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Open Orbits in Satellite Dynamics

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Recent investigations into the effects of a constant atmospheric drag force on an essentially circular satellite orbit have led to an analytical expression for the resulting orbit, which is obviously not exactly circular, but so close to it that profitable simplifications can be made in its derivation. In the present paper several of the resulting type orbits are analyzed and compared.

1 Closed and Open Satellite Orbits

For a point satellite of constant mass m in the gravitational field μ of a point master, the Kepler force (Chobotov, 1996) is of magnitude

$$K = \frac{\mu m}{r^2} \tag{1}$$

If in additon to the Kepler force (1) there acts a negative transverse force of magnitude D, which we will assume to be constant, then Newton's second law requires that

$$-\frac{\mu m}{r^2} = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{2a}$$

$$-D = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \tag{2b}$$

Depending on the magnitude of the term $\ddot{r}$ in equation (2a) we may obtain for the orbit shape an ellipse, a circle, or a spiral. We shall refer to ellipses and circles as closed orbits, and to spirals as open orbits.

The solution of equations (2) is exact if

$$D = 0 \qquad \text{and} \qquad \ddot{r} \neq 0 \tag{3}$$

and results in an ellipse as orbit. Conditions (3) by the way, may also result in a parabola or a hyperbola; but strictly speaking, they are not orbits (*Latin*: orbis = ring).

The solution is also exact if

$$D = 0 \quad \text{and} \quad \ddot{r} = 0 \quad \text{and} \quad \dot{r} = 0 \tag{4}$$

and results in a circle as orbit (Figure 1).

The solution obtainable if

$$D \neq 0 \quad \text{and} \quad 0 \leq m\ddot{r} << K \tag{5}$$

is approximate and results in a spiral orbit (Figure 2).

The present paper is devoted to an investigation of the spiral orbits which result from conditions (5).

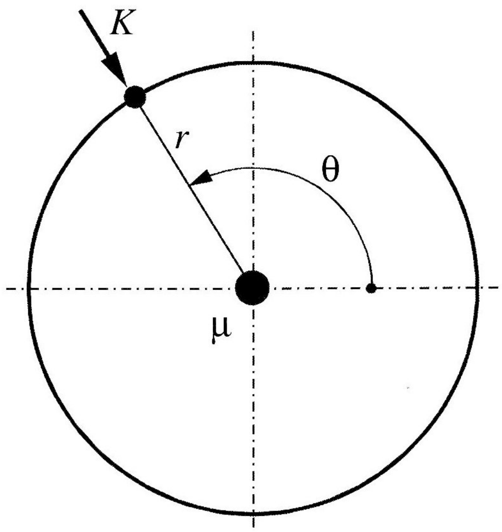

Figure 1. A Circular Closed Orbit

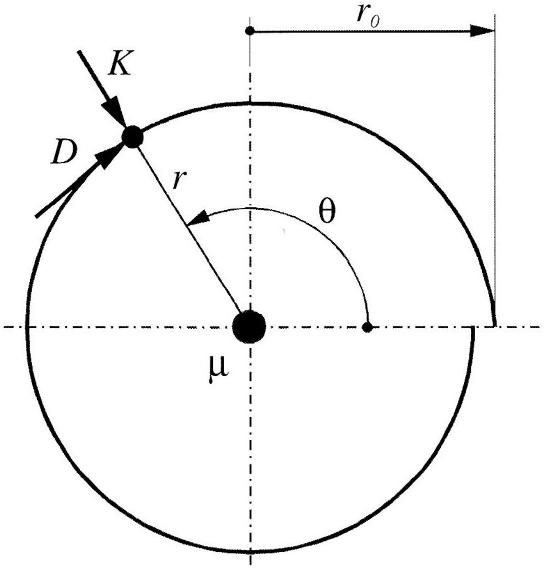

Figure 2. A Spiral Open Orbit

2 The Ward Spiral

By assuming that $\ddot{r}$ can be neglected in equation (2a), equations (2a) and (2b) can be combined into

$$\dot{r} = \frac{-2D}{\sqrt{\mu}\, m} r^{3/2} \tag{6}$$

Equation (6) can be integrated in closed form. With the limits $r = r$ and $r = r_o$ for $t = t$ and $t = 0$, the Ward spiral (Ward, 2000) results (line 1, Table 1)

$$r = \frac{r_o}{\left(1 + \frac{D}{m}\sqrt{\frac{r_o}{\mu}}\, t\right)^2} \tag{7}$$

This parameter equation can be changed into polar form (Figure 2) by using another approximation, viz.

$$r_o\theta = v_o t \tag{8}$$

as a consequence of which the Ward spiral takes on the form (line 1, Table 2)

$$r = \frac{r_o}{\left(1+\frac{D}{K_o}\theta\right)^2} \tag{9}$$

A point satellite on a Ward spiral has an orbital energy, with eccentricity $\varepsilon \approx 0$ and semi-major axis $a \approx r$, of

$$E = -\frac{\mu m}{2a} = -\frac{\mu m}{2r_o}\left(1+\frac{D}{m}\sqrt{\frac{r_o}{\mu}}t\right)^2 = -\frac{\mu m}{2r_o}\left(1+\frac{D}{K_o}\theta\right)^2 \tag{10}$$

It changes by approximately

$$\Delta E = -\frac{\mu m}{2r_o}\frac{4\pi D}{K_o} = -2\pi D r_o \tag{11}$$

per orbit, i.e., it decreases as the orbit spirals inward, with

$$K_o = \frac{\mu m}{r_o^2} \tag{12}$$

It was Dr. C.A. Ward's unique contribution to recognize the significance of the insignificance of the term $\ddot{r}$ in equation (2a), which indeed is often so small that it can be successfully neglected. This situation is e.g. typical for satellites at high altitude on essentially circular orbits that are subjected to a constant atmospheric drag, which causes only minor orbital changes, i.e., such satellites remain on essentially circular orbits (Rimrott and Salustri, 2001). Ward's approach leads to an integrable expression, which eventually results in the Ward spiral (7). In the derivation process for this spiral there are numerous approximations, and the reader is well advised to keep this in mind.

These approximations may be of a cumulative nature and lead eventually to a result that represents a poor approximation over all, or they may be of a corrective nature leading to a relatively good approximation eventually.

We can change equation (6) into polar form if we approximate the angular speed either by

$$\dot{\theta} = \sqrt{\frac{\mu}{r^3}} \tag{13}$$

or if we allow for the fact that the radius vector changes by only a small amount, by the somewhat simpler

$$\dot{\theta} = \sqrt{\frac{\mu}{r_0^3}} \tag{14}$$

With equation (13) we obtain instead of equation (6)

$$\frac{dr}{d\theta} = -\frac{2D}{\sqrt{\mu m}}\frac{r^{3/2}}{\dot{\theta}} = -\frac{2D}{\mu m}r^3 \tag{15}$$

Integrated between the limits $r = r$ and $r = r_0$ for $\theta = \theta$ and $\theta = 0$ we have

$$r = \frac{r_o}{\sqrt{1+\frac{4D}{K_o}\theta}} \tag{16}$$

which appears in line 3 of Table 2.

If on the other hand we employ equation (14) then equation (6) becomes

$$\frac{dr}{d\theta} = -\frac{2Dr_o^{3/2}}{\mu m}r^{3/2} \tag{17}$$

which results in

$$r = \frac{r_o}{\left(1+\frac{D}{K_o}\theta\right)^2} \tag{18}$$

the same as equation (9). Equation (18) appears in line 1 of Table 2.

Equation (6) can be differentiated with respect to time and then used to supply us with an idea as to the magnitude of $m\ddot{r}$. We have then

$$m\ddot{r} = 6\frac{D}{K}D \tag{19}$$

an amount which fulfils condition (5) if the drag force D is small enough.

Equations (16) and (18) are a first indication that the Ward spiral (7) may have competition from other spirals, as we will see in the subsequent section.

3 Other Spirals

The whole concept of open orbits is based on the assumption that $\ddot{r}$ is so small that it can be neglected. This concept also obviously includes the case $\ddot{r} = 0$. This in turn means that $\dot{r}$ = constant, which we can bring about by letting

$$\dot{r} = -\frac{2D}{\sqrt{\mu}\, m}r_o^{3/2} \tag{20}$$

instead of equation (6). Integration of equation (20) leads to

$$r = r_o\left(1-\frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t\right) \tag{21}$$

which is entered in line 2 of Table 1. Using equation (8) we obtain the polar form

$$r = r_o\left(1-\frac{2D}{K_o}\theta\right) \tag{22}$$

which appears in line 2 of Table 2.

With the help of equation (13) we have

$$\frac{dr}{d\theta} = -\frac{2Dr_o^{3/2}}{\mu m} r^{3/2} \tag{23}$$

which when integrated supplies us with

$$r = \frac{r_o}{\left(1+\frac{D}{K_o}\theta\right)^2} \tag{24}$$

which is entered in line 1 of Table 2.

If on the other hand we choose to make use of equation (14) then

$$\frac{dr}{d\theta} = -\frac{2Dr_o^3}{\mu m} \tag{25}$$

and

$$r = r_o\left(1-\frac{2D}{K_o}\theta\right) \tag{26}$$

Equation (26) is entered in line 2 of Table 2.

Of the many other possibilities to approximate the radius change rate let us begin with

$$\dot{r} = -\frac{2D}{\sqrt{\mu m r_o^{3/2}}} r^3 \tag{27}$$

Upon integration we obtain the spiral

$$r = \frac{r_o}{\sqrt{1+\frac{4D}{m}\sqrt{\frac{r_o}{\mu}}t}} \tag{28}$$

and for the force

$$m\ddot{r} = \frac{12D\,r^3}{K\,r_o^3} D \tag{29}$$

Equations (27), (28) and (29) have been entered in line 3 of Table 1. If we change the independent variable we can write

$$\frac{dr}{d\theta} = -\frac{2D}{\mu m} r^3 \tag{30}$$

and integrate to get

$$r = \frac{r_o}{\sqrt{1+\frac{4D}{K_o}\theta}} \tag{31}$$

Equations (30) and (31) appear as line 3 in Table 2.

On the other hand if we begin with

$$\dot{r} = -\frac{2D}{\sqrt{\mu}m\sqrt{r_o}}r^2 \tag{32}$$

we get

$$r = \frac{r_o}{1+\frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t} \tag{33}$$

and

$$m\ddot{r} = \frac{8Dr}{Kr_o}D \tag{34}$$

These three expressions are entered in line 4 in Table 1.
If we change the independent variable and write

$$\frac{dr}{d\theta} = -\frac{2Dr_o}{\mu m}r^2 \tag{35}$$

we obtain

$$r = \frac{r_o}{1+\frac{2D}{K_o}\theta} \tag{36}$$

a hyperbolic spiral. Equations (35) and (36) have been entered in line 4 of Table 2.

Lastly, let us use the approximation

$$\dot{r} = -\frac{2D\sqrt{r_o}}{\sqrt{\mu}\,m}r \tag{37}$$

which leads to

$$r = r_o \exp\left(-\frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t\right) \tag{38}$$

and to

$$m\ddot{r} = \frac{4Dr_o}{Kr}D \tag{39}$$

Equations (37), (38) and (39) appear in line 5 of Table 1. If we write, instead of equation (37),

$$\frac{dr}{d\theta} = -\frac{2Dr_o^2}{\mu m}r \tag{40}$$

and integrate we get

$$r = r_o \exp\left(-\frac{2D}{K_o}\theta\right) \tag{41}$$

a logarithmic spiral. Equations (40) and (41) are entered in line 5 of Table 2.

	1 **Time Derivative**	2 **Force**	3 **Radius**	4 **Series Expansion**
1	$\dot{r} = -\frac{2D}{\sqrt{\mu m}} r^{3/2}$	$m\ddot{r} = \frac{6D}{K} D$	$r = \frac{r_o}{\left(1 + \frac{D}{m}\sqrt{\frac{r_o}{\mu}}t\right)^2}$	$r = r_o\left(1 - \frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t + \ldots\right)$
2	$\dot{r} = -\frac{2Dr_o^{3/2}}{\sqrt{\mu m}}$	$m\ddot{r} = 0$	$r = r_o\left(1 - \frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t\right)$	
3	$\dot{r} = -\frac{2D}{\sqrt{\mu m} r_o^{3/2}} r^3$	$m\ddot{r} = \frac{12Dr^3}{Kr_o^3} D$	$r = \frac{r_o}{\sqrt{1 + \frac{4D}{m}\sqrt{\frac{r_o}{m}}t}}$	$r = r_o\left(1 - \frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t + \ldots\right)$
4	$\dot{r} = -\frac{2D}{\sqrt{\mu m}\sqrt{r_o}} r^2$	$m\ddot{r} = \frac{8Dr}{Kr_o} D$	$r = \frac{r_o}{1 + \frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t}$	$r = r_o\left(1 - \frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t + \ldots\right)$
5	$\dot{r} = -\frac{2D\sqrt{r_o}}{\sqrt{\mu m}} r$	$m\ddot{r} = \frac{4Dr_o}{Kr} D$	$r = r_o \exp\left(-\frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t\right)$	$r = r_o\left(1 - \frac{2D}{m}\sqrt{\frac{r_o}{\mu}}t + \ldots\right)$

$$K = \frac{\mu m}{r^2}$$

Table 1. Parameter Equations

	1 **Derivative**	2 **Radius**	3 **Type**	4 **Series Expansion**
1	$\frac{dr}{d\theta} = -\frac{2Dr_o^{3/2}}{\mu m} r^{3/2}$	$r = \frac{r_o}{\left(1 + \frac{D}{K_o}\theta\right)^2}$	Ward	$r = r_o\left(1 - \frac{2D}{K_o}\theta + \ldots\right)$
2	$\frac{dr}{d\theta} = -\frac{2Dr_o^3}{\mu m}$	$r = r_o\left(1 - \frac{2D}{K_o}\theta\right)$	Archimedes	
3	$\frac{dr}{d\theta} = -\frac{2D}{\mu m} r^3$	$r = \frac{r_o}{\sqrt{1 + \frac{4D}{K_o}\theta}}$		$r = r_o\left(1 - \frac{2D}{K_o}\theta + \ldots\right)$
4	$\frac{dr}{d\theta} = -\frac{2Dr_o}{\mu m} r^2$	$r = \frac{r_o}{1 + \frac{2D}{K_o}\theta}$	Hyperbolic	$r = r_o\left(1 - \frac{2D}{K_o}\theta + \ldots\right)$
5	$\frac{dr}{d\theta} = -\frac{2Dr_o^2}{\mu m} r$	$r = r_o \exp\left(-\frac{2D}{K_o}\theta\right)$	Logarithmic	$r = r_o\left(1 - \frac{2D}{K_o}\theta + \ldots\right)$

$$K_o = \frac{\mu m}{r_o^2}$$

Table 2. Polar Equations

4 Assessment

Tables 1 and 2 have each a column 4, where the series expansions of the five different spirals are given. These series expansions turn out to be the same for all spirals, an indication that they are essentially equivalent for small values of the drag force D. Thus at first sight all five orbit spirals appear of equal suitability. Amongst them the Archimedean spiral appeals because of its simplicity. However, there is one property of the Ward spiral that makes it stand out from all the others, in that the Ward spiral predicts the applied torque correctly, as is shown in the following. The others do not, as the reader is invited to confirm!

The satellite's angular momentum, with $\varepsilon \approx 0$ and the Ward spiral's equation (7), is

$$H = m\sqrt{\mu r} = m\sqrt{\mu r_o}\frac{1}{1+\frac{D}{m}\sqrt{\frac{r_o}{\mu}}t} \tag{42}$$

Its derivative with respect to time is the torque

$$\dot{H} = -\frac{r_o}{\left(1+\frac{D}{m}\sqrt{\frac{r_o}{\mu}}t\right)^2}D = -rD \tag{43}$$

q.e.d.

5 Conclusion

The insignificance of the term $m\ddot{r}$ in Newton's second law in certain circumstances can be successfully exploited to obtain an analytic expression for open satellite orbits. In the present paper it has been shown that the spiral obtained by Ward shares characteristics with many other spirals, all of which lead to the same expression, when expanded into a series and broken off after the second term. As to the question which spiral represents an open satellite orbit the best, one might be inclined to give the nod to the Archimedean spiral because of its simplicity, until one realizes that the Ward spiral stands out from all others by a significant property.

Literature

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