system (4.1). For this purpose we shall show, that for any solutions $X(t) = (x(t), y(t), s(t))$ of the system (4.1) the inequality

$$\overline{\lim_{t\to+\infty}} W_1(X(t)) \leq \Gamma_1 \tag{4.9}$$

is valid. Really,

$$\dot{W}_1 + 2\lambda W_1 = -\alpha_r{}^2(\alpha_r - \lambda)\,x^2 + \alpha_r{}^2 x - \alpha_r{}^2(\alpha_r - \lambda)\,y^2 \leq \frac{\alpha_r{}^2}{4(\alpha_r - \lambda)} = 2\lambda\Gamma_1$$

That is why

$$(W_1 - \Gamma_1)^\bullet + 2\lambda(W_1 - \Gamma_1) \leq 0$$

whence it follows, that for all $t \geq 0$

$$W_1(X(t)) - \Gamma_1 \leq \big[W_1(X(0)) - \Gamma_1\big]e^{-2\lambda t}$$

i.e. (4.9) is true. From (4.9) positive invariance of a set D_0 follows.

Let us prove, that the set D_1 is positively invariant for solutions of the system (4.1). It is obvious, that there is a number $s = s_3$ such, that $s_3 < s_1$ and

$$\int_{s_3}^{s_2} \varphi(s)\,ds = 0$$

The level surfaces of function V_1 are defined by the equations

$$\delta^2\alpha_r^2 b^2\left[\left(x - \frac{\alpha_r b - M_H s}{\alpha_r^2 b}\right)^2 + \left(y + \frac{M_H}{\alpha_r b}\right)^2\right] = \Phi(s, C)$$

where $\Phi(s, C)$ is a one-parameter family of functions

$$\Phi(s, C) = C - 2\int_{s_1}^{s} \varphi(s)\,ds$$

The domain D_1 is bounded by the closed part of a surface $\Phi(s, p_0{}^2)$ and contains a stable stationary point O_1. Under the condition (3.5) $\dot{V}_1 \leq 0$, and $\dot{V}_1 < 0$ in all points ∂D_1, except for points

$$O_i\left(\frac{\alpha_r b - M_H s_i}{\alpha_r^2 b}, -\frac{M_H}{\alpha_r b}, s_i\right) \qquad i = 1, 2$$

Hence by virtue of continuous dependence of solutions on initial data the positive invariance of the set D_1 follows.

Let us consider the cuts S_1 and S_2 of sets D_1 and D_2 respectively by the plane $s = s_1$

$$S_1 = \left\{X :\ \delta^2 b^2 \alpha_r^2\left(x - \frac{M s_2}{\alpha_r^2 b}\right)^2 + \delta^2 b^2 \alpha_r^2 (y - M/\alpha_r b)^2 \leq p_0^2\right\}$$
$$S_2 = \{X :\ \alpha_r^2 x^2 + \alpha_r^2 y^2 \leq 2\Gamma,\ \ s = s_1\}$$

Under the condition (4.8) there exists $\Gamma_1 = \frac{\Gamma_0^2}{2}$ such, that the circle S_2 touches the boundary of a circle S_1 from within. Then we have the inclusion $S_2 \subset S_1 \subset D_1$, from which owing to positive invariancy of the sets D_0 and D_1 we receive a positive invariance of the set G.

So, G is positively invariant, and G contains only one stationary point of the system (4.1) — the point O_1. For function V_1 in the domain G the conditions 1) - 5) are executed. Really, $V_1 > 0$ in all points of G, except for the point O_1, in which $V_1 = 0$, i.e. the conditions 1) and 2) are executed. From the relation (4.6) we have the conditions 3) and 4). Further, it is easy to see, that $\lim\limits_{s\to-\infty} \int_{s_1}^{s} \varphi(s)\,ds = +\infty$. Therefore, condition 5) is executed. Consequenly, any trajectory $X(t, X_0)$ of the system (4.1) such, that $X_0 \in G$, tends to the point O_1. The theorem 4.2 is proved.

Theorem 4.3 *Let $M_H = \bar{\kappa}(1-s)$, $\bar{\kappa} = const$. Then, if the condition (3.7) is executed, the system (4.1) is globally asymptotically stable.*

Proof. Let us make the change of variables

$$\bar{x} = -\alpha_r x + 1 \qquad \bar{y} = \alpha_r y \qquad \bar{s} = s$$

and preserve their names. Then the system (4.1) is reduced to the form

$$\begin{aligned} \dot{x} &= -\alpha_r x + sy \\ \dot{y} &= -\alpha_r y + s - xs \\ \dot{s} &= -\nu y - \kappa s + \kappa \end{aligned} \tag{4.10}$$

where $\nu = \delta b$, $\kappa = \bar{\kappa}\delta$.

Let us introduce the function

$$W_2 = \frac{1}{2}[x - (1-\nu)]^2 + \frac{1}{2}y^2 + \frac{1}{2}s^2$$

and the designation

$$\Gamma_2 = \frac{(\alpha_r - 2\lambda)^2(1-\nu)^2}{8\lambda(\alpha_r - \lambda)} + \frac{\kappa^2}{8\lambda(\kappa - \lambda)}$$

where $\lambda \in [0, \lambda_0]$, $\lambda_0 = \min\{\alpha_r, \kappa\}$. Let us show that for any solution $X(t) = (x(t), y(t), s(t))$ of the system (4.10) the evaluation

$$\overline{\lim_{t\to+\infty}}\, W_2(X(t)) \le \Gamma_2 \tag{4.11}$$

is valid. Really,

$$\begin{aligned} &\dot{W}_2 + 2\lambda W_2 = -\alpha_r x^2 + \alpha_r(1-\nu)x - \kappa s^2 + \kappa s + \lambda x^2 + \\ &+\lambda(1-\nu)^2 - 2\lambda(1-\nu)x + \lambda y^2 + \lambda s^2 = -(\alpha_r - \lambda)x^2 + (\alpha_r - 2\lambda)(1-\nu)x - \\ &-(\kappa - \lambda)s^2 + \kappa s - (\alpha_r - \lambda)y^2 = (\alpha_r - \lambda)\left[x^2 - \frac{(\alpha_r - 2\lambda)(1-\nu)}{\alpha_r - \lambda}x\right] - \\ &-(\kappa - \lambda)\left[s^2 + \frac{\kappa s}{\kappa - \lambda}\right] - (\alpha_r - \lambda)y^2 = \\ &= -(\alpha_r - \lambda)\left[x^2 - \frac{(\alpha_r - 2\lambda)(1-\nu)}{2(\alpha_r - \lambda)}x\right]^2 + \frac{(\alpha_r - 2\lambda)^2(1-\nu)^2}{4(\alpha_r - \lambda)} - \\ &-(\kappa - \lambda)\left[s + \frac{\kappa}{2(\kappa - \lambda)}\right]^2 + \frac{\kappa^2}{4(\kappa - \lambda)} - (\alpha_r - \lambda)y^2 \le \\ &\le \frac{(\alpha_r - 2\lambda)^2(1-\nu)^2}{4(\alpha_r - \lambda)} + \frac{\kappa^2}{4(\kappa - \lambda)} = 2\lambda\Gamma_2 \end{aligned}$$

Hence we have the estimate

$$W_2(X(t)) - \Gamma_2 \le [W_2(X(0)) - \Gamma_2]\, e^{-2\lambda t}$$

which implies (4.11). From (4.11) the dissipativity of (4.10) follows. Consequently the system (4.1) is dissipative as well.

Using the change of variables

$$\sigma = s \qquad \eta = -\nu y - \kappa s + \kappa \qquad z = x - \frac{\kappa}{\nu}(s - s^2)$$

we reduce the system (4.10) to the system

$$\begin{aligned} &\dot{\sigma} = \eta \qquad \dot{\eta} = -(\alpha_r + \kappa)\eta + \nu\sigma z - \psi(\sigma) \\ &\dot{z} = -\alpha_r z + \left(\frac{2\kappa}{\alpha_r \nu} - \frac{1}{\nu}\right)\sigma\eta - \frac{\kappa}{\alpha_r \nu}\eta \end{aligned} \tag{4.12}$$

where $\psi(\sigma) = \frac{\kappa}{\alpha_r}(\sigma_3 - \sigma) + (\alpha_r \kappa + \nu)\sigma - \alpha_r \kappa$.

The stationary set of the system (4.12) consists of one, two or three points.

Let us introduce the function

$$V_2 = \frac{\alpha_r \nu^2}{2(\alpha_r - 2\kappa)} z^2 + \frac{1}{2}\eta^2 + \int_0^\sigma \psi(\sigma)\, d\sigma$$

and calculate the derivative of function V_2 by virtue of the system (4.12)

$$\begin{aligned} \dot{V}_2 = &-\frac{\alpha_r \nu^2}{\alpha_r - 2\kappa} z^2 + \frac{\alpha_r \nu^2}{\alpha_r - 2\kappa}\left(\frac{2\kappa}{\alpha_r \nu} - \frac{1}{\nu}\right)\sigma\eta z - \\ &-\frac{\nu\kappa}{\alpha_r - 2\kappa}\eta z - (\alpha_r + \kappa)\eta^2 + \nu\sigma\eta z - \psi(\sigma)\eta + \\ &+\psi(\sigma)\eta = -\frac{\alpha_r \nu^2}{\alpha_r - 2\kappa} z^2 - \frac{\nu\kappa}{\alpha_r - 2\kappa}\eta z - (\alpha_r + \kappa)\eta^2 \end{aligned} \tag{4.13}$$

If the conditions (3.7) are fulfilled then the quadratic form in right-hand side of equation (4.13) is definitely negative. Repeating the argument used when proving the theorem, 4.2 after the relation (4.6) was obtained, we come to the conclusion that the system (4.12) is dychotomous and consequently the system (4.1) is dychotomous as well. Since the system (4.1) is dissipative and dychotomous it is globally asymptotically stable. The theorem 4.3 is proved.

5 Proof of the Theorems 3.1 and 3.2

Proof of the Theorem 3.1.

1) Under the condition (3.5) the stationary set of the system (3.3), (3.4) consists of two points - the stable point O_1 and the saddle O_2: $O_i\,(0, 0, x_i, y_i, s_i)$, $i = 1, 2$,

$$x_i = \frac{\alpha_r}{\alpha_r^2 + s_i^2} \qquad y_i = -\frac{s_i}{\alpha_r^2 + s_i^2}$$

$$s_{1,2} = \frac{\alpha_r}{2}\left(b \mp \sqrt{B^2 - 4M_H^2}\right)$$

Let the trajectory $X(t,p) = (\psi_x(t,p), \psi_y(t,p), x(t,p), y(t,p), s(t,p))$ of the systems (3.3), (3.4) be bounded on $(0, +\infty)$. Then a set Ω_p of ω - limit points of p is not empty. Let $q \in \Omega_p$, $q = (q_i)$, $i = 1, \ldots, 5$. By virtue of asymptotic stability of the linear system (3.3) we have $q_1 = q_2 = 0$.

Let us introduce the set $\Psi = \{X = X(t,q) : t \in \mathbf{R}^1\}$. It is known (Nemytskij, 1949), that $\Psi \subset \Omega_p$, i.e. all points $X(t,q)$ are ω - limit ones for the point p. Obviously, the system (3.3), (3.4) on the set Ψ can be represented as equations (4.1).

By virtue of equation (4.13) functions $V_2(x(t))$ do not increase with respect to t on some $(\tau, +\infty)$. From here and from the boundedness of $V_2(X(t,q))$ the existence of finite limits

$$\lim_{t\to+\infty} V_2(X(t,q)) = L_1$$
$$\lim_{t\to-\infty} V_2(X(t,q)) = L_2$$

follows. It follows from the boundedness of the trajectory $X(t,q)$, that there exists an ω - limit point q_ω of $X(t,q)$. As it was marked above the trajectory $X(t,q_\omega) \in \Omega_q$ for all , $t \in \mathbf{R}^1$ (Ω_q is the ω - limit set of the point q). Therefore, $V_2(X(t,q_\omega)) = L_1$ for all $t \in \mathbf{R}^1$. Then, using equations (4.13) and (4.12), we receive the identities $\eta(t,q_\omega) \equiv 0$, $z(t,q_\omega) \equiv 0$. From the system (4.12) and $\eta(t,q_\omega) \equiv 0$ it follows, that $\sigma(t,q_\omega) \equiv \text{const}$. So, $\lim\limits_{t\to+\infty} X(t,q) = c_1$, where c_1 — stationary point.

The similar reasoning implies that $\lim\limits_{t\to-\infty} X(t,q) = c_2$, where c_2 is a stationary point. If $c_1 \neq c_2$, then $X(t,q)$ is a doubly asymptotic trajectory, going out from a saddle O_2 and entering either the stable stationary point O_1, or the saddle O_2. Let $\hat{q} \in \Psi$. It means, that there is a sequence $\{t_n\}$, $t_n \to +\infty$, such, that $\lim\limits_{t\to\infty} X(t_n,p) = \hat{q}$, i.e. for any $\varepsilon > 0$ there exists N, such, that for all $n > N$ the inequality $|X(t_n,p) - \hat{q}| < \varepsilon$ is true. Then by virtue of continuous dependence of solutions on initial data we have $\lim\limits_{n\to\infty} X(t_n,p) = O_i$, $i = 1,2$. Hence it follows, that $\hat{q} = O_i$, $i = 1,2$. So, $\Omega_p \subset \Lambda$, i.e. Ω_p consists of isolated points.

The statement 2) of the theorem can be proved just in the same way as the statement 3) of the theorem 4.1. The theorem 3.1 is proved.

Proof of the theorem 3.2.

Under the conditions of the theorem the stationary set of the system (3.3), (3.4) can consist of one, two or three points.

Repeating the argument, used in the teorem 3.1 when proving that the system (3.3), (3.4) is dychotomous, and using the theorem 4.2 and function V_2, we establish that the system (3.3), (3.4) is dychotomous.

Let us prove the dissipativity of the system (3.3), (3.4) for $M_H = \bar{\kappa}(1-s)$. Let us introduce the function

$$W = \frac{1+\nu}{2}(x^2+y^2) + \frac{1}{2}s^2 - x$$

Let us prove, that for any solution $X(t) = \big(\psi_x(t), \psi_y(t), x(t), y(t), s(t)\big)$ of the system (3.3), (3.4) the inequality

$$\overline{\lim_{t\to+\infty}} W(X(t)) \leq \Gamma \tag{5.1}$$

where Γ is a constant, is true. Really

$$\begin{aligned}
&\dot{W} + 2\lambda W = -[(\alpha_r - \lambda)x^2 + (\alpha_r - 2\lambda + \alpha_r(1-\mu)(1+\nu)\psi_x x] - \\
&-[(\alpha_r - \lambda)y^2 - \alpha_r(1+\nu)(1-\mu)\psi_y y] - [(\alpha_r - \lambda)s^2 - (\kappa - \nu\varphi_2(z))s] - \\
&- \left[\nu(\alpha_r - \lambda)x^2 - \alpha_r\nu(1-\mu)\psi_y xs + \frac{\lambda_1}{2}s^2\right] - \\
&- \left[\nu(\alpha_r - \lambda)y^2 - \alpha_r\nu(1-\mu)\psi_x ys + \frac{\lambda_1}{2}s^2\right] - \\
&-\alpha_r^2(1-\mu)\psi_x \leq 2\lambda\varphi(\psi_x,\psi_y) - (x,s)H_x(x,s)^T - (y,s)H_y(y,s)^T
\end{aligned} \tag{5.2}$$

where, we remind, $\nu = \delta b$, $k = \delta\bar{\kappa}$. Here

$$H_i = \begin{pmatrix} \nu(\alpha_r - \lambda) & -\frac{\alpha_r\nu(1-\mu)\psi_i}{2} \\ -\frac{\alpha_r\nu(1-\mu)\psi_i}{2} & \frac{\lambda_1}{2} \end{pmatrix} \qquad i = x, y,$$

$$\begin{aligned}
\varphi(\psi_x,\psi_y) &= \frac{(\alpha_r - 2\lambda + \alpha_r(1+\nu)(1-\mu)\psi_x)^2}{8\lambda(\alpha_r - \lambda)} + \\
&+ \frac{\alpha_r^4(1-\mu)^2(1+\nu)^2\psi_y^2}{8\lambda(\alpha_r - \lambda)} + \frac{(\kappa - \alpha_r\nu(1-\mu)\psi_y)^2}{8\lambda(\kappa - \lambda_2)} + \frac{\alpha_r^2(1-\mu)\psi_x}{2\lambda}
\end{aligned}$$

$$\lambda = \lambda_1 + \lambda_2 \qquad \lambda_1 > 0 \qquad \lambda_2 > 0 \qquad \lambda \in [0,\lambda_0] \qquad \lambda_0 = \min\{\alpha_r, k\}$$

Then

$$\det H_i = \frac{\lambda_1\nu(\alpha_r - \lambda)}{2} - \frac{\alpha_r^2\nu^2(1-\mu)^2\psi_i^2}{4} \qquad i = x, y$$

By virtue of asymptotic stability of the system (3.3) there exists a number $M > 0$ such, that

$$|\psi_x(t)| \le Me^{-\alpha_s t} \qquad |\psi_y(t)| \le Me^{-\alpha_s t} \qquad \text{for all} \quad t > 0$$

It is easy to see, that there exists a number $T > 0$ such, that for all $t \ge T$ we have

$$\det H_i(t) > 0 \qquad i = x, y$$

Hence and from relations (5.2) it follows, that for all $t \ge T$ the relation

$$\dot{W} + 2\lambda W \le 2\lambda\psi(t) \tag{5.3}$$

where $\psi(t) = \varphi(Me^{-\alpha_s t},\ Me^{-\alpha_s t})$, holds. It is obvious, that there is a finite $\lim_{t\to+\infty} \psi(t) = \Gamma$. Therefore, numbers $T_0 \ge T$, k_0, $\alpha_0 > 0$ exist such, that for all $t \ge T_0$

$$\psi(t) \le \Gamma + k_0 e^{-\alpha_0 t}$$

Therefore, from relations (5.2) and (5.3) the evaluation

$$W(X(t)) - \Gamma \le [W(X(0)) - \Gamma]\, e^{-2\lambda t} + \frac{2\lambda k_0}{\alpha_0 - 2\lambda}\left(e^{-2\lambda t} - e^{-\alpha_0 t}\right)$$

follows, whence the inequality (5.1) follows.

From the inequality (5.1) and asymptotic stability of the linear system (3.3) the dissipativity of the system (3.3), (3.4) follows.

Since the system (3.3),(3.4) is dissipative and dychotomous it is globally asymptotically stable. The theorem 3.2 is proved.

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