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About the Heat Transfer of a Moving Viscous Liquid in Canals of Various Shapes

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The paper deals with a problem of heat transfer of a viscous liquid moving between two plates or in a cylindrical pipe. Exact solutions are obtained and compared with the results from literature.

The problem of heat transfer of a viscous liquid moving between two stationary plates was considered by Leibenson (1955). Mekhtiev and Mamedov (1966) have undertaken the effort to solve the problem when the upper plate moves with the constant speed while the lower one is fixed. In the present paper the exact solution of this problem is obtained and the results are compared with those mentioned above.
Let us assume that the coordinate system origin is located at the point O, the axis z is directed parallel to the plates and the axis y is orthogonal to them. Let $2h$ be the distance between the plates and the upper plate moves with the constant velocity V_0 (Figure 1).

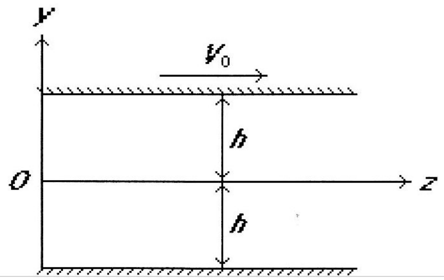

Figure 1. Scheme of the Plates Location

It is assumed that the liquid viscosity and thermal constants do not depend on the temperature, the dissipation of energy is absent, the heat transfer is steady. Let the temperature in the inlet cross-section of the flat pipe be constant and equal to T_1. According to Leibenson (1955) the Fourier-Kirchhoff equation of the temperature distribution, neglecting the term $\frac{\partial^2 T}{\partial z^2}$, has the form

$$\frac{\partial^2 T}{\partial y^2} - \frac{c_p \rho V}{\lambda}\frac{\partial T}{\partial z} = 0 \qquad |y| \le h \tag{1}$$

and the boundary conditions are

$$\begin{aligned} &T(\pm h, z) = T_2 \\ &T(y,0) = T_1 \qquad |y| \le h \end{aligned} \tag{2}$$

where T is the liquid temperature; ρ is the liquid density; c_p is the average heat capacity at constant pressure; λ is the thermal conductivity. Assuming that the motion is parallel to the axis z and the flow is laminar (Couette flow) we use the expression (Schlichting, 1956) for the velocity distribution

$$V(x) = \frac{3w}{2+3u}\left(1+u+ux-x^2\right) \tag{3}$$

where

$$u = \frac{V_0}{\frac{\Delta p h^2}{l\eta}} \qquad x = \frac{y}{h}$$

η is the dynamic viscosity; Δp is the pressure loss by friction on the distance l; $w = \dfrac{\Delta p h^2 (2+3u)}{6l\eta}$ is the average velocity. The dimensionless plate velocity u is varied from 0 to 2/3.
Thus the problem reduces to the solution of the differential equation

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{3 c_p \rho w h}{\lambda (2+3u)} (1+u+ux-x^2) \frac{\partial \theta}{\partial \xi} = 0 \qquad |x| \le 1 \tag{4}$$

$$\theta(\pm 1, \xi) = 0$$

$$\theta(x,0) = 1 \qquad |x| \le 1$$

where

$$\theta(x,\xi) = \frac{T - T_2}{T_1 - T_2} \qquad \xi = \frac{z}{h}$$

The general solution of the equation (4) is searched in the form

$$\theta(x,\xi) = \sum_{n=1}^{\infty} C_n t_n(x) e^{-\beta_n \xi}$$

As a result we come to the Sturm-Liouville problem with coefficients C_n and β_n.

$$\frac{d^2 t_n(x)}{dx^2} + k_n (1+u+ux-x^2) t_n(x) = 0 \tag{5}$$

$$t_n(\pm 1) = 0 \tag{6}$$

where

$$k_n = \frac{3 w c_p \rho h \beta_n}{\lambda (2+3u)}$$

In Mekhtiev and Mamedov (1966), the solution of the equation (5) was approximately accomplished by the Ritz method with the function $t_n(x)$ chosen in the form

$$t_n(x) = A_n (1+u+ux-x^2)$$

The exact solution of the equation (5) can be presented in the form (Eishinskii, 1999):

$$t_n(x) = e^{-\frac{\sqrt{k_n}(x-\frac{u}{2})}{2}} \left[C_1 \left(x - \frac{u}{2} \right) \Phi \left(\frac{3}{4} - \frac{\sqrt{k_n}}{4} \left(\frac{u+2}{2} \right)^2, \frac{3}{2}, \sqrt{k_n} \left(x - \frac{u}{2} \right)^2 \right) + \right.$$

$$\left. + C_2 \Phi \left(\frac{1}{4} - \frac{\sqrt{k_n}}{4} \left(\frac{u+2}{2} \right)^2, \frac{1}{2}, \sqrt{k_n} \left(x - \frac{u}{2} \right)^2 \right) \right] \tag{7}$$

where Φ is the confluent hypergeometric function.

Taking into account the boundary conditions (6) we obtain the transcendental equation for the determination of the eigenvalues k_n

$$\left(1-\frac{u}{2}\right)\Phi\left(\frac{3}{4}-\frac{\sqrt{k_n}}{4}\left(\frac{u+2}{2}\right)^2,\frac{3}{2},\sqrt{k_n}\left(1-\frac{u}{2}\right)^2\right)\Phi\left(\frac{1}{4}-\frac{\sqrt{k_n}}{4}\left(\frac{u+2}{2}\right)^2,\frac{1}{2},\sqrt{k_n}\left(1+\frac{u}{2}\right)^2\right)+$$

$$\left(1-\frac{u}{2}\right)\Phi\left(\frac{3}{4}-\frac{\sqrt{k_n}}{4}\left(\frac{u+2}{2}\right)^2,\frac{3}{2},\sqrt{k_n}\left(1-\frac{u}{2}\right)^2\right)\Phi\left(\frac{1}{4}-\frac{\sqrt{k_n}}{4}\left(\frac{u+2}{2}\right)^2,\frac{1}{2},\sqrt{k_n}\left(1+\frac{u}{2}\right)^2\right)=0 \quad (8)$$

In Figure 2 the dependence of the lowest eigenvalue k_1 on the parameter u is presented as well as the same value calculated in the following form

$$k_n=\frac{35(3u^2+u)}{3(70u^3+84u^2+56u+16)}$$

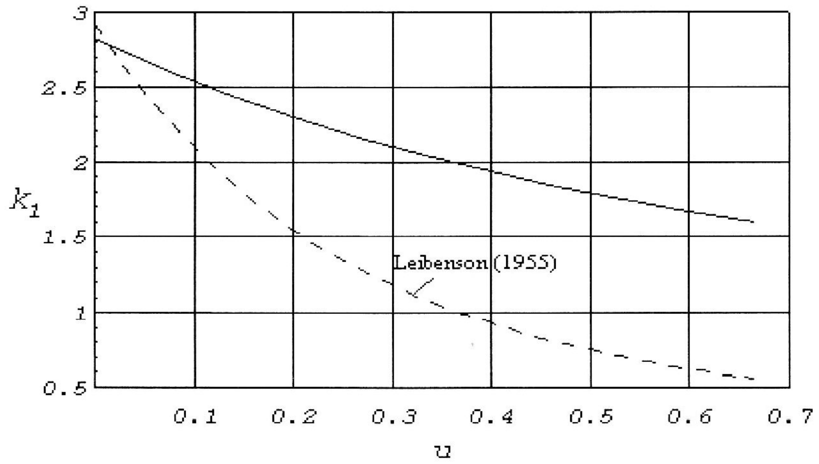

Figure 2. The Values of k_1 Depending on u

The average temperature θ^* in the cross-section of the flat pipe is obtained as the quotient of the total heat amount and the total heat capacity of the liquid transfering through the cross-section in unit time (Leibenson, 1955).
Thus

$$\theta^*=\frac{\int_{-1}^{1}V(x)\theta(x,\xi)dx}{\int_{-1}^{1}V(x)dx} \quad (9)$$

Now we introduce the Nusselt number, which characterizes the heat exchange from the moving liquid to the upper and lower plates

$$Nu_1=-\frac{2}{\theta^*}\left(\frac{\partial\theta}{\partial x}\right)_{x=1} \qquad Nu_2=\frac{2}{\theta^*}\left(\frac{\partial\theta}{\partial x}\right)_{x=-1} \quad (10)$$

When $\xi \to \infty$ the limit equations are

$$Nu_1 = -\frac{2t_1'(1)\int_{-1}^{1} V(x)dx}{\int_{-1}^{1} V(x)t_1(x)dx} \qquad Nu_2 = \frac{2t_1'(-1)\int_{-1}^{1} V(x)dx}{\int_{-1}^{1} V(x)t_1(x)dx} \tag{11}$$

Assuming u=0, i.e. considering the flow between the fixed plates, we obtain k_1=2.827 from equation (8), and from relations (11) follows that

$$Nu_1 = Nu_2 = \frac{4\sqrt{k_1}e^{-\frac{\sqrt{k_1}}{2}}(1-\sqrt{k_1})\Phi\left(\frac{5-\sqrt{k_1}}{4},\frac{3}{2},\sqrt{k_1}\right)}{3\int_0^1 (1-x^2)e^{-\frac{\sqrt{k_1}x^2}{2}}\Phi\left(\frac{1-\sqrt{k_1}}{4},\frac{1}{2},\sqrt{k_1}x^2\right)dx} = 3.77035$$

This is in good agreement with the approximate value 3.78743 obtained in (Leibenson, 1955).
The results of computations with formulas (11) are plotted in Figure 3. Here the results obtained by the Ritz method (Mekhtiev and Mamedov, 1966) are presented too. Obvious non-coincidence of them shows that the approximation $t_n(x) = A_n(1+u+ux-x^2)$ is inadequate to give the satisfactory problem solution.

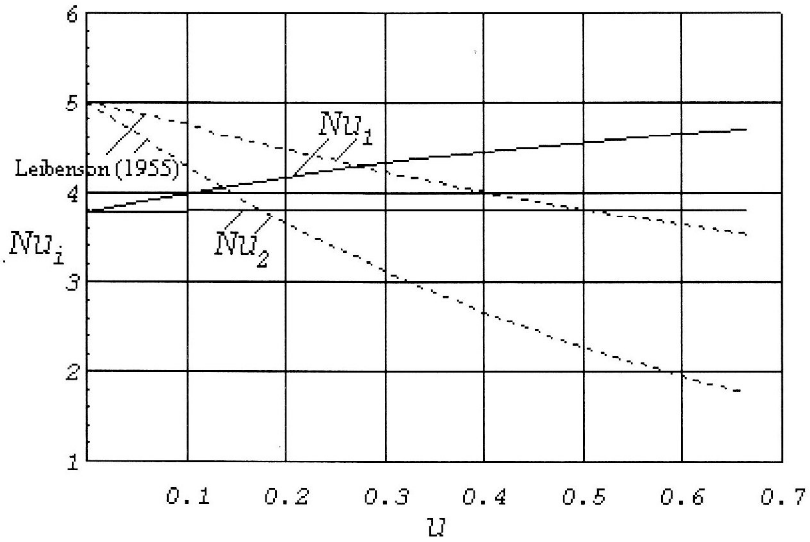

Figure 3. The Values of Nu_i Depending on u

We turn now to the consideration of an axi-symmetric flow in a cylindrical pipe of radius a, which also has been considered in Leibenson (1955). The temperature distribution can be presented by the Fourier-Kirchhoff equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} - \frac{c_p \rho V}{\lambda}\frac{\partial T}{\partial z} = 0 \qquad 0 \le r \le a \tag{12}$$

and the boundary conditions are

$$T\big|_{r=a} = 0$$

$$T\big|_{z=0} = T_0 \qquad\qquad 0 \le r \le a$$

where T is the excess of temperature of the fluid with respect to that of the wall, which is thus the origin point; r is the distance from the pipe axis (axis z).
Stokes law of the velocity distribution through the cross-section can be taken in the form (Leibenson, 1955) and (Schlichting, 1956)

$$V = 2w\left(1-\zeta^2\right) \qquad\qquad \zeta = \frac{r}{a}$$

where w is the average liquid velocity.
The general solution of the equation (12) is searched in the form

$$T(\zeta, z) = \sum_{n=1}^{\infty} C_n T_n(\zeta) e^{-\beta_n z}$$

which leads to the Sturm-Liouville problem

$$\frac{d^2 T_n(\zeta)}{d\zeta^2} + \frac{1}{\zeta}\frac{dT_n(\zeta)}{d\zeta} + k_n(1-\zeta)T_n(\zeta) = 0 \qquad\qquad k_n = \frac{2c_p \rho u \beta_n a^2}{\lambda} \tag{13}$$

$$T_n(1) = 0 \tag{14}$$

The exact solution of the differential equation (13) can be presented in the form (Eishinskii, 1999)

$$T_n(\zeta) = Ce^{-\frac{\sqrt{k_n}\zeta^2}{2}} \Phi\left(\frac{2-\sqrt{k_n}}{4}, 1, \sqrt{k_n}\zeta^2\right) \tag{15}$$

from which, taking into account boundary conditions (14), we can obtain the transcendental equation for the determination of k_n:

$$\Phi\left(\frac{2-\sqrt{k_n}}{4}, 1, \sqrt{k_n}\right) = 0 \tag{16}$$

The lowest eigenvalue k_1 found from the equation (16) is equal to $k_1 = 7.3136$ and is in good agreement with the value 7.33 obtained in Leibenson (1955).
The Nusselt number is

$$Nu = -\frac{2}{T^*}\left(\frac{\partial T}{\partial \zeta}\right)_{\zeta=1}$$

where

$$T^* = \frac{\int_0^1 T(\zeta)V(\zeta)\zeta d\zeta}{\int_0^1 V(\zeta)\zeta d\zeta}$$

is the average temperature through the cross-section of cylindrical pipe.
When $z \to \infty$ the limit value is

$$Nu = \frac{\left(2-\sqrt{k_1}\right)\sqrt{k_1}\, e^{-\frac{\sqrt{k_1}}{2}} \Phi\left(\frac{6-\sqrt{k_1}}{4}, 2, \sqrt{k_1}\right)}{4\int_0^1 \left(1-\zeta^2\right) e^{-\frac{\sqrt{k_1}\zeta^2}{2}} \Phi\left(\frac{2-\sqrt{k_1}}{4}, 1, \sqrt{k_1}\zeta^2\right)\zeta\, d\zeta} = 3.65679$$

which is in good agreement with the value 3.65220 obtained approximately in Leibenson (1955).
Thus the classic results of Leibenson (1955) obtained by the approximation method are proved on the basis of the exact solution of the problem.

Literature

1. Eishinskii, A.M.: Torsion of the anisotropic and non-homogeneous bodies. Dnepropetrovsk: Polygraphist, (1999), S. 395.

2. Leibenson, L.S.: Collection of works, Vol. 3. Publishing of Academy of Science of the USSR, (1955).

3. Mekhtiev, V.M.; Mamedov, M.A.: To the question about the heat exchange for the moving viscous liquid. Izvestia vusov. Oil and gas, (1966), N 11, 79-81.

4. Schlichting, G.: Theory of the boundary layer. Moscow, IL, (1956).

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