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A Note on Hohmann Transfer Velocity Kicks

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A Hohmann transfer is a well-known spacecraft manoeuvre, initiated by a horizontal velocity kick Δv_1 which effects a change from an original, say, circular orbit to the Hohmann transfer ellipse in its perigee, and completed by a second horizontal velocity kick Δv_2 in the apogee, to effect a change from the transfer ellipse to a final, say, larger circular orbit.
A velocity kick as mentioned above is apparently instantaneous, and free of any side effects, a very idealized concept, which, as it turns out, is far removed from reality.
Recent investigations into Ward spirals have shed some light into how velocity changes can be brought about. It is shown that a vertical impulse component must be present to accompany a horizontal impulse in order to assure that the altitude remains constant during a horizontal velocity change.

1 Introduction

In Figure 1 a typical Hohmann transfer between two circular orbits is depicted. The required velocity kick (Rimrott, 1989) in juncture 1 is

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) \tag{1}$$

The question as to how to produce this velocity kick turns out to be fraught with unexpected complications.

To simplify matters we will assume that satellite as well as master, are masses of point size. That makes it possible to refer to the radius vector r also as altitude, a somewhat more descriptive designation.

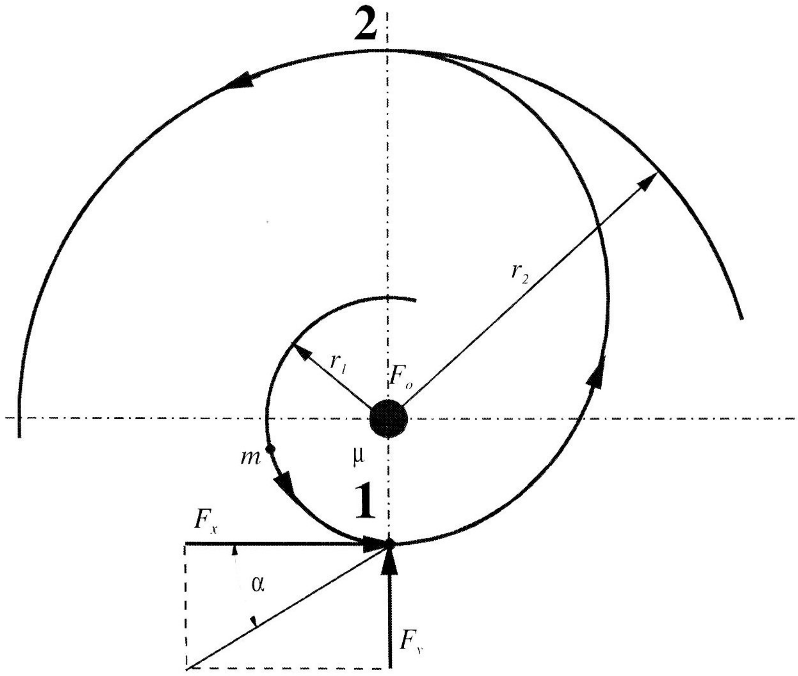

Figure 1. A Hohmann Transfer between two Circular Orbits

There will also be talk of circular motion and of altitude changes, which are obviously terms that exclude each other. It is, however, convenient to retain the concept of circular motion and the associated equations. Thus, when altitude and velocity changes are discussed, they refer more accurately to near-circular orbits.

Lastly, we assume that the satellite mass m remains constant throughout, an assumption that implies that the mass losses due to the firing of the on-board minirockets to produce the impulses $F_x \Delta t$ and $F_y \Delta t$ are negligible.

As far as the present investigation is concerned, conditions at junctures 1 and 2 are essentially similar, such that it suffices to restrict the study and look solely at juncture 1.

Understandably the first thought would be that the fundamental relationship

impulse = momentum change

must somehow apply, i.e.

$$F_x \Delta t = m \Delta v_1 \tag{2}$$

It will be shown that in orbit dynamics equation (2) supplies only part of the answer, with some serious side effects, including the possibility of a result that is the opposite to what might be expected.

The following paper is devoted to an in-depth analysis of the effects of an impulse (2) on the orbit of a point satellite in the gravitational field of a point master.

2 A Horizontal Impulse

The apparently obvious answer to achieve a horizontal velocity change is the application of a horizontal impulse (2). A closer look, however , reveals that a horizontal impulse alone is insufficient. Recent investigations into the Ward spiral (Rimrott and Salustri, 2001) show that a horizontal force F_x on a point satellite m can be taken into consideration by writing Newton's second law equations as

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\mu\, m}{r^2} \tag{3a}$$

$$m\left(r\ddot{\theta} + 2\,\dot{r}\dot{\theta}\right) = F_x \tag{3b}$$

Equations (3a) and (3b) can be combined to remove the variable θ. The result is

$$\frac{d}{dt}\sqrt{r^3\ddot{r} + \mu r} = \frac{F_x}{m} r \tag{4}$$

For near-circular orbits equation (4) can be simplified by realising that the first term on the left side can be neglected, thus leading to

$$\frac{d}{dt}\sqrt{\mu\, r} = \frac{F_x}{m} r \tag{5}$$

which can be rewritten as

$$\dot{r} = \frac{2\,F_x}{\sqrt{\mu}\; m} r^{3/2} \tag{6}$$

i.e. there will be an increase of altitude as long as the force F_x is acting. We introduce $\dot{r} = \Delta r / \Delta t$ and obtain

$$\Delta r(F_x) = \frac{2F_x \Delta t}{\sqrt{\mu}\, m} r^{3/2} \tag{7}$$

Using for the magnitude K of the Kepler force

$$K = \frac{\mu\, m}{r^2} \tag{8}$$

and with

$$v_0^2 = \frac{\mu}{r_1} \tag{9}$$

we can eventually write for the altitude change

$$\Delta r(F_x) = \frac{2v_0 \Delta t}{K_1} F_x = \frac{2mv_0 \Delta v_1}{K_1} \tag{10}$$

and conclude that a horizontal impulse $F_x \Delta t$ produces a positive altitude change (10) as a side effect.

Velocity and altitude on a circular orbit are related by the vis-viva integral

$$v^2 = \frac{\mu}{r} \tag{11}$$

Differentiation with respect to time and re-arranging results in

$$\dot{v} = -\frac{\mu}{2v\, r^2} \dot{r} \tag{12}$$

and together with equation (6) one obtains

$$\dot{v} = -\frac{F_x}{m} \tag{13}$$

i.e., there will be a *decrease* of orbital velocity as long as the thrust force F_x is acting. The relationship looks deceptively like Newton's second law except for the sign, an apparent paradox caused by the fact that F_x is not the whole force acting but only a superimposition upon an already established orbital motion within a central force field.

We now introduce

$$\dot{v} = \frac{\Delta v}{\Delta t} \tag{14}$$

and can then write, with the help of equations (13) and (2),

$$\Delta v(F_x) = -\frac{F_x}{m} \Delta t = -\Delta v_1 \tag{15}$$

We conclude that the horizontal impulse $F_x \Delta t$ produces not only the side effect (10) of an attitude climb but also a negative velocity change, i.e. just opposite of what we are looking for.

The results obtained in this section appear in line 1 of Table 1.

No.	F_x	F_y	Δr	Δv	$\tan\alpha$
1	$m\frac{\Delta v_1}{\Delta t}$	0	$\frac{2mv_0\Delta v_1}{K_1}$	$-\Delta v_1$	0
2	$m\frac{\Delta v_1}{\Delta t}$	$\frac{2mv_0\Delta v_1}{r_1}$	$\frac{mv_0\Delta v_1}{K_1}$	0	$\frac{2\, v_0\Delta t}{r_1}$
3	$m\frac{\Delta v_1}{\Delta t}$	$\frac{4mv_0\Delta v_1}{r_1}$	0	Δv_1	$\frac{4\, v_0\Delta t}{r_1}$
4	$m\frac{\Delta v_1}{\Delta t}$	$\frac{6mv_0\Delta v_1}{r_1}$	$-\frac{mv_0\Delta v_1}{K_1}$	$2\Delta v_1$	$\frac{6\, v_0\Delta t}{r_1}$
5	0	F_y	$-\frac{r_1}{2K_1}F_y$	$\frac{r_1}{2mv_0}F_y$	∞

Table 1. The Influence of a Vertical Thrust on Altitude Change and Velocity Change

3 A Vertical Impulse

The magnitude K of the Kepler force (Rimrott, 1989) and the altitude r are related by equation (8). The equation's partial derivative is

$$\frac{\partial K}{\partial r} = -2\frac{\mu m}{r^3} = -2\frac{K}{r} \tag{16}$$

or, if we set $\frac{\partial K}{\partial r} = \frac{\Delta K}{\Delta r}$ we obtain

$$\Delta r = -\frac{r_1}{2K_1}\Delta K_1 \tag{17}$$

for the altitude change in juncture 1. Now we introduce

$$F_y = \Delta K_1 \tag{18}$$

to get eventually

$$\Delta r(F_y) = -\frac{r_1}{2K_1}F_y \tag{19}$$

and conclude that the altitude change due to an additional vertical thrust F_y (in addition to the Kepler force) is negative.

The velocity v on a circular orbit is given by the vis-viva integral (11)

$$v^2 = \frac{\mu}{r} = \frac{K\,r}{m} \tag{20}$$

from which we obtain

$$\frac{\partial v}{\partial K} = \frac{r}{2mv} \tag{21}$$

or with $$\frac{\partial v}{\partial K} = \frac{\Delta v}{\Delta K} = \frac{\Delta v}{F_y}$$

$$\Delta v\left(F_y\right) = \frac{r_1\,F_y}{2mv_0} \tag{22}$$

representing the velocity change in juncture 1 due to a vertical thrust force F_y. The results of this section appear as line 5 in Table 1.

4 Both Impulses

Now let us stipulate that it is possible to remove the side effect (10) of an altitude rise by a simultaneous application of both, the horizontal impulse $F_x \Delta t$ and a vertical impulse $F_y\,\Delta t$ (Figure 1).

We specify that the application of both impulses should not lead to a change of altitude, i.e. that

$$\Delta r = \Delta r(F_x) + \Delta r(F_y) = 0 \tag{23}$$

From equations (10) and (19) we can write

$$\Delta r = \frac{2v_0}{K_1} F_x\,\Delta t - \frac{r_1}{2K_1} F_y = 0 \tag{24}$$

or

$$F_y = \frac{4v_0 \Delta t}{r_1} F_x \tag{25}$$

It is interesting to note that the vertical thrust force is a function of the horizontal impulse. Invoking equation (2) we may thus write

$$F_y = \frac{4mv_0 \Delta v_1}{r_1} \tag{26}$$

an equation which shows that the thrust F_y turns out to be independent of the duration Δt of the horizontal impulse.

Equations (2) and (25) lead to

$$\tan\alpha = \frac{F_y}{F_x} = \frac{4v_0\Delta t}{r_1} \tag{27}$$

For the velocity change we form

$$\Delta v = \Delta v(F_x) + \Delta v(F_y) \tag{28}$$

and obtain, with the help of equations (15) and (22)

$$\Delta v = -\frac{F_x\Delta t}{m} + \frac{F_y r_1}{2mv_0} \tag{29}$$

or with equation (25)

$$\Delta v = \frac{F_x\Delta t}{m} = \Delta v_1 \tag{30}$$

i.e. exactly the velocity change (2) that is required.

The results obtained above appear in line 3 of Table 1.

The thrust required to be applied to a point satellite to achieve a velocity change (30) is thus

$$F = \sqrt{F_x^2 + F_y^2} \tag{31}$$

which, from equation (27), results in

$$F = \frac{F_x}{\cos\alpha} \tag{32}$$

Equation (27) shows that the smaller the minirocket burn duration Δt, the smaller the angle α. While equation (32) shows that the smaller the angle α, the closer the resultant thrust F to its horizontal component F_x.
To carry the investigation a little further we could try and satisfy the condition $\Delta v = 0$ and calculate the magnitude of the vertical thrust F_y necessary to effect this.

We make the ansatz

$$\Delta v(F_y) = \Delta v(F_x) \tag{33}$$

or, from equations (15) and (22)

$$\frac{r_1}{2mv_0}F_y = \frac{\Delta t}{m}F_x \tag{34}$$

giving us an

$$F_y = \frac{2v_0F_x\Delta t}{r_1} = \frac{2mv_0\Delta v_1}{r_1} \tag{35}$$

The associated altitude change is then

$$\Delta r = \Delta r(F_x) + \Delta r(F_y) \tag{36}$$

and, from equations (10) and (19)

$$\Delta r = \frac{v_0 F_x \Delta t}{K_1} = \frac{m v_0 \Delta v_1}{K_1} \tag{37}$$

with a

$$\tan\alpha = \frac{F_y}{F_x} = \frac{2 v_0 \Delta t}{r_1} \tag{38}$$

The results (35), (37) and (38) are entered as line 2 in Table 1. Table 1 displays very instructively which influence an increase of the vertical impulse has on the horizontal velocity change that results.

We summarize the preceding as follows:

1. In order to achieve the required Δv_1, equation (1), a horizontal impulse of

$$F_x \Delta t = m \Delta v_1 \tag{39}$$

   must be applied.

2. To ensure that Δv_1 is actually attained and that there is no side effect (10), the horizontal impulse (39) has to be accompanied by a vertical thrust force of

$$F_y = \frac{4 m v_0 \Delta v_1}{r_1} \tag{40}$$

3. The resultant force of

$$F = \sqrt{F_x^2 + F_y^2} = \frac{m \Delta v_1}{\Delta t \cos\alpha} \tag{41}$$

   is to be applied at an angle α (Figure 1), with α from

$$\tan\alpha = \frac{4 v_0 \Delta t}{r_1} \tag{42}$$

5 Numerical Example

Let us look at a Hohmann transfer manoeuvre of a point satellite of mass m = 100 000 kg on a circular orbit of radius r_1 = 6 600 km about the Earth (μ = 398 601.19 km³/s²) to be lifted into a circular orbit of r_2 = 7 000 km by means of a Hohmann transfer. The required first velocity kick (1) is then

$$\Delta v_1 = \sqrt{\frac{398\ 601.19}{6\ 600}} \left(\frac{\sqrt{2(7\ 000)}}{6\ 600 + 7\ 000} - 1 \right) \text{km/s} = 0.113\ \text{km/s} \tag{43}$$

The on-board mini-rockets can produce a thrust of 400 kN. The burn duration (2) is thus

$$\Delta t = \frac{100\,000\,(0.113)}{400}\,\text{s} = 28.25\ \text{s} \tag{44}$$

The horizontal impulse (2) is

$$F_x \Delta t = 400\,(28.25)\,\text{kNs} = 11\,300\ \text{kNs} \tag{45}$$

The Kepler force (8) has a magnitude of

$$K_1 = \frac{398\,601.19\,(100\,000)}{6\,600^2}\,\text{kN} = 915\ \text{kN} \tag{46}$$

while the orbital speed (9) is

$$v_0 = \sqrt{\frac{398\,601.19}{6\,600}}\,\text{km/s} = 7.771\ \text{km/s}$$

Unless precautions are taken the impulse (45) has as a side effect an altitude climb (10) of

$$\Delta r(F_x) = \frac{2(7.771)11\,300}{915}\,\text{km} = 192\ \text{km} \tag{47}$$

In order to prevent the altitude climb (47), a vertical mini-rocket burn of thrust force (26)

$$F_y = \frac{4(100\,000)\,7.771\,(0.113)}{6\,600}\,\text{kN} = 53.22\ \text{kN} \tag{48}$$

is required. The resultant thrust force (31) is

$$F = \sqrt{400^2 + 53.22^2}\ \text{kN} = 403.52\ \text{kN} \tag{49}$$

It must act for 28.25 s at an angle from the horizontal obtained from equation (27) and amounting to

$$\alpha = 7{,}58° \tag{50}$$

6 Conclusions

We find that the desired Δv_1 for a Hohmann transfer can be produced by a horizontal impulse only if the altitude is held constant. This in turn means that a simultaneous vertical impulse must be provided. Similar conditions apply, of course, also for the second velocity change Δv_2, which completes the Hohmann transfer.

Literature

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List of Symbols

E = orbital energy, J

F = thrust, N

K = Kepler force, N

a = semi-major axis, m

m = mass, kg

r = radius, m

t = time, s

v = speed, m/s

Δr = altitude change, m, of satellite

Δv = speed change, m/s, of satellite

Δv_1 = speed change, m/s, required for Hohmann transfer

α = angle from horizontal

μ = gravitational attraction parameter, m^3/s^2

Subscripts

$_x$ = horizontal (parallel to Earth surface)

$_y$ = vertical (perpendicular to Earth surface)

$_0$ = on circle 1

$_1$ = at juncture 1

$_2$ = at juncture 2

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