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Buckling Analysis of Axially Compressed Square Elastic Tubes with Weakly Supported Edges

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The local buckling of weakly supported thin-walled square tubes under axial compression is studied. By means of asymptotic methods, an expression for the critical pressure is found. The asymptotic results agree well with the numerical results obtained with the Finite Element Method (FEM) and with the sweep method. The dependence of the critical loading on the shell length and on the type of edge supporting is analyzed.

1 Introduction

The buckling analysis of thin-walled cylindrical shells subjected to axial compressive forces by means of the classical buckling theory usually delivers a significantly higher value for the critical load, than that obtained experimentally. One of the main reasons for the disagreement between the theoretical and the experimental results is the sensitivity of the critical loading to the imperfections of the shell. Another reason for the disagreement is the effect of the real boundary conditions. Nachbar and Hoff (1962) have reported that the ratio of the axial critical load value for a circular cylindrical shell with a free edge to the classical critical value for that shell with freely supported edges (Timoshenko and Woinowsky-Krieger, 1959) is equal to 0.37. The difference between the boundary conditions imposed in the theoretical analysis and the real boundary conditions may be a reason for the significant disagreement between the theoretical and the experimental results.

In the present paper we consider the case of an elastic tube with square cross-sectional area compressed between two rigid plates. Only friction forces act in the planes of contact between the ends of the tube and the plates.

If the friction forces are large enough to prevent any shell edge displacements in the plane of the plates, the boundary conditions can be modeled as freely supported edges. The critical load for such boundary conditions has been first obtained by F. Bleich (1952). In this case the buckling mode covers the entire shell uniformly.

Assuming that the friction forces are equal to zero we obtain the other limiting case — a shell with weakly supported edges. In this paper, it is for such boundary conditions that we construct a new approximate solution of the buckling problem. The value obtained for the critical loading is less than that corresponding to a freely supported tube, and the buckling mode is localized near one of the edges of the tube. Similar results have been obtained for a circular cylindrical shell with a weakly supported edge (Nachbar and Hoff, 1962, Tovstik, 1995).

2 Formulation of the Problem

We consider a tube with a square cross-sectional area, each four faces of which are rectangular plates of width a and length b, as shown in Figure 1.

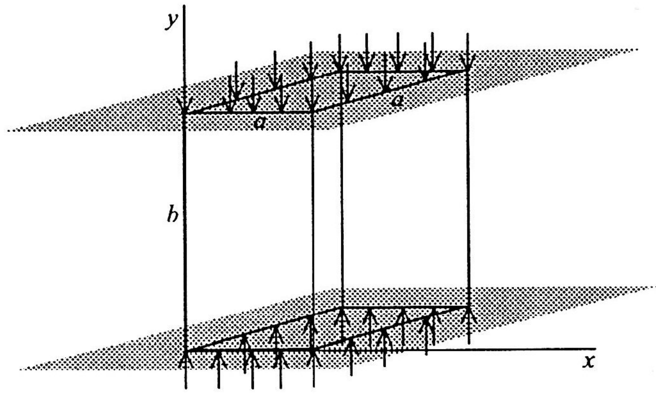

Figure 1. A Thin Square Tube Compressed Between Two Plates.

Both plate edges $y = 0$ and $y = b$ are subjected to the uniformly distributed load $t\sigma$, where t is the plate thickness, and σ is a pressure. The differential equation for the deflection $w^{(k)}$ of the k^{th}-plate has the form (Bleich, 1952)

$$D\Delta\Delta w^{(k)} + t\sigma w^{(k)}_{,yy} = 0 \qquad k = 1,2,3,4 \tag{1}$$

where

$$\Delta w = w_{,xx} + w_{,yy} \qquad D = \frac{Et^3}{12(1-\nu^2)} \tag{2}$$

E is Young's modulus, and ν is Poisson's ratio.

We consider the initial problem under the following assumptions. First, we neglect the longitudinal compression-stretching of the plates and suppress any rigid body motion of the tube. Then, we suppose that the plates are rigidly joint along the line where they meet each other. And finally, we assume that the bending moments at the line where the plates meet are equal to each other.

As a result of these assumptions we obtain the following boundary conditions for each plate (Figure 2).

$$\begin{aligned} &w^{(k)}(0,y) = w^{(k)}(a,y) = 0 \\ &w^{(k)}_{,x}(a,y) = w^{(k+1)}_{,x}(0,y) \qquad k = 1,2,3,4 \\ &w^{(k)}_{,xx}(a,y) = w^{(k+1)}_{,xx}(0,y) \end{aligned} \tag{3}$$

In equations (3) we use a sum modulo 4, i.e., for example, $4 + 1 = 1 \pmod 4$. The second condition in equations (3) prevents the flattening of the tube.

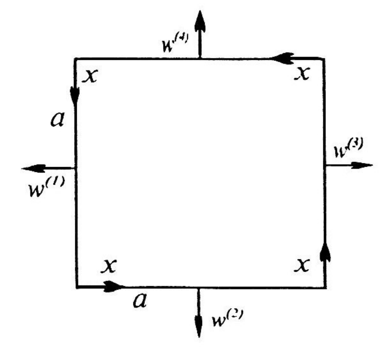

Figure 2. Local Coordinates for the Tube Faces.

The solution $w^{(k)}(x,y) = (-1)^{(k)} W^{(k)}(y) \sin(\pi x/a)$ satisfies the boundary conditions (3). Substituting this solution into equation (1) we find the ordinary differential equation for the function $W^{(k)}(y)$:

$$\frac{d^4 W^{(k)}}{dy^4} - 2\frac{\pi^2}{a^2}\frac{d^2 W^{(k)}}{dy^2} + \frac{\pi^4}{a^4} W^{(k)} + \lambda \frac{\pi^2}{a^2}\frac{d^2 W^{(k)}}{dy^2} = 0 \tag{4}$$

where

$$\lambda = \frac{t\sigma a^2}{D\pi^2} \tag{5}$$

The solution $W(y)$ of equation (4) does not depend on k. Equation (4) also describes the buckling of a plate with the freely supported edges $x = 0$ and $x = a$. Therefore, the initial problem is reduced to the study of a plate two edges of which are freely supported, whereas for the other two the boundary conditions have to be imposed.

3 Solution of the Buckling Problem

We assume first that the edges of the plate $y = 0$ and $y = b$ are freely supported, i. e.

$$W = \frac{d^2 W}{dy^2} = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b \tag{6}$$

In this case the solution of equation (4) has the form $W(y) = \sin(\pi n y/b)$, $n = 1, 2, \ldots$. Substituting this solution into equation (4) we find the following expression for the eigenvalues:

$$\lambda_n = \frac{b^2 n^2}{a^2} + 2 + \frac{a^2}{b^2 n^2} \tag{7}$$

Minimizing λ_n by n we obtain the critical eigenvalue $\lambda_{c1} = \min_n \lambda_n$, and from equation (5) the corresponding critical pressure σ_{cr}. To find the critical eigenvalue we solve the equation $d\lambda_n/dn = 0$ and find $n_{c1} = b/a$. If the ratio b/a is an integer, then $\lambda_{c1} = 4$ and

$$\sigma_{c1} = 4D\pi^2/(ta^2) \tag{8}$$

otherwise $\lambda_{c1} > 4$ and $\lambda_{c1} \to 4$ as $b/a \to \infty$ (Bleich, 1952).

Now we consider the boundary conditions

$$\frac{d^2 W}{dy^2} - \nu \frac{\pi^2}{a^2} W = 0 \qquad \frac{d^3 W}{dy^3} - (2-\nu)\frac{\pi^2}{a^2}\frac{dW}{dy} + \lambda \frac{\pi^2}{a^2}\frac{dW}{dy} = 0 \qquad \text{for} \quad y = 0 \quad \text{and} \quad y = b \tag{9}$$

corresponding to the case when the square tube is compressed between two rigid plates without friction, and therefore has weekly supported edges. The last term in the second condition is the component of the compressive force in the direction of the normal to the deformed plate.

We seek the solution of equation (4) in the form $W(y) = e^{\beta \pi y/a}$. After substituting this solution into (4) the following equation is obtained for β

$$\beta^4 + (\lambda - 2)\beta^2 + 1 = 0 \tag{10}$$

We seek the eigenvalue λ satisfying the inequality $0 < \lambda < 4$. Equation (10) has two roots

$$\beta_{1,2} = \frac{1}{2}\left(\pm i\sqrt{\lambda} - \sqrt{4-\lambda}\right) \tag{11}$$

with negative real parts, and two roots

$$\beta_{3,4} = \frac{1}{2}\left(\pm i\sqrt{\lambda} + \sqrt{4-\lambda}\right) \tag{12}$$

with positive real parts. The solution of equation (4) that decreases away from the edge $y = 0$ has the form

$$W(y) = C_1 e^{\beta_1 \pi y/a} + C_2 e^{\beta_2 \pi y/a} \tag{13}$$

where C_1 and C_2 are unknown constants. We choose the constants to make solution (13) satisfy the boundary conditions (9) for $y = 0$. Condition (9) for $y = b$ will be fulfilled approximately in the assumption that the ratio b/a is large enough.

Substituting equation (13) into the boundary conditions (9) for $y = 0$, we obtain the system of two linear algebraic equations

$$\begin{aligned} &C_1(\beta_1^2 - \nu) + C_2(\beta_2^2 - \nu) = 0 \\ &C_1[\beta_1^3 + \beta_1(\lambda - 2 + \nu)] + C_2[\beta_2^3 + \beta_2(\lambda - 2 + \nu)] = 0 \end{aligned} \tag{14}$$

System (14) has nontrivial solution if its determinant is equal to zero, i.e.

$$(\beta_1^2 - \nu)[\beta_2^3 + \beta_2(\lambda - 2 + \nu)] = (\beta_2^2 - \nu)[\beta_1^3 + \beta_1(\lambda - 2 + \nu)] \tag{15}$$

Using formulas $\beta_1^2 + \beta_2^2 = 2 - \lambda$, $\beta_1^2\beta_2^2 = 1$, following from equation (10), we find

$$\lambda_{c2} = 3 - 2\nu - \nu^2 \tag{16}$$

The approximate value of the critical axial pressure is then

$$\sigma_{c2} = \pi^2 D\lambda_{c2}/(ta^2) = \frac{\pi^2 E t^2 (3 - 2\nu - \nu^2)}{12a^2(1-\nu^2)} = \frac{\pi^2 E t^2}{12a^2}\frac{3+\nu}{1+\nu} \tag{17}$$

Considering that Poisson's ratio ν satisfies the inequality $0 \leq \nu \leq 0.5$ leads to

$$\frac{7}{3}\frac{\pi^2 E t^2}{12a^2} \leq \sigma_{c2} \leq 3\frac{\pi^2 E t^2}{12a^2} \tag{18}$$

For arbitrary values of ν and b/a the inequality $0.4375 \leq \sigma_{c2}/\sigma_{c1} \leq 0.75$ is satisfied. For example, for a tube made of steel $\nu = 0.3$ and $\sigma_{c2}/\sigma_{c1} \leq 0.5775$.

Since β_1 and β_2 are complex conjugate it follows from the first equation in (14) that the coefficients C_1 and C_2 are also complex conjugate. We seek them in the form

$$C_1 = A + iB, \quad C_2 = A - iB \tag{19}$$

Substituting these coefficients in equations (14) we obtain

$$A = C\frac{1+\nu}{2}\sqrt{\lambda_{c2}} \qquad B = -C\frac{1-\nu^2}{2} \tag{20}$$

where C is an arbitrary constant. Substituting equations (11), (19) and (20) into the solution (13) we obtain the buckling mode

$$w(x,y) = De^{[-(1+\nu)\pi y/(2a)]}\sin[\sqrt{\lambda_{c2}}\pi y/(2a) - \alpha]\sin(\pi x/a) \tag{21}$$

where $\alpha = \arctan[\sqrt{\lambda_{c2}}/(1-\nu)]$, and D is an arbitrary constant.

In Figure 3 the function $w(a/2, y)$ is plotted for $D = 1\ m$, $a = 1\ m$, and $\nu = 0.3$.

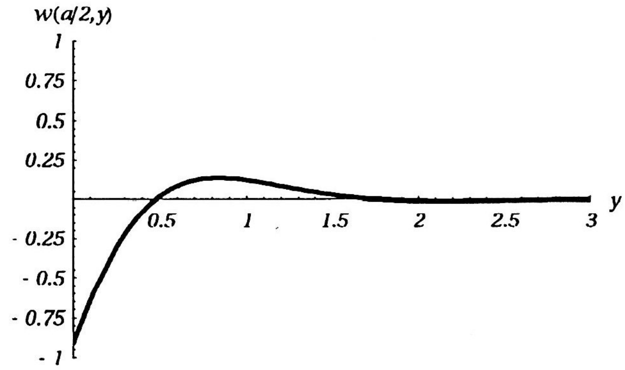

Figure 3. Function $w(a/2, y)$ vs. y.

The 3D plot for the function $w(x, y)$ for $D = 1\ m$, $b = 3\ m$, $a = 1\ m$, $\nu = 0.3$ is shown in Figure 4. The buckling mode oscillates as y varies and decreases rapidly away from the tube edge $y = 0$.

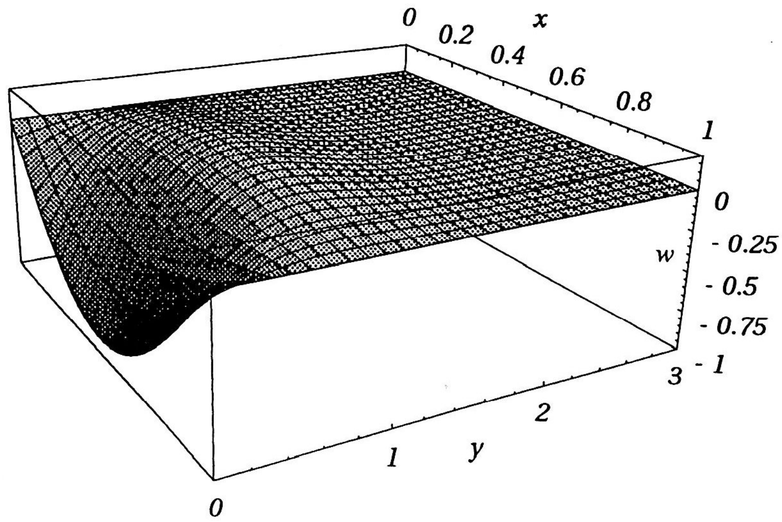

Figure 4. Buckling Mode $w(x, y)$.

The same approximate solution may be constructed near the other plate edge $y = b$. It is observed that the initial problem is symmetrical with respect to the transverse midplane of the tube.

It is interesting to compare the results for tubes with square cross-sectional area to those for tubes with circular cross-secional area of the same perimeter and the same generatrix length. Then, $4a = 2\pi R$, where R is the radius of the circular tube and a is the side of the square tube. We denote as ξ the ratio t/a, and consider first the freely supported tubes. Then, for a circular tube the critical load is $\sigma_{c1}^{cir} = \frac{E\pi}{2\sqrt{3(1-\nu^2)}}\xi$ (Timoshenko and Woinowsky-Krieger, 1959), and from equation (8), for a square tube the critical load is $\sigma_{c1}^{sq} = \frac{\pi^2 E}{3(1-\nu^2)}\xi^2$. It is usually assumed that the thin shell theory is applicable if $\xi \leq 0.1$. It means that for any ν the inequality $\xi < \frac{\sqrt{3(1-\nu^2)}}{2\pi}$ is satisfied, and therefore $\sigma_{c1}^{sq} < \sigma_{c1}^{cir}$.

For the same tube with weakly supported edges the critical load decreases. For a square tube it follows from equation (17) that $\sigma_{c2}^{sq} = K_{sq}\sigma_{c1}^{sq}$, where $0.4375 \leq K_{sq} \leq 0.75$, and for a circular tube $\sigma_{c2}^{cir} = K_{cir}\sigma_{c1}^{cir}$, where $K_{cir} = 0.5$ (Tovstik, 1995). In this case the inequality $\sigma_{c2}^{sq} < \sigma_{c2}^{cir}$ is also valid for thin shells.

As it might be predicted the circular tube resists better to axial compression than the corresponding square tube. Increasing the number of the tube's faces to $n > 4$, we might expect that the critical load for such a multifaced tube converges to that of the circular one, as $n \to \infty$, but the character of convergence as n increases requires special considerations.

A similar method may be used to study the vibration of the tube. Indeed, instead of equation (10) the characteristic equation for the vibration of the tube is given by

$$\beta^4 - 2\beta^2 + 1 - \lambda = 0 \tag{22}$$

where $\lambda = \frac{\rho t a^4 \omega^2}{D\pi^4}$ and ρ is the tube thickness. So, for $\lambda < 1$, equation (22) has two roots with negative real parts $\beta_{1,2} = -\sqrt{\pm\sqrt{\lambda}+1}$, and therefore there may be a solution decreasing away from the edge.

4 Numerical Considerations

The buckling of a steel tube with square cross-sectional area and weakly supported edges has been analyzed with the finite element package ANSYS 5.4. Here we consider a tube with $\xi = t/a = 0.01$, $E = 2.07 \times 10^{11}\ N/m^2$, and $\nu = 0.29$.

The buckling of a long square tube under axial compression reminds of the buckling of an axially compressed beam, and the localization does not occur. The critical pressure for beam-like buckling under axial compression of a square tube with one clamped edge and one free edge is given by

$$\begin{aligned} \sigma_{beam} &= \frac{EJ\pi^2}{16atb^2} \\ &= \frac{E(a^2+t^2)\pi^2}{24b^2} \end{aligned} \tag{23}$$

This pressure is less than the pressure obtained from equation (17) if

$$r > \sqrt{\frac{1+\nu}{2(3+\nu)}}\sqrt{\frac{1}{\xi^2}+1} \tag{24}$$

where $r = b/a$. Since for thin shells $\xi \ll 1$, we find that for tubes of the length

$$b < \frac{a}{\xi}\sqrt{\frac{1+\nu}{2(3+\nu)}} \tag{25}$$

the buckling mode localizes near the free edge. For example, for the square tube under consideration beam-like buckling prevails only if $b > 44.3a$.

The asymptotic expression (17) may be used to find the critical load for not too long a tube with either one or two weakly supported edges.

First we consider the case when one edge of the tube is weakly supported and the other edge is clamped. For this case the buckling mode is localized near the weakly supported edge, as shown in Figure 5.

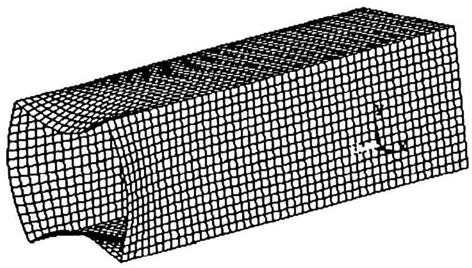

Figure 5. Buckling Mode (FEM) for a Square Tube with One Weakly Supported Edge.

The comparison of the results of asymptotic and numerical analyses by FEM and sweep method are given in Table 1 and in Figure 6. The accuracy of formula (17) increases with r. For $r > 2$ the results obtained with the asymptotic expression (17) and the numerical results found with the sweep method and FEM practically coincide, as shown in Figure 6.

r	$\sigma \times 10^{-7}\ N/m^2$ (Asymptotic)	$\sigma \times 10^{-7}\ N/m^2$ (FEM)	$\sigma \times 10^{-7}\ N/m^2$ (Sweep)
1	4.342	4.528	4.501
2	4.342	4.352	4.342
3	4.342	4.351	4.342
5	4.342	4.351	4.342
10	4.342	4.351	4.342

Table 1. Comparison of Results for Critical Pressure Obtained by Asymptotic and Numerical Methods for a Square Tube with One Weakly Supported Edge.

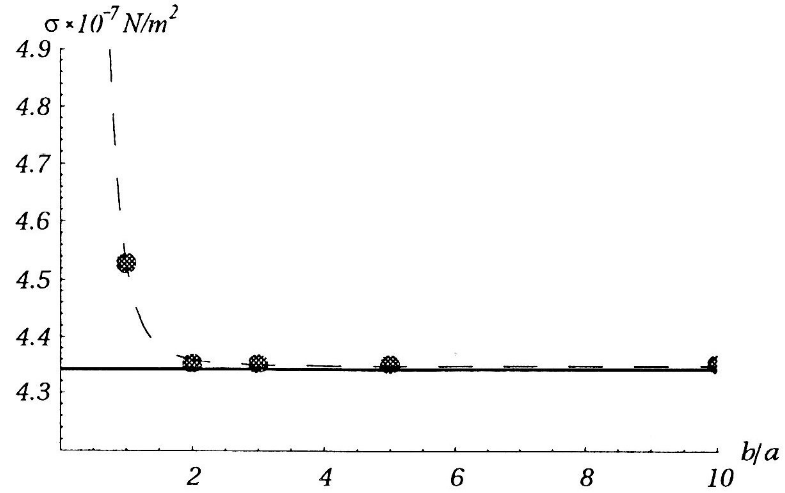

Figure 6. Critical Pressure vs. b/a for a Square Tube with One Weakly Supported Edge. Asymptotic (Solid Line) and FEM (Dots) Results.

The results of the asymptotic and of the numerical analyses by FEM and sweep method for a tube with two weakly supported edges are given in Table 2. In this case the critical load is asymptotically doubled and the buckling mode has the form $W(y) = C_1 W(y) + C_2 W(-y)$. The asymptotic analysis does not allow us to find the constants C_1 and C_2 (Tovstik, 1995). Due to the symmetry of the initial problem one can expect that $C_1 = C_2$ or $C_1 = -C_2$. The buckling mode $W(y) = W(y) + W(b-y)$ is symmetric with respect to the transverse midplane $y = b/2$ and is called even(e), whereas the buckling mode $W(y) = W(y) - W(b-y)$ is asymmetric with respect to the transverse midplane and is called odd (o) (Figure 7).

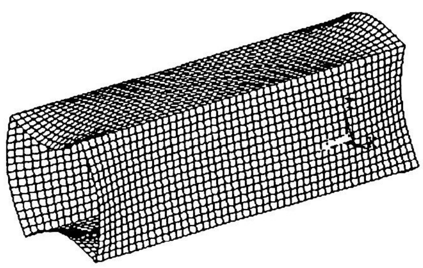

Figure 7. Odd Buckling Mode (FEM) for a Square Tube with Two Weakly Supported Edges.

The numerical analysis gives two close critical loads corresponding to even and odd modes.

r	$\sigma \times 10^{-7}\, N/m^2$ (Asymptotic)	$\sigma \times 10^{-7}\, N/m^2$ (FEM)		$\sigma \times 10^{-7}\, N/m^2$ (Sweep)	
1	4.342	3.860 (o)	4.866 (e)	3.847(o)	4.852 (e)
2	4.342	4.257 (e)	4.457 (o)	4.249(e)	4.449 (o)
3	4.342	4.341 (o)	4.362 (e)	4.333(o)	4.353 (e)

Table 2. Comparison of Results for Critical Pressure Obtained by Asymptotic and Numerical Methods for a Square Tube with Two Weakly Supported Edges.

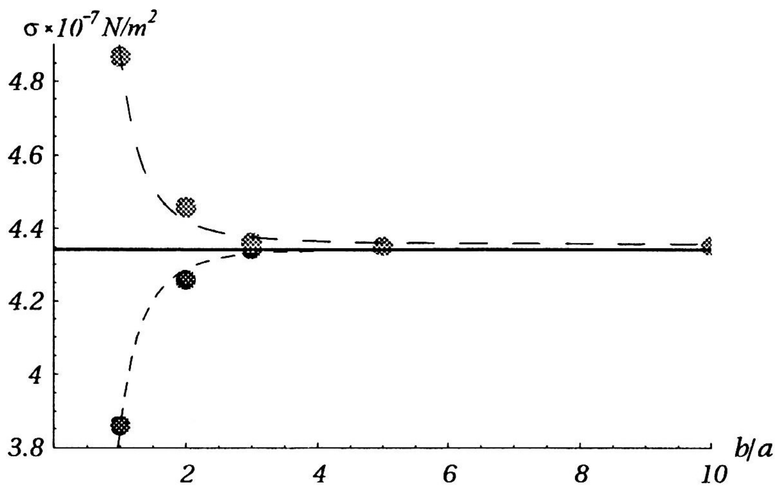

Figure 8. Critical Pressure vs. b/a for a Square Tube with Two Weakly Supported Edges. Asymptocic (Solid Line) and FEM (Dots) Results.

Again, two close critical loads converge rapidly to the asymptotic value as r increases (Figure 8).

Acknowledgment

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