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On the Extended Notion of Material Homogeneity

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The theory of continuous distributions of inhomogeneities is shown to be applicable beyond its original context, to certain classes of material bodies that are not necessarily uniform.

1 Introduction

The mathematical theory of continuous distributions of inhomogeneities is one of the great, if not widely known, achievements of modern continuum mechanics. At its foundation lies the notion of a constitutively induced material geometry, expressed in terms of a reduction of the principal bundle of frames of the body to a sub-bundle governed by the material symmetry group. The emergence of this geometrical apparatus, however, is based on the assumption that the body points are pairwise materially isomorphic or, in other words, that the mechanical responses of all the body points are essentially the same. This uniformity assumption, which is thus a precondition for the very definition of the inhomogeneities, is violated by an increasingly important class of artificial materials arising in many technological applications: the so-called functionally graded materials (Suresh and Mortensen, 1998; Yamanouchi et al., 1990), which are manufactured so as to have smoothly varying elastic properties.

Motivated, on the one hand, by this particular application and, on the other hand, by the natural curiosity of exploring the range of applicability of any well formulated theory, this note presents a preview of the methodology that may be used to achieve a generalization of the theory of inhomogeneities. In a companion paper (Epstein and de León, 2000), further aspects of the theory are discussed, while this presentation emphasizes the underlying structural fabric of the theory.

2 The Mathematical Backbone

Given an n-dimensional differentiable manifold, such as a material body $\mathcal{B}$, there exist several canonically defined structures that can be considered. Three of them in particular are of interest for the theory of inhomogeneities in continuous media: the *tangent bundle* $T\mathcal{B}$, the *principal bundle of linear frames* $\mathcal{F}\mathcal{B}$, and the *Lie groupoid of linear isomorphisms* $\Pi\mathcal{B}$. These three structures are mutually related, and are all characterized by the same *structural group*, namely, the *general linear group* $GL(n, R)$. The essence of the theory of inhomogeneities within the context of continuum mechanics is the *reduction* of these canonical structures to counterparts which are characterized by a proper subgroup of the general linear group. This concept can be best explained in terms of the groupoid $\Pi\mathcal{B}$.

The groupoid $\Pi\mathcal{B}$ of linear isomorphisms of $\mathcal{B}$ is the collection of all non-singular linear maps between the tangent spaces of all ordered pairs of points (X, Y) of $\mathcal{B}$. Thus, a typical element $z \in \Pi\mathcal{B}$ has two natural projections, $pr_1 : \Pi\mathcal{B} \to \mathcal{B}$ and $pr_2 : \Pi\mathcal{B} \to \mathcal{B}$, providing, respectively, the source point, $X = pr_1(z)$, and the target, $Y = pr_2(z)$, such that z represents a linear map between their tangent spaces: $z : T_X\mathcal{B} \to T_Y\mathcal{B}$. Taking any atlas of $\mathcal{B}$, and expressing these linear maps in terms of their matrix components in the natural bases thereof, it is not difficult to see that $\Pi\mathcal{B}$ is a fibre bundle over the product manifold $\mathcal{B} \times \mathcal{B}$.

As already pointed out, the groupoid $\Pi\mathcal{B}$ is canonically defined and bears, therefore, no information about the mechanical response of the body on which it is based. Assume now, however, that the body is *materially uniform* (Noll, 1967) or, in plain language, that it is made of the same material at all of its points. In particular, if the body is made of a *first grade elastic material* with constitutive equation:

$$\mathbf{t} = \mathbf{t}(\mathbf{F}, X) \tag{1}$$

where $\mathbf{t}$ is the *Cauchy stress tensor* and $\mathbf{F}$ the *deformation gradient* at the body point X, uniformity means that for every pair of points $X, Y \in \mathcal{B}$ there exists an isomorphism

$$P_{XY} : T_X\mathcal{B} \to T_Y\mathcal{B} \tag{2}$$

such that

$$\mathbf{t}(\mathbf{F}P_{XY}, X) = \mathbf{t}(\mathbf{F}, Y) \tag{3}$$

for all non-singular $\mathbf{F}$.

These *material isomorphisms* P_{XY} constitute the essential ingredient of the theory of inhomogeneities as developed in Noll (1967) and Wang (1967). It is not difficult to show that the freedom inherent in the choice of material isomorphisms is governed precisely by the *material symmetry groups* of the different points. Specifically, if P_{XY} is a given material isomorphism between the points X and Y, then the set of all possible material isomorphisms, consistent with the constitutive equation of the body, is given by:

$$\mathcal{P}_{XY} = P_{XY}\mathcal{G}_X = \mathcal{G}_Y P_{XY} = \mathcal{G}_Y P_{XY}\mathcal{G}_X \tag{4}$$

where $\mathcal{G}_X$ and $\mathcal{G}_Y$ are, respectively, the symmetry groups of X and Y. If we now equip the body with the collection of all material isomorphisms, we clearly obtain a subset $\mathcal{GB}$ of the groupoid $\Pi\mathcal{B}$. Moreover, this subset is itself a groupoid, since it satisfies the following properties (Mackenzie, 1987):

(*i*) *Partial multiplication*

A *partial multiplication* (or grupoid operation) wz between elements w, z of $\mathcal{GB}$ is defined whenever $pr_2(z) = pr_1(w)$. Moreover, $pr_1(wz) = pr_1(z)$ and $pr_2(wz) = pr_2(w)$.

(*ii*) *Associativity*

The groupoid operation is *associative*, namely:

$$v(wz) = (vw)z \tag{5}$$

for all $v, w, z \in \mathcal{GB}$ such that $pr_1(v) = pr_2(w)$ and $pr_1(w) = pr_2(z)$.

(*iii*) *Local unit*

For each $X \in \mathcal{B}$ there exists an element $e_X \in \mathcal{GB}$ which acts as a *unit* at X in the sense that:

$$pr_1(e_X) = pr_2(e_X) = X \tag{6}$$

and

$$ze_{pr_1(z)} = e_{pr_2(z)}z = z \tag{7}$$

for all $z \in \mathcal{GB}$. (In our case, e_X is simply the identity isomorphism of $T_X\mathcal{B}$).

(*iv*) *Inverse*

Each $z \in \mathcal{GB}$ has a two-sided inverse $z^{-1} \in \mathcal{GB}$ satisfying:

$$pr_1(z^{-1}) = pr_2(z) \qquad pr_2(z^{-1}) = pr_1(z) \tag{8}$$

$$z^{-1}z = e_{pr_1(z)} \qquad zz^{-1} = e_{pr_2(z)} \tag{9}$$

These properties simply reflect the fact that, with the restricted collection of maps afforded by the material isomorphisms, the body manifold is still endowed with pairwise point operations which can be composed and inverted. We may say, then, that $\mathcal{GB}$ is a *reduction* of $\Pi\mathcal{B}$. Its structural group is the symmetry group of any one of its points (these groups are all mutually conjugate). We call this object $\mathcal{GB}$ the *material groupoid* of $\mathcal{B}$ relative to the given elastic constitutive law.

The question of *local homogeneity* boils down to the determination of whether or not this material groupoid is *integrable*. Integrability of a groupoid can be expressed as the condition of existence of local charts such that the pull-back of the canonical unit section of $\Pi\mathcal{R}^n$ to $\Pi\mathcal{B}$ belongs to the reduced groupoid $\mathcal{GB}$. This, in a nutshell, is the theory of inhomogeneities of Noll and Wang rephrased in terms of its most natural setting, the theory of Lie groupoids.

3 Extended Notions

It should be clear from the previous presentation that homogeneity is a property which can only be defined in uniform bodies. Indeed, in the absence of uniformity, how can one define a materially based groupoid, whose integrability will be the judge of local homogeneity? It is a remarkable fact that, even in the absence of material uniformity, certain types of material bodies may exhibit enough structure to enable the existence of a materially based groupoid.

Indeed, consider the case of an elastic material body such that the material symmetry groups of all of its points are mutually conjugate within the general linear group. We call this property *unisymmetry*. It follows from their very definition that

$$uniformity \rightarrow unisymmetry \tag{10}$$

but the arrow in this statement cannot be reversed in general. For it may well happen that a functionally graded solid body is, say, fully isotropic at every point but with smoothly varying elastic properties from point to point. Having ascertained, through the constitutive equation, that a body is unisymmetric, it follows that a collection $\mathcal{C}_{XY}$ of non-singular linear maps exists between the tangent spaces of every pair of points (X, Y) of $\mathcal{B}$, with the property that $C_{XY} \in \mathcal{C}_{XY}$ if, and only if,

$$\mathcal{G}_Y = C_{XY}\mathcal{G}_X C_{XY}^{-1} \tag{11}$$

We call C_{XY} a *symmetry isomorphism*. The set $\mathcal{C}_{XY}$ of all symmetry isomorphisms between X and Y may be called the *conjugator* between those two points.

What is the degree of freedom in the choice of C_{XY}? It is not difficult to see that

$$\mathcal{C}_{XY} = C_{XY}\mathcal{N}_X = \mathcal{N}_Y C_{XY} = \mathcal{N}_Y C_{XY}\mathcal{N}_X \tag{12}$$

where $\mathcal{N}_X$ and $\mathcal{N}_Y$ are, respectively, the *normalizers* of the groups $\mathcal{G}_X$ and $\mathcal{G}_Y$ within the general linear group. We recall that the normalizer $\mathcal{N}(\mathcal{G})$ of a subgroup $\mathcal{G}$ of a group $\mathcal{H}$ is the set

$$\mathcal{N}(\mathcal{G}) = \{H \in \mathcal{H} | HGH^{-1} \forall G \in \mathcal{G}\} \tag{13}$$

It is now a matter of direct verification to ascertain that a unisymmetric material body equipped with the collection of all its symmetry isomorphisms constitutes a subgroupoid $\mathcal{CB}$ of $\Pi\mathcal{B}$. In principle, therefore, we can use this weaker material structure to define the concept of local *homosymmetry* by identifying it with the integrability of $\mathcal{CB}$. We have thus extended not only the notion of uniformity, but also that of homogeneity.

4 Physicality

The geometrical object $\mathcal{CB}$ just defined makes perfect mathematical sense, and so does its integrability. Does this definition, however, make any physical sense? One of the worries is that the normalizer of a group within the general linear group includes automatically all homogeneous ("spherical") dilatations, since they commute with every matrix. These undesirable elements of $\mathcal{CB}$ cannot be eliminated canonically, unless there is some extra information available, beyond pure unisymmetry. The type of information necessary is contained in the answer to the following question: do the material points exhibit any preferred states? Such states could be ones in a preferential density or, even better, in a natural relaxed state. In both these cases, it is possible to sharpen the groupoid $\mathcal{CB}$ by further reducing its structural group to, respectively, the normalizer of the symmetry group within either the *special linear group* or the *orthogonal group*.

In the latter case, it can be shown (Epstein and de León, 2000) that, for a number of solid symmetries, the notions of homosymmetry and homogeneity coincide. Namely, if a body is uniform but its integrability is determined only on the basis of its unisymmetry, the same result is obtained as if it had been determined on the basis of its uniformity. This surprising property is not true in general, but it is true, for example, in the following important classes of elastic solids: fully isotropic, transversely isotropic and orthotropic. In conclusion, we have obtained a means for gauging homogeneity for functionally graded bodies exhibiting just the relatively weak property of having the same type of material symmetries at all points.

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