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The Maximum Principle of Pontryagin in the Heat-mass Transfer Problem

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A flow of a viscous heat-conducting Newtonian fluid in a long and narrow rectangular basin is studied. The heat-physical characteristics depend substantially on the temperature, which, in turn, is a function of coordinates and time. The problem on a flow optimal control with a quadratic quality functional is considered. The heat flow to a free surface of a fluid is assumed to be a control parameter. By the maximum principle of Pontryagin some conditions of a control optimality are found. The problem considered has arosen from studying one of the glass melting problems.

1 Problem Setting

Consider the three-dimensional flow of a viscous heat-conducting fluid in the Cartesian coordinate system $x = (x_1, x_2, x_3)$. We assume that the viscosity and heat-conductivity of the medium depend substantially on the temperature, which is a function $T(x,t)$ of coordinates and time. The fluid is assumed to be Newtonian with some "effective" dependence of the viscosity on the temperature $\nu(T) = \mu(T)/\rho$, where ρ is the density. The general system of equations of motion and energy can be found, for example, in Smyshlyaev et al. (1989).

We also assume that the mass forces take the form of an Archimedean lift only (the Oberbek – Boussinesq approximation) and in the other addends $\rho = \text{const}$. In the equation of energy the dissipative terms are neglected, and besides, we assume that the fluid moves in the channel that strongly extends along the axis ox_1, and therefore the change of speed in the ox_2-direction may also be neglected. In this case we have

$$\frac{\partial u}{\partial x_2} \sim 0 \qquad \frac{\partial v}{\partial x_2} \sim 0 \qquad \frac{\partial v}{\partial x_3} \sim 0 \qquad \frac{\partial w}{\partial x_2} \sim 0$$

where $V(u, v, w)$ is the velocity vector of a fluid particle. The equations of motion, energy, and continuity can be written as

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + 2\frac{\partial}{\partial x_1}\left(\nu\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\nu\frac{\partial v}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\nu\left(\frac{\partial u}{\partial x_3} + \frac{\partial v}{\partial x_1}\right) \tag{1}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \frac{\partial}{\partial x_1}\left(\nu\frac{\partial v}{\partial x_1}\right) \tag{2}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \beta g(T - T_0) + 2\frac{\partial}{\partial x_3}\left(\nu\frac{\partial w}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\nu\frac{\partial v}{\partial x_1}\right) + \frac{\partial}{\partial x_1}\nu\left(\frac{\partial u}{\partial x_3} + \frac{\partial w}{\partial x_1}\right) \tag{3}$$

$$\frac{dT}{dt} = \frac{\partial}{\partial x_1}\left(k\frac{\partial T}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\frac{\partial T}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(k\frac{\partial T}{\partial x_3}\right) \tag{4}$$

$$\frac{\partial w}{\partial x_3} + \frac{\partial u}{\partial x_1} = 0 \tag{5}$$

Here p is the pressure; β is the cubical dilatation coefficient divided by the fluid density, g is the gravity, $k(T)$ is the thermal diffusivity. The left-hand sides of equations (1) – (4) are total derivatives. The initial and boundary conditions take the form

$$u = u_0 \qquad v = v_0 \qquad w = w_0 \qquad T = T_0 \qquad p = p_0 \qquad \text{for} \qquad t = 0 \tag{6}$$

$$u = v = w = 0 \quad \text{for} \quad x \in \Gamma \quad \text{(solid boundary)} \tag{7}$$

$$\partial V_\tau / \partial n = 0 \qquad V_n = 0 \qquad \text{for} \qquad x \in \gamma \quad \text{(free surface)} \tag{8}$$

$$T(x,t) = T_0(s) \qquad p = p_a + \rho g(H - x_3) \qquad \text{for} \quad x = s \in \Gamma \tag{9}$$

$$\partial T / \partial n = q(s,t) \qquad p = p_a \qquad \text{for} \quad x = s \in \gamma \tag{10}$$

where V_τ, V_n are the projections of the velocity on the tangent τ and the normal n to the surface, s are points (coordinates) of the surface, H is the fluid depth; $x_3 = 0$ corresponds to the lower boundary point, p_a is the pressure in the external medium (atmosphere); $q(s,t)$ is the heat flow of the input fluid due to heat conductivity (it is a control parameter of the process).

The optimal control problem consists of the need for the best choice of the heat flow $q(s,t)$. We assume that the quality of such a choice is characterized by the degree of approximation of the solutions to the given characteristics of the process under consideration. In the following, the approximation is described by the value of the integral difference (i. e., a functional) between the velocities and temperatures of the fluid particles, which can be found from both the solutions and the real process. In addition it is desired that the minimal deviation shows minimal energy consumption (the quantity of heat).

For a simple application of the maximum principle of Pontryagin (Pontryagin et al.,1976; Syrazetdinov, 1977) we need some new notations of the variables: $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = (u, v, w, T, p/\rho)$ is the vector of the required functions, $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ are the right-hand sides of equations (1) – (4) after the transposition of the terms of inertia and the convective terms from the left-hand sides of the equations to the right-hand ones.

Denoting derivatives by indices, we obtain:

$$\Phi_1 = -\varphi_{5x_1} + 2(\nu\varphi_{1x_1})_{x_1} + (\nu\varphi_{2x_1})_{x_2} + (\nu\varphi_{1x_3} + \varphi_{2x_1})_{x_3} - \varphi_1\varphi_{1x_1} - \varphi_3\varphi_{1x_3}$$

$$\Phi_2 = -\varphi_{5x_2} + (\nu\varphi_{2x_1})_{x_1} - \varphi_1\varphi_{2x_1}$$

$$\Phi_3 = -\varphi_{5x_3} + 2(\nu\varphi_{3x_3})_{x_3} + (\nu\varphi_{2x_1})_{x_2} + (\nu(\varphi_{1x_3} + \varphi_{3x_1}))_{x_1} - \\ -\varphi_1\varphi_{3x_1} - \varphi_3\varphi_{3x_3} + \beta g(\varphi_4 - T_0)$$

$$\Phi_4 = (k\varphi_{4x_1})_{x_1} + (k\varphi_{4x_2})_{x_2} - \varphi_1\varphi_{4x_1} - \varphi_2\varphi_{4x_2} - \varphi_3\varphi_{4x_3}$$

$$\Phi_5 = \varphi_{1x_1} + \varphi_{3x_3} = 0$$

In this case equations (1) – (5) can be written in the more compact form

$$\varphi_{it} = \Phi_i \qquad i = 1, ..., 4 \qquad \Phi_5 = 0 \tag{11}$$

The initial and boundary conditions (6) – (10) can be transformed in just the same way.

Let the above-mentioned functional of the quality control have the form:

$$I(\theta, \varphi, \varphi_x, q) = \int_0^\theta \left[\int_G F_1(t, x, \varphi, \varphi_x) dx + \int_S F_2(t, x, \varphi, q) dS \right] dt \tag{12}$$

where θ is the process time (of heating); G is the volume with the boundary $S = \Gamma \cup \gamma$; F_1, F_2 are nonnegative functions, defined in the domain G and on surface S respectively (for example, the sums of $\alpha_i(\varphi_i - \varphi_{i0})^2$, $\beta_i(\varphi_{ix} - \varphi_{ix_0})^2$, for $i = 1, ..., 5$. In this case $\varphi_{i0}, \varphi_{ix_0}$ are some given ("desired") values of velocities, temperature, and pressure of the fluid and α_i, β_i are the "weight" coefficients, determined by the peculiarities of the process). Functional (12) takes into account the difference of prescribed values, which are found from the systems (1) – (5) and from those given (for example, obtained from the experiment), and the value of energy supplied. Obviously, for the process to be optimal it is necessary that the functional I takes a minimal value. In this case the corresponding solutions of equation (11) are called optimal.

2 The Optimal Control Conditions for the Process of Fluid Motion and Fluid Heating

We assume that in G the function $F_1(t, x, \varphi, \varphi_x)$ has square integrable derivatives with respect to the variables x, φ, φ_x up to second order, and has derivatives of first order with respect to t. In addition, we assume that the function $F_2(t, x, \varphi, q)$ admits an extension from S to G such that it has smooth properties similar to F_1. In this case, functional (12) can be represented as

$$I(\theta, \varphi, \varphi_x, q) = \int_0^\theta dt \int_G (F_1 + F_{21x_1} + F_{22x_2} + F_{23x_3}) dx \tag{13}$$

where $3F_{2j} = F_2 / cos(n, x_j), j = 1, ..., 3$.

We consider a conjugate system of functions $\psi_i(t, x)$, having sufficient smoothness, $H = -F_1 - F_{2jx_j} + \psi_i \Phi_i$ (for $j = 1, ..., 3,\ i = 1, ..., 5$ the summation is assumed to be over i, j), and that of equations

$$\psi_{it} = -H_{\varphi_i} + (H_{\varphi_i x_j x_k})_{x_j x_k} \qquad i = 1, ..., 5 \tag{14}$$

where the summation is over $j, k = 1, ..., 3$, and the mixed derivatives do not depend on the order of differentiation.

The boundary conditions have the form

$$\psi_i(\theta, 0) = 0 \qquad \psi_i(t, x) = 0 \qquad \text{for} \quad x = s \in S \tag{15}$$

Now we construct the variation of functional (13). By definition of the quantity $H(t, x, \varphi, \varphi_x, \varphi_{xx}, \psi, q)$, the variation of the integrand term can be written as

$$-\Delta H + (\psi, \Delta\Phi) = -\Delta H + \psi_i \Delta\Phi_i \qquad i = 1, ..., 5 \tag{16}$$

where

$$\Delta H = H(t, x, \varphi + \Delta\varphi, ..., \psi, q + \Delta q) - H(t, x, \varphi, ..., \psi, q) = \Delta_q H + \Delta_\varphi H$$

The partial variations $\Delta_q H$ and $\Delta_\varphi H$ take the form

$$\Delta_q H = H(t, x, \varphi, \varphi_x, \varphi_{xx}, \psi, q + \Delta q) - H(t, x, \varphi, \varphi_x, \varphi_{xx}, \psi, q) \tag{17}$$

$$\begin{aligned} \Delta_\varphi H = H(t, x, \varphi + \Delta\varphi, ..., \psi, q + \Delta q) - H(t, x, \varphi, ..., \psi, q + \Delta q) = \\ = (H_\varphi, \Delta\varphi) + (H_{\varphi_x}, \Delta\varphi_x) + (H_{\varphi_{xx}}, \Delta\varphi_{xx}) + \varepsilon \end{aligned} \tag{18}$$

where the difference of the values of function is represented by the Taylor formula with the residual ε in Lagrangian form. The symbol (,) denotes the scalar products of the vectors $H_\varphi, H_{\varphi_x}, H_{\varphi_{xx}}$ and the corresponding vectors of variation. The expression for ε can be written in the following form

$$2\varepsilon = (H_{i\varphi_j\varphi_k}, \Delta\varphi_j \Delta\varphi_k) \qquad j, k = 0, ..., 3$$

where the summation is over $i = 1, ..., 6$. The representation $\varphi_0 = \varphi$ is a vector of the required quantities, $\Delta\varphi_0 = \Delta\varphi$ is its variation, $\varphi_j = \varphi_{x_j}$ are derivatives with respect to x_j, the mixed derivatives do not depend on their order. H_i are as following

$$H_i = H(t, x, \varphi + \theta_i \Delta\varphi, ..., \psi, q + \Delta q) \qquad 0 \leq \theta_i \leq 1 \qquad i = 1, ..., 6$$

With equation (11) we have

$$\Delta\Phi = \Delta\varphi_t \qquad \Delta\Phi_5 = 0$$

Let us rewrite equation (16) by using equations (17) and (18):

$$-\Delta H + (\psi, \Delta\Phi) = -\Delta_q H - (H_\varphi, \Delta\varphi) - (H_{\varphi_x}, \Delta\varphi_x) - (H_{\varphi_{xx}}, \Delta\varphi_{xx}) + (\psi, \Delta\varphi_t) \tag{19}$$

We transform the last addend in the right-hand side of equation (19) in the following way

$$(\psi, \Delta\varphi_t) = (\psi, \Delta\varphi)_t - (\psi_t, \Delta\varphi)$$

Similarly, we rewrite the third and the fourth addends:

$$(H_{\varphi_x}, \Delta\varphi_x) = (H_{\varphi_x}, \Delta\varphi)_x - ((H_{\varphi_x})_x, \Delta\varphi)$$
$$(H_{\varphi_{xx}}, \Delta\varphi_{xx}) = (H_{\varphi_{xx}}, \Delta\varphi)_{x^2} - ((H_{\varphi_{xx}})_{x^2}, \Delta\varphi)$$

where the symbols $(...)_x$ and $(...)_{x^2}$ are the sums of the first and the second derivatives with respect to the same coordinates $x = (x_1, x_2, x_3)$ respectively. Substituting the last relations into equation (19) and unifying the second addends in each expression, by virtue of system (14), we obtain

$$(H_\varphi + \psi_t - (H_{\varphi_x})_x - (H_{\varphi_{xx}})_{x^2}, \Delta\varphi) = 0$$

Taking into account the integration in time in the variations of functional (13) and the boundary conditions (6) and (15), we get

$$(\psi, \Delta\varphi)|_{t=\theta} - (\psi, \Delta\varphi)|_{t=0} = 0$$

In this case, we assume that the vector of the desired functions φ is given at the initial instant and therefore its variation is zero

$$\Delta\varphi|_{t=0} = 0$$

Passing to the variations of functional (13), we transform the remaining first addends in the right-hand side of equation (19) by the Ostrogradsky formula:

$$\int_G (H_{\varphi_x}, \Delta\varphi)_x dx = \int_S ((H_{\varphi_x})_n, \Delta\varphi) ds$$
$$\int_G (H_{\varphi_{xx}}, \Delta\varphi)_{x^2} dx = \int_S (A, n) ds$$

where $(H_{\varphi_x})_n$ are the projections of components of the vector H_{φ_x} on the normal n to the surface S; $A = (A_1, A_2, A_3)$; $A_k = (H_{\varphi_{x_k x_k}}, \Delta\varphi)_{x_k} + (H_{\varphi_{x_k x_{k+1}}}, \Delta\varphi)_{x_{k+1}}$; $k = 1, ..., 3$; $k+1$ denotes the cyclic transposition of indices by the rule: $1+1=2$, $2+1=3$, $3+1=1$.

Finally, the variation of functional (13) is

$$\Delta I = -\int_0^\theta \left\{ \int_G (\Delta_q H - \varepsilon) dx + \int_S [((H_{\varphi_x})_n, \Delta\varphi) + (A, n)] \right\} dt$$

Since the value ε is of second order (a product of variations), we can conclude that the maximum of the function $H(t, x, \varphi, \varphi_x, \varphi_{xx}, \psi, q)$ under the given control $q_0(t, x)$ is attained under the condition of a minimum of this functional.

This statement is the maximal principle of Pontryagin for the process of heat-mass transfer. Since the problem is considered in a sufficiently general form, the conditions for an extremum are rather lengthy. In the partial cases the problem can substantially be simplified substantially, by using a stronger requirement for the conjugate state (14), (15) and the initial model.

Literature

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