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Gauge Momentum Map and Spatial Control of Extended Objects

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Equations of motion for a system with a free and proper action of a symmetry group on the configuration space decompose into the shape space equations, gauge momentum equations and reconstruction equations. The reconstruction equations do not depend on the shape space momenta. This allows for a control in the group space in terms of control forces acting in the shape space.

1 Introduction

The best example of the problem of a directional control of extended objects is provided by the ability of cats to twist in a fall so that they land on their paws. Its mechanics has been studied by Kane (1969). A comprehensive exposition in terms of the symmetry group of the system and the corresponding geometric phases can be found in Montgomery (1989). See also Marsden (1992). The geometric phases appear when one lifts the reduced motion to the full phase space by solving the reconstruction equations.

I got interested in this subject when, in a recent work with Richard Cushman (Cushman and Śniatycki, 1999), we discovered gauge momentum map and the corresponding splitting of the reduced equations of motion into non-autonomous Hamiltonian equations in the cotangent bundle of the shape space and autonomous gauge momentum equations.

2 Setting

We consider here a Hamiltonian system with configuration space Q, phase space T^*Q and Hamiltonian of the form $H = K + \pi_Q^* V$, where $K : T^*Q \to \mathbb{R}$ is the kinetic energy, $V : Q \to \mathbb{R}$ is the potential energy, and $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection. The kinetic energy K determines a Riemannian metric k on Q such that

$$K(p) = \frac{1}{2} k^*(p, p)$$

where k^* is the induced metric on the fibres of the cotangent bundle projection $\pi_Q : T^*Q \to Q$.

We assume that the symmetry group of our Hamiltonian system is a Lie group G acting properly and freely on the configuration space Q. This implies that Q is the total space of a principal bundle with structure group G and projection $\pi : Q \to M$. Since the lifted action of G lifted to T^*Q is a symmetry of our Hamiltonian H, the kinetic and potential energy are G-invariant. Therefore the Riemannian metric k on Q is G-invariant.

The tangent bundle TQ of Q can be decomposed into the vertical distribution

$$\mathrm{ver}TQ = \ker T\pi$$

and the horizontal distribution defined as the k-orthogonal complement of $\mathrm{ver}TQ$,

$$\mathrm{hor}TQ = (\mathrm{ver}TQ)^\perp$$

Both distributions are G-invariant, and $\mathrm{ver}TQ \oplus \mathrm{hor}TQ = TQ$. Hence, the horizontal distribution $\mathrm{hor}TQ$ is a connection in the principal bundle $\pi : Q \to M$.

3 Decomposition of the Reduced Space

The *reduced space* of the system is the space $(T^*Q)/G$ of G-orbits in T^*Q. We denote by $\rho : T^*Q \to (T^*Q)/G$ be the orbit map. The reduced phase space has the structure of a fibre bundle over M isomorphic to the direct sum of the coadjoint bundle $Q[\mathfrak{g}^*]$ and the cotangent bundle T^*M over M :

$$(T^*Q)/G = Q[\mathfrak{g}^*] \oplus_M T^*M$$

The Hamiltonian H on T^*Q gives rise to functions H_Δ be a function on T^*M and H_Γ function on $Q[g^*]$ such that $H = \Gamma^* H_\Gamma + \Delta^* H_\Delta$. For every $y \in T^*_x M$,

$$H_\Delta(y) = \frac{1}{2} g^*(x)(y, y) + V(x)$$

where g^* is the metric on the fibres of T^*M induced by the G-invariant Riemannian metric k on Q. Similarly, for each $\tau \in Q[\mathfrak{g}^*]_x$

$$H_\Gamma(\tau) = \frac{1}{2} \kappa^*(x)(\tau, \tau)$$

where κ^* is the metric on the fibres of $Q[\mathfrak{g}^*] \to M$ induced by k.

Let ω be the canonical symplectic form on T^*Q. vector field X_H on (T^*Q, ω) corresponding to the Hamiltonian H is defined by

$$X_H \lrcorner \omega = dH$$

Since ω_Q ad H are G-invariant, so is X_H. Thus it projects to a vector field $\rho_* X$ on the reduced phase space $(T^*Q)/G$. The Hamiltonian equations of motion decompose into the reduced equations describing $\rho_* X$ and the reconstruction equations describing the lift to T^*Q of the integral curves of $\rho_* X$.

4 Gauge Momentum Map

The group $\mathrm{Aut}(Q)$ of automorphisms of the principal bundle Q consists of diffeomorphisms of Q which commute with the action of G. It is an infinite dimensional Lie group, usually called the *gauge group* of the theory. For the purpose of this lecture we need not be concerned with details of the topology of $\mathrm{Aut}(Q)$. It suffices to observe that the Lie algebra $\mathrm{aut}(Q)$ of $\mathrm{Aut}(Q)$ consists of G-invariant vector fields X on Q tangent to the fibres of the principal bundle projection $\pi : Q \to M$. It is isomorphic to the space of sections of the adjoint bundle $Q[\mathfrak{g}] \to M$.

The action of $\mathrm{Aut}(Q)$ on Q lifts a Hamiltonian action on T^*Q. For each X in $\mathrm{aut}(Q)$, the action of the one parameter group $\exp tX$ on T^*Q is given by the flow of the Hamiltonian vector field of a function $\mathcal{J}_X$, which we call the *gauge momentum associated to* X. Since the Lie algebra $\mathrm{aut}(Q)$ is isomorphic to the space of sections of the adjoint bundle $Q[\mathcal{G}] \to M$, it follows that its dual $(\mathrm{aut}(Q))^*$ is a space of distributions. The momentum map $\mathcal{J} : T^*Q \to (\mathrm{aut}(Q))^*$ for the action of $\mathrm{Aut}(Q)$ on T^*Q is defined by $\mathcal{J}_X = \langle \mathcal{J} \mid X \rangle$. Here the pairing $\langle \mid \rangle$ is to be understood in the sense of distributions.

Let Γ be the map from T^*Q to the coadjoint bundle $Q[\mathfrak{g}^*]$ formed from the composition of the projection $T^*Q \to \mathrm{ver}T^*Q$, the reduction map $\mathrm{ver}T^*Q \to (\mathrm{ver}T^*Q)/G$ followed by the bundle isomorphism of $(\mathrm{ver}T^*Q)/G$ onto $Q[\mathfrak{g}^*]$. For each X in $\mathrm{aut}(Q)$,

$$\mathcal{J}_X = \langle \Gamma \mid \zeta_X \rangle$$

where ζ_X is a section of the adjoint bundle $Q[\mathcal{G}] \to M$ corresponding to X and the pairing $\langle \cdot \mid \cdot \rangle$ is taken pointwise. We call Γ the *gauge momentum map* for the action of $\mathrm{Aut}(Q)$ on T^*Q.

5 Decomposition of Equations of Motion

Let $\Delta : T^*Q \to T^*M$ be the map formed from composition of the projection $T^*Q \to \text{hor}T^*Q$ and reduction map $\rho : \text{hor}T^*Q \to (\text{hor}T^*Q)/G$ followed by the bundle isomorphism of $(\text{hor}T^*Q)/G$ with T^*M.

Let $t \mapsto p(t)$ be an integral curve of X_H, $t \mapsto \gamma(t) = \Gamma(p(t))$ its projection to $Q[\mathfrak{g}^*]$, $t \mapsto y(t) = \Delta(p(t))$ its projection to T^*M, and $t \mapsto x(t) = \pi(\gamma(t))$ its projection to M. We denote by $\sigma : x(t) \mapsto \gamma(t)$ the section of $\pi : Q[\mathfrak{g}^*] \mapsto M$ along the curve $t \mapsto x(t)$ such $\sigma(x(t)) = \gamma(t)$. The equations of motion split as follows.

Shape phase space equations

$$\dot{y} \lrcorner \omega_M = -\dot{y} \lrcorner \left(\pi_M^* \left\langle \gamma \mid \hat{\Omega} \right\rangle\right) + \dot{y} \lrcorner dH_\Delta + \Theta$$

where

- ω_M is the canonical symplectic form of T^*M.
- $\hat{\Omega}$ is the 2- form on M with values in the fibres of $Q[\mathfrak{g}]$ induced by the curvature form Ω of the connection $\text{hor}TQ$ on Q. The term $\left(\pi_M^* \left\langle \gamma \mid \hat{\Omega} \right\rangle\right)$ is a gauge momentum dependent magnetic term.
- Θ is a gauge momentum dependent 1-form (generalized force).

Gauge momentum equations

$$\nabla_{\dot{x}} \sigma = T\Gamma(\omega_Q^\sharp(\text{verd}H_\Gamma))$$

where

- ∇ is the covariant derivative operator on sections of $Q[g^*] \to M$ corresponding to the connection $\text{hor}TQ$.
- The right hand side depends only on $\gamma(t) = \sigma(x(t))$ and not on $y(t)$. That is the gauge momentum eqautions decouple from the shape phase space equations

Reconstruction equations

In a trivialization

$$G \times \mathfrak{g}^* \to T^*G : (C, \alpha) \mapsto TL_C\alpha$$

where L_C is the left translation by $C \in G$, the reconstruction equations read

$$\dot{C} = TL_C\{\nabla_{\dot{x}}\sigma + Ad_{C^{-1}}(\hat{A}(x)\dot{x})\}$$

where

- $\hat{A} : TM \to \mathfrak{g}$ is a trivialization representation of the connection form on Q.
- It should be noted that the reconstruction equations decouple from the shape phase space equations.

6 Application to Control

Given a curve $t \mapsto x(t)$ in the shape space M, the gauge momentum equations determine its lifts $t \mapsto \gamma(t)$ to $Q[\mathfrak{g}^*]$. If $t \mapsto \gamma(t)$ is given, the reconstruction equations determine the lift $t \mapsto q(t)$ of $t \mapsto x(t)$ to a curve in Q.

The decomposition of equations of motion given here is also valid in the presence of additional time dependent generalized forces F acting on the system, providing that

- the generalized forces are invariant under the action of the group G,
- the work of generalized forces on virtual displacements tangent to G-orbits vanishes.

The first condition implies that G is a symmetry group of the system. The second condition implies conservation of the equivariant momentum map $J : T^*Q \to \mathfrak{g}^*$ corresponding to the action of G on T^*Q.

Under these conditions the generalized forces push forward by Δ to a 1-form F_Δ on T^*M. In other words

$$F = \Delta^* F_\Delta$$

The form F_Δ appears on the right hand side of the shape phase space equations. The gauge momentum equations and the reconstruction equations remain unchanged.

The generalized forces $F = \Delta^* F_\Delta$ can serve the role of control forces used to determine a trajectory in the shape space M. Its lift to a curve in Q may give rise to a required change of orientation of the system.

7 Extended Objects

Let $B \subset \mathbb{R}^3$ be a compact co-dimension zero submanifold with boundary. It describes the reference configuration of our extended object. Configurations of the body are described by embeddings $q : B \to \mathbb{R}^3$. Since we are interested only in the orientation of the body in space, the configurations under consideration map the centre of mass of the body to the origin. If we denote by ρ a positive smooth function on B describing the mass distribution in the body, the condition that the centre of mass is at the origin reads

$$\int_B q\rho d_3 b = 0$$

where $d_3 b$ is the Lebesgue measure on B given by its original embedding into $\mathbb{R}^3$. The total mass of the body is

$$m = \int_B \rho d_3 b > 0$$

The embeddings $q : B \to \mathbb{R}^3$ considered here are of Sobolev class H^k, $k \geq 3$. It ensures that q and its derivatives up to the order $k-2$ are continuous.

The actual configuration space of an extended system is a closed submanifold Q_c of the extended configuration space

$$Q = \left\{ (q : B \to \mathbb{R}^3) \in H^k \mid \int_B q\rho d_3 b = 0 \right\} \tag{1}$$

The conditions specifying the Q_c are given by the constitutive law of the system. If we do not know the exact constitutive law for our extended system, we can work with the extended configuration space Q. The conclusions about the actual system can be discussed in terms of assumptions about the constitutive law.

The space Q given by (1) is a smooth manifold modeled on a Hilbert space. For each embedding $q \in Q$, a tangent vector in $T_q Q$ is a mapping $u : B \to \mathbb{R}^3$ of Sobolev class H^k such that $\int_B u\rho d_3 b = 0$.

We denote by T^*Q the L^2-cotangent bundle space of Q. For each $q \in Q$, the space T_q^*Q consists of H^k maps $p : B \to R^3$ such that $\int_B p\rho d_3 b = 0$. The evaluation map is

$$T_q^*Q \times T_qQ \to \mathbb{R} : (p, u) \mapsto \langle p \mid u \rangle = \int_B (p \cdot u) d_3 b \tag{2}$$

where the dot $\cdot$ denotes the dot product in $\mathbb{R}^3$. In other words, for every $b \in B$, $(p \cdot u)(b) = p(b) \cdot u(b)$ is the dot product of $p(b)$ and $u(b)$.

The kinetic energy metric k on Q is

$$\mathrm{k}(u, v) = \int_B u \cdot v \rho d_3 b \tag{3}$$

Since ρ is strictly positive on B, for each $q \in Q$, the map $\mathrm{k}^\flat : T_qQ \to T_q^*Q : u \mapsto \mathrm{k}^\flat u$ such that $\langle \mathrm{k}^\flat u \mid v \rangle = \mathrm{k}(u, v)\ \forall\ v \in T_qQ$, is an isomorphism. Eqs. (2) and (3) yield

$$\mathrm{k}^\flat(u) = \rho u$$

The inverse of $\mathrm{k}^\flat$ is denoted by $\mathrm{k}^\sharp : T^*Q \to TQ$. For each $p \in T^*Q$,

$$\mathrm{k}^\sharp(p) = \rho^{-1} p$$

We have The pull-back of the kinetic energy metric k by $\mathrm{k}^\sharp$ yields a metric k^* on T^*Q. In other words,

$$\mathrm{k}^*(p, p') = \int_B p \cdot p' \rho^{-1} d_3 b$$

The kinetic energy $K : T^*Q \to \mathbb{R}$ of the system is

$$K(p) = \frac{1}{2}\mathrm{k}^*(p, p)$$

The L^2-cotangent bundle space T^*Q is weakly symplectic. Let $\pi_Q : T^*Q \to Q$ be the cotangent bundle projection. The canonical 1-form θ of T^*Q is given by

$$\langle \theta(q, p) \mid w \rangle = \langle p \mid T\pi_Q(w) \rangle$$

for each $w \in T_{(q,p)}T^*Q$. The symplectic form is $\omega = -d\theta$.

The action of the connected component G of SO(3) on $\mathbb{R}^3$ yields an action $G \times Q \to Q : (C, q) \mapsto Cq$, where $(Cq)(b) = Cq(b)$ for every $b \in B$. This action of G on Q is free and proper. Hence, the orbit space $M = Q/G$ is a manifold, and Q has the structure of a principal bundle over M with structure group G, and the projection map $\pi : Q \to M$. The base space M is called the *shape space* of the body [Shapere and Wilczek].

The Lie algebra so(3) of SO(3) consists of skew symmetric matrices. To each $\xi \in$ so(3) the infinitesimal action of ξ on $\mathbb{R}^3$ is given by the matrix multiplication $z \mapsto \xi z$ for each $z \in \mathbb{R}^3$. The fundamental vector field X_ξ corresponding to $\xi \in$ so(3) is given by

$$X_\xi(q) = \xi q \tag{4}$$

The lift of the action of G on Q to T^*Q is given by

$$G \times T^*Q \to T^*Q : (C, (q, p)) \mapsto (Cq, pC^{-1})$$

It is a Hamiltonian action on T^*Q with the equivariant momentum map $J : T^*Q \to \mathrm{so}(3)^*$ such that, for every $\xi \in$ so(3),

$$\langle J(q, p) \mid \xi \rangle = \langle p \mid X_\xi(q) \rangle = \int_B (p \cdot \xi q) d_3 b$$

Hence,

$$J(q,p) = \int_B (q \wedge p) d_3 b$$

where $q \wedge p = \frac{1}{2}(q \otimes p - p \otimes q)$. For each $(q,p) \in T^*Q$, $J(q,p)$ is the usual angular momentum of the system in the state (q,p).

The kinetic energy metric k is G-invariant. Hence, the distribution $\mathrm{hor}TQ$ on Q, perpendicular to $\ker T\pi \subset TQ$ with respect to the kinetic energy metric k, is G-invariant and it is a connection in the principal bundle Q. A vector $u \in T_qQ$ is in $\mathrm{hor}TQ$ if, for all $\xi \in \mathrm{so}(3)$,

$$\mathrm{k}(u, X_\xi(q)) = \int_B (u \cdot \xi q)\rho d_3 b = 0$$

This is equivalent to $\int_B (q \wedge \rho u) d_3 b = 0$. Hence, $u \in T_qQ$ is horizontal if and only $(q, \rho u) \in J^{-1}(0)$.

As in Marsden (1992), for each $q \in Q$, we introduce the map $\mathbb{I}(q) : \mathrm{so}(3) \to \mathrm{so}(3)^*$ as follows. For each $\xi, \zeta \in so(3)$,

$$\langle \mathbb{I}(q)\xi \mid \zeta \rangle = \mathrm{k}(X_\xi(q), X_\zeta(q)) \tag{5}$$

Eqs. (3) and (4) yield

$$\begin{aligned} \langle \mathbb{I}(q)\xi \mid \zeta \rangle &= \int_B X_\xi(q) \cdot X_\zeta(q) \rho d_3 b = \int_B (\xi q) \cdot (\zeta q) \rho d_3 b \\ &= (\xi^T \zeta) I(q) = -\mathrm{tr}\,(\xi \zeta I(q)) \end{aligned}$$

where

$$I(q) = \int_B q \otimes q \rho d_3 b$$

is the inertia tensor of the body. Since ρ is positive, the principal moments of inertia of $I(q)$ are positive. Hence, $\mathbb{I}(q)$ is invertible for every $q \in Q$. It is called the *locked inertia tensor.* For each $C \in \mathrm{SO}(3)$,

$$\mathbb{I}(Cq) = Ad^T_{C^{-1}} \mathbb{I}(q) Ad_{C^{-1}}$$

Let α be the so(3)-valued form on Q given by

$$\alpha(q,u) = \mathbb{I}(q)^{-1} J(q, k^\flat u) = \mathbb{I}(q)^{-1} J(q, \rho u) \tag{6}$$

For each $\xi, \zeta \in \mathrm{so}(3)$,

$$\langle \mathbb{I}(q)\alpha(q, X_\xi(q)) \mid \zeta \rangle = \langle \mathbb{I}(q)\xi \mid \zeta \rangle$$

Hence,

$$\alpha(q, X_\xi(q)) = \xi \quad \forall \quad q \in Q \text{ and } \xi \in \mathrm{so}(3)$$

Moreover, for every $u \in \mathrm{hor}T_qQ$ and $\varphi \in \mathrm{so}(3)^*$,

$$\begin{aligned} \langle \varphi \mid \alpha(q,u) \rangle &= \left\langle \varphi \mid \mathbb{I}(q)^{-1} J(q, k^\flat u) \right\rangle = \left\langle J(q, k^\flat u) \mid \mathbb{I}(q)^{-1} \varphi \right\rangle \\ &= \int_B (\rho u \cdot (\mathbb{I}(q)^{-1}\varphi) q) d_3 b = k(u, X_{\mathbb{I}(q)^{-1}\varphi}(q)) = 0 \end{aligned}$$

since $\mathrm{hor}TQ$ is k-orthogonal to all fundamental vector fields. Hence α is the connection form of the connection $\mathrm{hor}TQ$.

8 Collective Property of Reconstruction

Let $t \mapsto q(t)$ be a curve in Q and $t \mapsto \pi(q(t))$ its projection to the shape space M. The horizontal lift of $t \mapsto \pi(q(t))$ to Q is a curve of the form $t \mapsto C(t)q(t)$ such that the tangent vector $\dot{C}(t)q(t) + C(t)\dot{q}(t)$ is horizontal, that is, $\alpha(C(t)q(t), \dot{C}(t)q(t) + C(t)\dot{q}(t)) = 0$. It follows from the properties of a connection that the reconstruction equation reads

$$\dot{C}(t) = -C(t)\alpha(q(t), \dot{q}(t))$$

Since

$$\alpha(q, u) = \mathbb{I}(q)^{-1}J(q, \rho u)$$

it follows that in order to solve the reconstruction equation we need only to know the curve $\mathbb{I}(q(t))^{-1}J(q(t), \rho\dot{q}(t))$

in the Lie algebra of SO(3). That is, it suffices to know how the inertia tensor I the angular momentum J vary with t. The details of the dynamics of the extended object are not important. Hence, we can replace the actual dynamics of the extended object by a simple model.

9 A Model

We consider a simplified model consisting of two axially symmetric rigid bodies attached by a universal joint at the origin of a Cartesian coordinate system. We assume that the joint is placed at the centre of mass of the combined system (this corresponds to the limit when the mass of the point of the joint goes to infinity).

The principal moments of inertia of one body are $\vec{i}, \vec{j}, \vec{k}$, and the corresponding moments of inertia are $I_x, I_y, I_z = I_x$. The tensor of inertia of the first body is

$$I' = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_x \end{pmatrix}$$

The second body is rotating around the x-axis with angular velocity ω. At $t = 0$ the principal moments of inertia of the second body are

$$\cos\alpha\vec{i} + \sin\alpha\vec{k}, \vec{j}, -\sin\alpha\vec{i} + \cos\alpha\vec{k}$$

That is, the second frame at time t is

$$\begin{aligned} \vec{u} &= \cos\alpha\vec{i} + \sin\omega t \sin\alpha\vec{j} + \cos\omega t \cos\alpha\vec{k} \\ \vec{w} &= \cos\omega t\vec{j} - \sin\omega t\vec{k} \\ \vec{w} &= -\sin\alpha\vec{i} + \sin\omega t\cos\alpha\vec{j} + \cos\omega t\cos\alpha\vec{k} \end{aligned}$$

Denoting by $I_u, I_v, I_w = I_v$ the principal moments of inertia corresponding to the axes $\vec{u}, \vec{v}, \vec{w}$, the tensor of inertia of the second body is given by

$$I'' = \begin{pmatrix} I_u\cos^2\alpha + I_v\sin^2\alpha & 0 & (I_v - I_u)\sin\alpha\cos\alpha \\ 0 & I_v & 0 \\ (I_v - I_u)\sin\alpha\cos\alpha & 0 & I_u\sin^2\alpha + I_v\cos^2\alpha \end{pmatrix}$$

This shows that I'' is independent of t. The total inertia tensor is

$$I = I' + I'' = \begin{pmatrix} I_x + I_u\cos^2\alpha + I_v\sin^2\alpha & 0 & (I_v - I_u)\sin\alpha\cos\alpha \\ 0 & I_y + I_v & 0 \\ (I_v - I_u)\sin\alpha\cos\alpha & 0 & I_x + I_u\sin^2\alpha + I_v\cos^2\alpha \end{pmatrix}$$

Since

$$\det I = (I_v + I_y)(I_x^2 + I_x I_u + I_x I_v + I_u I_v) \neq 0$$

we can compute the inverse of the locked inertia tensor $\mathbb{I}$.

The rotation

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t \\ 0 & -\sin\omega t & \cos\omega t \end{pmatrix}$$

corresponds to the rotation vector

$$\vec{\Omega} = \omega \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The total angular momentum equals to the angular momentum of the second body

$$J = I''\Omega = \omega \begin{bmatrix} I_u \cos^2\alpha + I_v \sin^2\alpha \\ 0 \\ (I_v - I_u)\sin\alpha\cos\alpha \end{bmatrix}$$

Since $\xi = \mathbb{I}^{-1}J''$ is independent of t, the reconstruction equation

$$C^{-1}\dot{C} = \mathbb{I}^{-1}J'' = \xi$$

has solution

$$C(t) = \exp t \begin{pmatrix} 0 & \xi_z & 0 \\ -\xi_z & 0 & \xi_x \\ 0 & -\xi_x & 0 \end{pmatrix}$$

where

$$\begin{aligned}
\xi_x &= \frac{\omega}{\det I}\{(I_y + I_v)(I_x + I_u \sin^2\alpha + I_v \cos^2\alpha)(I_u \cos^2\alpha + I_v \sin^2\alpha) \\
&\quad +(I_y + I_v)(I_v - I_u)^2 \sin^2\alpha\cos^2\alpha\} \\
\xi_y &= 0 \\
\xi_z &= \frac{\omega}{\det I}\{(I_y + I_v)(I_v - I_u)\sin\alpha\cos\alpha(I_u \cos^2\alpha + I_v \sin^2\alpha) + \\
&\quad +(I_y + I_v)(I_x + I_u \cos^2\alpha + I_v \sin^2\alpha)(I_v - I_u)\sin\alpha\cos\alpha\} \\
&= \frac{\omega}{\det I}(I_y + I_v)(I_v - I_u)(I_x + I_u \cos^2\alpha + I_v \sin^2\alpha)\sin\alpha\cos\alpha
\end{aligned}$$

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