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Theoretical Models of Micro-cracked Continua: Discontinuity of Scalar and Vector Fields

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In order to represent discontinuities in the deformation of a material and its consequences on the energy dissipation during micro-crack kinetics, a model of micro-cracked continuum is derived. The Micro-crack density is represented in terms of the non-metric connection on a manifold. Torsion and curvature of the non-metric connection represent a non-topological deformation and explicitly include mesoscopic discontinuities. The developed model includes both the non-equilibrium thermodynamic processes of micro-crack creation and the micro-crack growth. This approach contrasts to the empirical methodology of continuum mechanics that seeks a phenomenological description. An illustrative example of the model application is presented for the uniaxial vibration test, providing representations for wave propagation within a micro-cracked solid. The result of this example highlights the importance of rigorously revisiting the conservation laws in the framework of non-metric connected manifolds.

1 Introduction

Micro-crack distribution in brittle materials is a matter of concern regarding the reliability of these materials under mechanical stresses. Brittle materials such as glass, ceramics, and PMMA always contain various amounts of micro-cracks and crack-like flaws, which are introduced either during processing or during surface machining (Green, 1998). Toughness and strength of these materials are strongly dependent on the amount and structural orientation of internal micro-cracks. Nucleation and growth of distributed micro-cracks also play an important role during the macroscopic failure process of brittle materials. Global failure of brittle material is usually attributed to a single macroscopic crack propagation. However, a single crack propagation model has been shown to not fully predict the experimental results particularly for brittle materials such as PMMA (Sharon and Fineberg, 1996). These authors have pointed out that dense sets of micro-cracks appear around the single crack, resulting from dynamic instability. Moreover, the micro-branching instability is proposed as the main mechanism for the increase of energy dissipation due to a rapidly propagating crack. At the extreme, micro-cracking in the vicinity of a macroscopic crack edge has been shown advantageous in controlling and even arresting a single macroscopic crack propagation (Clegg, 1999). In any case, in the neighbourhood of the propagating crack, classical continuum mechanics does neither provide a physical description of the discontinuities nor assign correct internal energy contributions for various thermodynamic processes during the micro-crack kinetics. Despite recent advances in the fracture dynamic and in continuum damage mechanics, the existence of numerous theoretical models of brittle micro-cracked materials based on different choices of damage variables merely shows the absence of consensus in this domain (Rabier, 1989; He and Curnier, 1995).

Physically, micro-cracks are displacement and/or velocity discontinuities in an otherwise intact material. Most continuum mechanical models of micro-cracked solids intend a phenomenological description by assuming internal variables (Vakulenko and Kachanov, 1971; Chaboche, 1988; He and Curnier, 1995). These internal variables are used to simulate the change of material properties and do not have any influence on the formulation of the conservation laws. To be close to the physical phenomenon, some micro-mechanical models are based on the physical discontinuity of matter and then assume the description of micro-cracks with contacting lips and dry friction at these lips. Each micro-crack is then included into a cell, which is its direct neighbourhood. The cell is the smallest unit that allows the bulk material properties to be quantified after homogenisation. The most important property of the basic cell is the ability to describe the relative translation of contacting lips (cohesion-decohesion) (Broberg, 1997). This kind of model requires the mathematical technique of homogenisation, which is cumbersome and practically difficult to apply in presence of a disordered distribution of micro-crack orientations. The crack opening modes (components of relative displacement of the crack lips) are the internal variables for these models (Maugin, 1992). In fact, at least two types of models may be proposed to capture the behaviour of the clouds of cracks: a) models with discontinuity of matter and b) models with discontinuity of fields. The micro-cracked continuum model (based on discontinuity of fields) is preferred in this paper. Discontinuous scalar and vector fields are considered (Rakotomanana, 1998). A previous study has shown that the micro-cracked continuum model is more general than a model based on the discontinuity of matter and assuming micro-cracks with contacting lips and dry friction (Ramaniraka and Rakotomanana, 2000). The goal of this paper is to develop a micro-cracked continuum model based on the discontinuities of scalar and vector fields on the continuum.

2 Model based on Discontinuity of Vector and Scalar Fields

2. 1 Geometry Structure of the Model

Consider a brittle material under a rapid traction test (Sharon and Fineberg, 1996). Physically, each new micro-crack results in a discontinuity of the deformation between atoms (microscopic level) or between grains (mesoscopic level) in the material. A change in the thermodynamic energy content of the material is thus expected. The model of a micro-cracked continuum has been modelled by an affinely connected manifold and is solely based on the discontinuity of fields (displacement, velocity) rather than the discontinuity of matter. The use of a path integral-like method (Schouten, 1954) allows us to obtain the geometrical variables capturing the jump of scalar and vector fields within a continuum: a) for any discontinuous scalar field, the torsion tensor and b) for any discontinuous vectorial field, the torsion and the curvature tensor. The field of micro-cracks is entirely characterised by the tensors of torsion and curvature, considered as constitutive primal variables (Kröner, 1981; Maugin, 1993). It follows that the geometry of a continuum with micro-cracks is defined for any vectorial basis (e_1, e_2, e_3) by

1. a metric tensor and a volume-form (usual variables of classical continuum)

$$g = g_{ab}\, e^a \otimes e^b \qquad \omega_0 = \det(e_1, e_2, e_3) e^1 \wedge e^2 \wedge e^3$$

2. an affine connection characterised by the torsion and curvature tensors (additional variables for a continuum with continuous distribution of micro-cracks)

$$\aleph = \left[\left(\Gamma^c_{ac} - \Gamma^c_{ba}\right) - \aleph^c_{0ab}\right] e^a \otimes e^b \otimes e_c$$

$$\mathfrak{R} = \left[e_b\left(\Gamma^c_{da}\right) - e_a\left(\Gamma^c_{db}\right) + \Gamma^e_{da}\Gamma^c_{eb} - \Gamma^e_{db}\Gamma^c_{ea} - \aleph^e_{0ab}\Gamma^c_{ed}\right] e^a \otimes e^b \otimes e^d \otimes e_c$$

where

$$\Gamma^c_{ab} = e^c\left(\nabla_{e_a} e_b\right) \qquad \aleph^c_{0ab} e_c \equiv \left[e_a, e_b\right]$$

Symbol $[\,,\,]$ denotes the usual LIE-JACOBI bracket. The deformation of a micro-cracked continuum includes a transformation of g and ω_0 (metric change) and a transformation of ∇ (topology change), the two deformations being "observed internally by the micro-cracked continuum" meaning that they are projected onto an embedded basis (e_1, e_2, e_3), which deforms with the continuum. Constants of structure defined by $\aleph^c_{0ab} e_c \equiv [e_a, e_b]$ (a and b vary from 1 to 3), which may be resumed into the 1-form $\aleph_0 = \aleph^b_{0ab} e^a$, include the three modes of each micro-crack opening (Ramaniraka and Rakotomanana, 2000). This 1-form field was originally proposed in the framework of general relativistic mechanics (Cartan, 1986).

2. 2 Differential Operators on Micro-cracked Continuum

The divergence of a vector field projected on any coordinate or non coordinate basis (e_1, e_2, e_3) has been derived previously and takes the following form in a micro-cracked continuum (Rakotomanana, 1998)

$$div\, v = \frac{1}{J}\nabla_{e_a}\left[Jv\left(e^a\right)\right] + \sum_{(abc)} \aleph^d_{0ab}\varepsilon_{dce} v\left(e^e\right)$$

In the previous relationship, we have $J = \omega_0(e_1, e_2, e_3)$, $\aleph^c_{0ab} e_c = [e_a, e_b]$ and $\varepsilon_{dce} = J\omega_0(e_d, e_c, e_e)$. A more concise formulation of the divergence operator is obtained by applying the circular permutation and by introducing the previous 1-form field $\aleph_0 = \aleph^b_{0ab} e^a$. Thus, the divergence is split as a summation of a classical divergence, denoted $Div\, v$ hereafter, and of a contribution of the singularity distribution

$$div\, v = Div\, v + \aleph_0(v) \qquad Div\, v = \frac{1}{J}\nabla_{e_a}\left(Jv^a\right)$$

The operator Div reduces to the usual material divergence, which is extensively used in the framework of elastic large deformations of continua in the absence of a micro-crack distribution. In the same way, the divergence of any 1-form ω may be decomposed as follows

$$div\, \omega = Div\, \omega + g^{-1}(\aleph_0, \omega) \qquad Div\, \omega = \frac{1}{J}\nabla_{e_a}\left(J_u g^{ab}\omega_b\right)$$

By means of the divergence of a 1-form, it is straightforward to derive the expression of the Laplacian of any scalar field Θ within a micro-cracked continuum. The Laplacian of a scalar Θ is defined by $\Delta\Theta \equiv div(\nabla\Theta)$. The use of the covariant derivative and the divergence of 1-form allow us to write the Laplacian operator in a coordinate-free formulation

$$\Delta\Theta = Div(\nabla\Theta) + g^{-1}(\aleph_0, \nabla\Theta) = \overline{\Delta}\Theta + g^{-1}(\aleph_0, \nabla\Theta)$$

The second term constitutes the contribution of the continuous distribution of micro-cracks. The definition for the divergence of a vector can be generalised for any second-order tensor by raising and lowering the tensor indices by the metric tensor. For our purpose, the divergence of a second-order tensor, 1-covariant, 1-contravariant, is given in a coordinate-free formulation

$$div\, \pi = Div\, \pi + \pi(\aleph_0)$$

Considering any vector field v on a micro-cracked continuum, the Laplacian of v is defined by $\Delta v \equiv div(\nabla v)$. Replacing the mixed tensor by the velocity gradient in the divergence formula and by analogy to the Laplacian of a scalar field, a more concise form is also obtained

$$\Delta v = \overline{\Delta} v + \nabla v(\aleph_0) \qquad \overline{\Delta} v = \frac{1}{J}\nabla\left(Jg^{ac}\nabla_c v^b e_b\right)$$

2. 3 Conservation Laws and Constitutive Functions for Micro-cracked Continuum

In the present paper, a micro-cracked continuum model is a medium, the distribution of material points of which remains continuous but in which the existence of continuous distributions of scalar and vectorial discontinuities is permitted. The field singularity (here scalar and vectorial discontinuity) is entirely characterised by tensors $\aleph$ and $\Re$ considered as constitutive primal variables. The material of the rate type we deal with in this work is assumed to have constitutive laws defined by tensorial functions

$$\Im = \Im\left(\omega_0, g, \aleph, \Re, \theta, \nabla\theta, \zeta_g, \zeta_\aleph, \zeta_\Re, \zeta_\theta, \zeta_{\nabla\theta}\right)$$

The arguments of these constitutive functions are the primal variables (6 is the temperature field and $\nabla 6$ its gradient) and the rates "ζ" that are their first order time derivatives with respect to the continuum (Rakotomanana, 1998). By applying the classical COLEMAN and NOLL's method (Coleman and Noll, 1963), it is shown that the free energy ϕ of such a material takes necessarily the form

$$\phi = \phi\left(\omega_0, g, \aleph, \Re, \theta\right)$$

For later use, let us define the following variables

$$\zeta_q \equiv \theta\nabla\left(\frac{1}{\theta}\right) \qquad J_g \equiv \sigma - \rho\frac{\partial\phi}{\partial\omega_0} : \omega_0 i - 2\rho\frac{\partial\phi}{\partial g} \qquad J_\aleph \equiv -\rho\frac{\partial\phi}{\partial\aleph} \qquad J_\Re \equiv -\rho\frac{\partial\phi}{\partial\Re}$$

The conservation laws have been derived in (Rakotomanana, 1998) by accounting for the topology (connection change) of the micro-cracked continuum. In a more concise and more convenient form, they may be rewritten as follows:

- Mass conservation

$$\frac{\partial\rho}{\partial t} + Div\left(\rho v\right) + \aleph_0\left(\rho v\right) = 0$$

- Balance of linear momentum

$$\rho\left(\frac{\partial v}{\partial t} + \nabla_v v\right) = Div\,\sigma + \sigma\left(\aleph_0\right) + \rho b$$

- Balance of angular momentum

$$\sigma^T = \sigma$$

- Energy conservation

$$\rho C\left(\frac{\partial\theta}{\partial t} + \nabla_v\theta\right) = -Div\left(J_q\right) - \aleph_0\left(J_q\right) + \rho\theta\frac{\partial}{\partial\theta}\left(\frac{\sigma}{\rho}\right) : \zeta_g + J'_g : \zeta_g + J'_\aleph : \zeta_\aleph + J'_\Re : \zeta_\Re + r$$

where:

$$J'_g \equiv J_g - \rho\theta\frac{\partial}{\partial\theta}\left(\frac{\sigma}{\rho}\right) \qquad J'_\aleph \equiv J_\aleph - \rho\theta\frac{\partial}{\partial\theta}\left(\frac{J_\aleph}{\rho}\right) \qquad J'_\Re \equiv J_\Re - \rho\theta\frac{\partial}{\partial\theta}\left(\frac{J_\Re}{\rho}\right)$$

- Entropy inequality

$$J_q \cdot \zeta_q + J_g : \zeta_g + J_\aleph : \zeta_\aleph + J_\Re : \zeta_\Re \geq 0$$

The double dot symbol denotes the scalar product of tensors, which are defined by the relations $J'_g : \zeta_g = J'^{ab}_g \zeta_{gab}$, $J'_{\aleph} : \zeta_{\aleph} = J'^{ab}_{\aleph c} \zeta^{c}_{\aleph ab}$ and $J'_{\Re} : \zeta_{\Re} = J'^{abd}_{\Re c} \zeta^{c}_{\Re abd}$. The above conservation laws extend the so-called NOLL equation of motion for non homogeneous materials (Noll, 1963; Wang, 1967;Kröner, 1981), to the full set of equations of mass, linear momentum, and angular momentum conservation together with heat propagation and the entropy inequality for micro-cracked materials. The entropy inequality includes the thermal, viscous, and micro-cracked dissipation. The last two terms of the entropy inequality express the dissipation due to the dense sets of micro-cracks and quantify the irreversibility induced by the evolution of the non-metric connection of the continuum (affine structure). This micro-crack dissipation is indeed related to the change of the local topology. This is probably the reason why the classical continuum may be successfully used to model single crack propagation at low velocity but basically fails to model the nucleation and growth of dense sets of micro-cracks at high propagation velocities.

2. 4 Normal Dissipation in Micro-cracked Continuum

To obtain more tractable models, the hypothesis of normal dissipation restricts the class of constitutive laws, although remaining a relatively general framework to continuum models satisfying the second principle of thermodynamics. For normal dissipative materials, constitutive laws of the continuum with field discontinuities may be entirely reconstructed from a free energy and a potential of dissipation (Germain et al., 1983; Ziegler and Wherli, 1987)

$$\phi = \phi(\omega_0, g, \aleph, \Re, \theta) \qquad \psi = \psi(\omega_0, g, \aleph, \Re, \theta, \zeta_g, \zeta_{\aleph}, \zeta_{\Re}, \zeta_q)$$

For notation conciseness, the parameters (not variables) $(\omega_0, g, \aleph, \Re, 6)$ are dropped to avoid lengthy equations hereafter in the formulation of the dissipation potential. For a material characterised by the existence of a stress threshold, constitutive functions depend on the history of external applied forces. At first approximation, on can observe macroscopically that the behaviour of such a material changes abruptly when the intensity of applied forces overpasses a certain critical value. This sudden variation requires a non-continuously differentiable model. The conjugate dissipation potential is defined by the partial LEGENDRE-FENCHEL transform (Rakotomanana, 1998)

$$\psi^*(\zeta_g, J_{\aleph}, J_{\Re}, \zeta) \equiv Sup_{\zeta_{\aleph}, \zeta_{\Re}} \left[J_{\aleph} : \zeta_{\aleph} + J_{\Re} : \zeta_{\Re} - \psi(\zeta_g, \zeta_{\aleph}, \zeta_{\Re}, \zeta_q) \right]$$

The evolution laws of the density of micro-cracks is therefore calculated by means of the sub-gradient of the discontinuous dissipation potential

$$\zeta_{\aleph} \in \partial \psi^*_{J_{\aleph}} (\zeta_g, J_{\aleph}, J_{\Re}, \zeta_q)$$
$$\zeta_{\Re} \in \partial \psi^*_{J_{\Re}} (\zeta_g, J_{\aleph}, J_{\Re}, \zeta_q)$$

These evolution laws express the nucleation and the growth of region where dense sets of micro-cracks appear in the defected material. Most classes of non-classical solids are obtained by choosing special functions for the free energy and for the potential of the dissipation. The simplest example of such materials is the linear isotropic elastic solid with micro-crack density. It is defined by the quadratic free energy potential

$$\phi = \frac{1}{2} \lambda(\omega_0, \aleph, \Re, \theta) tr^2 \left(\frac{g-1}{2} \right) + \frac{1}{2} \mu(\omega_0, \aleph, \Re, \theta) tr \left[\left(\frac{g-1}{2} \right)^2 \right]$$

To compute the micro-crack density evolution, it is convenient to define first a set C, which is a convex set of the dual space $\{J_{\aleph}, J_{\Re}\}$, where there is no evolution of the rates of the micro-crack density. Set C contains the null tensors. For dual variables $\{J_{\aleph}, J_{\Re}\}$ in the interior of C, the density of micro-cracks remains constant whereas for those on the boundary, the density increases. Then, it is convenient to introduce the indicator function of the set C defined by

$$I_C(J_\aleph, J_\Re) \equiv \begin{cases} 0 & if\ \{J_\aleph, J_\Re\} \in C \\ +\infty & otherwise \end{cases}$$

By analogy to classical dry friction and rate-independent plasticity theory (Moreau, 1976), the dissipation potential may be identified as the conjugate of the indicator function of the set C as $\psi(\zeta_\aleph, \zeta_\Re) = I^*_C(\zeta_\aleph, \zeta_\Re)$. Details and proofs supporting this identification for evolution laws with threshold may be found in (Moreau, 1970; Maugin, 1993; Rakotomanana, 1998). By applying the LEGENDRE-FENCHEL transform, it is straightforward to derive

$$I^*_C(\zeta_\aleph, \zeta_\Re) = Sup_{\{J_\aleph, J_\Re\} \in C}\left[J_\aleph : \zeta_\aleph + J_\Re : \zeta_\Re - I_C(\zeta_\aleph, \zeta_\Re)\right] = Max\left[Sup_{\{J_\aleph, J_\Re\} \in C}\left(J_\aleph : \zeta_\aleph + J_\Re : \zeta_\Re\right), -\infty\right]$$

We then deduce the dissipation potential

$$\psi(\zeta_\aleph, \zeta_\Re) = I^*_C(\zeta_\aleph, \zeta_\Re) = Sup_{\{\zeta_\aleph, \zeta_\Re\} \in C}\left(J_\aleph : \zeta_\aleph + J_\Re : \zeta_\Re\right)$$

The total dissipation potential includes quadratic dissipation in terms of heat conduction and viscosity and a homogeneous function of degree one in terms of micro-crack density rates

$$\psi = \frac{1}{2}\kappa\left\|\zeta_q\right\|^2 + \frac{1}{2}\lambda' tr^2(\zeta_g) + \frac{1}{2}\mu' tr\left(\zeta_g^2\right) + Sup_{\{\zeta_\aleph \zeta_\Re\} \in C}\left(J_\aleph : \zeta_\aleph + J_\Re : \zeta_\Re\right)$$

The last terms in brackets are positive and represent the internal dissipation due to a micro-cracks distribution. In this relation, LAMÉ's coefficients (λ, μ) depend on the amount of the micro-crack density. Viscosity parameters (λ', μ') depend on the volume-form, metric and temperature. For an anisotropic material, the tensor representation of the scalar-valued free energy function and the dissipation potential should be implemented as for anisotropic elastic plastic solids (Rakotomanana et al., 1991). A model of micro-cracked solids is associated to each appropriate couple of potentials.

3 Applications to Linear Elastic Solids with Micro-cracks

The material is assumed to have isotropic symmetry and to undergo small elastic displacements. The goal of this section is to develop the wave propagation equation in micro-cracked linear elastic solids, where micro-cracks are frozen ($\zeta_\aleph \equiv 0$ and $\zeta_\Re \equiv 0$) (Wang, 1967). At the same time, this example suggests to revisit the equations of motion for micro-cracked continua, even for the simplest case. Without loss of generality, we only consider isothermal deformations in this example.

3.1 Wave Propagation Equations in Micro-cracked Solids

Starting from CAUCHY's equation of motion and HOOKE's linear elastic stress-strain law, it is straightforward to obtain the NAVIER equations in the absence of body force

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\nabla(div\, u) + \mu\Delta u$$

This equation holds for both non-cracked and cracked continua. For analysing wave propagation in isotropic elastic media, it is usual to introduce the longitudinal and transversal velocities of sound

$$c_L^2 = \frac{\lambda + 2\mu}{\rho} \qquad\qquad c_T^2 = \frac{\lambda}{\rho}$$

The wave propagation equations in micro-cracked solids are then directly obtained by using of the differential operators previously developed and by combining these operators with the NAVIER equations

$$\frac{\partial^2 u}{\partial t^2} = \left(c_L^2 - c_T^2\right)\nabla(Div\, u) + c_T^2 \overline{\Delta} u + \left(c_L^2 - c_T^2\right)\nabla[\aleph_0(u)] + c_T^2 \nabla u(\aleph_0)$$

The additional terms represent the configurational forces and cannot be eliminated by a choice of an affine connection and a base vector (Rakotomanana, 1998). For the sake of simplicity, we consider now a Cartesian vector basis directed by the referential body. The projection of the wave propagation equations onto a Cartesian vector base gives then a more tractable form (summation for index a)

$$\frac{\partial^2 u_b}{\partial t^2} = \left(c_L^2 - c_T^2\right)\frac{\partial^2 u_a}{\partial x_b \partial x_a} + c_T^2 \frac{\partial^2 u_b}{\partial x_a^2} + \left(c_L^2 - c_T^2\right)\frac{\partial\left(\aleph_{0a} u_a\right)}{\partial x_b} + c_T^2 \aleph_{0a} \frac{\partial u_a}{\partial x_b}$$

For the particular case where the displacement vector $u = (u_1, u_2, u_3)$ is depending only on one coordinate $x_1 = x$ and on the time t, further simplification gives

$$\frac{\partial^2 u_1}{\partial t^2} = c_L^2 \frac{\partial^2 u_1}{\partial x^2} + c_L^2 \aleph_{0a} \frac{\partial u_a}{\partial x} + \left(c_L^2 - c_T^2\right)\frac{\partial \aleph_{0a}}{\partial x} u_a$$

$$\frac{\partial^2 u_2}{\partial t^2} = c_T^2 \frac{\partial^2 u_2}{\partial x^2} \qquad\qquad \frac{\partial^2 u_3}{\partial t^2} = c_T^2 \frac{\partial^2 u_3}{\partial x^2}$$

The first equation governs the longitudinal wave propagation and the other two describe the transverse wave propagation. The existence of continuously distributed micro-cracks implies a coupling between the wave propagation along the three directions. The first equation looks like a linear damped KLEIN-GORDON wave equation (Kneubühl, 1997). Solving of the last two equations under some boundary conditions is straightforward and gives transverse waves. The first equation is more complicated but could easily be solved after separating the variables (valid only under some boundary conditions).

3. 2 Example of Steady State Waves

For the experimental identification of material properties, it is often assumed that the material properties are not coordinate-dependent (homogeneous distribution) in the "small" piece of material to be tested. Let us recall that the micro-crack distribution is captured with the 1-form field $\aleph_0 = \aleph_{0ab}^b e^a$. For further simplification, let us assume that the wave propagation is uni-dirctional and the micro-crack distribution reduces to a scalar field $\aleph_{01} = \aleph_0$

$$\frac{\partial^2 u}{\partial t^2} = c_L^2 \frac{\partial^2 u}{\partial x^2} + c_L^2 \aleph_0 \frac{\partial u}{\partial x}$$

The forms of the steady-state solutions depend on the amount of the micro-cracks density within the solid. Three types of solutions exist according to the micro-defect density values. For convenience of physical interpretation, we define the characteristic defect length

$$d_{\aleph} \equiv \frac{2}{\aleph_0}$$

Let us also define the defect circular frequency and calculate the discriminant

$$\omega_{\aleph} \equiv \frac{c_L}{d_{\aleph}} \qquad \Delta_{\aleph} = \aleph_0^2 c_L^4 - 4c_L^2\omega^2 = 4c_L^2\left(\omega_{\aleph}^2 - \omega^2\right)$$

Consider now a simple example for illustration. Suppose that a plate of an elastic material of thickness d is subjected to the steady-state displacement boundary condition $u(0,t) = \bar{u}\cos(\omega t)$ at the left boundary, and the plate is bonded to a fixed support at the right boundary. Let us determine the steady-state motion of the material depending on the amount of micro-defects within the solid.

- (High density of singularity). This is the case when $\omega_{\aleph} \geq \omega$. We obtain the steady-state wave

$$u(x,t) = \bar{u}\frac{\sinh\left(-\sqrt{1-\frac{\omega^2}{\omega_{\aleph}^2}}\frac{(x-d)}{d_{\aleph}}\right)}{\sinh\left(\sqrt{1-\frac{\omega^2}{\omega_{\aleph}^2}}\frac{d}{d_{\aleph}}\right)} e^{-\frac{x}{d_{\aleph}}}\cos(\omega t)$$

This solution includes various contributions of the wave attenuation. Mainly, the wave amplitude exponentially attenuates with distance, which typically conforms to the usual absorption contribution (Breazeale et al., 1981). It is observed that there is no change of the frequency and no decay of the time function during the propagation. Thus, a disturbance of arbitrary form propagates with a decay of the amplitude without having its shape changed. Apparently, no mechanical resonance occurs. However, the amplitude of the space-dependent part of the function may rise to infinity when

$$\sinh\left(\sqrt{1-\frac{\omega^2}{\omega_{\aleph}^2}}\frac{d}{d_{\aleph}}\right) = 0 \Rightarrow \omega_{\aleph} = \omega$$

This frequency equation would be a starting point for experimental measurements of the singularity distribution when the density is sufficiently high. There is a resonance when $\omega_{\aleph} = \omega$, which occurs when the wavelength and the crack opening length are comparable in magnitude, since we have $\omega/\omega_{\aleph} = 2\pi d_{\aleph}/\lambda$, in which λ is the wavelength.

- (Low density of singularity). When $\omega_{\aleph} \leq \omega$, the steady-state wave is given by

$$u(x,t) = \bar{u}\frac{\sin\left(-\sqrt{\frac{\omega^2}{\omega_{\aleph}^2}-1}\frac{(x-d)}{d_{\aleph}}\right)}{\sin\left(\sqrt{\frac{\omega^2}{\omega_{\aleph}^2}-1}\frac{d}{d_{\aleph}}\right)} e^{-\frac{x}{d_{\aleph}}}\cos(\omega t)$$

In addition to the decay of amplitude already observed in the previous case, there is a resonance phenomenon when the denominator vanishes

$$\sin\left(\sqrt{\frac{\omega^2}{\omega_\aleph^2}-1}\frac{d}{d_\aleph}\right)=0$$

The resonance frequencies for this plate are

$$\omega_n=\omega_\aleph\sqrt{1+n^2\pi^2\frac{d_\aleph^2}{d^2}}$$

4 Conclusion

Classical micro-cracked continuum models are based on the thermodynamics of irreversible processes with a priori chosen internal variables. Usually, these models attempt to describe the micro-crack density empirically based on the expertise of the authors. There is a lack of uniformity and even a lack of rigor in the choice of the micro-crack density variables. The present paper constitutes an attempt to give a theoretical basis for elaborating a continuum model in presence of dense sets of micro-cracks. The present model is a geometry-oriented one. The originality of the model lies in the choice of internal variables capturing the interstitial dissipation: torsion and curvature of the affine connection in the continuum. This kind of continuum model spans all types of cracks since it involves more details than the micro-mechanical approach involving only the three relative translations of the contacting lips of each crack. Accounting for rotations allows us to extend the usual model to a "non local" behaviour of micro-cracks. Both relative translations and relative rotations occur in a three-dimensional cracking material. The evolution laws for micro-crack density have been derived using the concept of a normal dissipating mechanism. The study opens two following aspects that should be investigated in a more systematic way and with the support of experimental measurements. Firstly, the existence of evanescent waves in a micro-cracked solids seems to be interesting with regards to controlling and better understanding internal damping in engineering materials such as ceramics or polymers, in earthquake propagation science and in hard biological materials such as bone tissue. Secondly, experimental investigations on the determination of the practical form of the convex set defining the micro-cracks density evolution or alternatively the form of yield stress function should be undertaken. In this way, a recent study (Zioupos et al., 1995) seems promising by discovering the classical TSAI-WU criterion as a candidate yield function for an anisotropic cortical bone.

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