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Buckling of Cylindrical Shells of Variable Thickness, Loaded by External Uniform Pressure

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This paper is dedicated to the memory of Professor N.A. Alfutov.

From the mathematical standpoint one has a partial differential equation with variable coefficients. Perturbation procedure gives the possibility for an analytical solution of this eigenvalue problem. Self-adjoint equations and Padé approximants are used for improving the obtained results.

1 Introduction

The problem under consideration is important from an engineering standpoint. It may be solved by numerical procedures, for example finite element or finite differences method, matrix method, etc. On the other hand, changing of thickness in practice does not exceed 20-40 %, so, the perturbation procedure is very natural in this case for an analytical solution.

Let us consider a circular cylindrical shell (Figure 1), loaded by uniform external pressure. It is assumed that the external pressure keeps always the direction towards the cylinder principal axis (dead loading). In the linear case one obtains a conservative eigenvalue problem (Alfutov, 1978; Grigolyuk and Kabanov, 1978). It is worth noting two essentially different limiting cases. For $L_1 << L$ one may suppose an isotropic ring-stiffened shell, for $(L - L_1) << L$ one has practically a shell of constant thickness. Here we are interested in cases $L_1 \sim L$, then the shell may be subdivided into three parts (or two parts, if $x_1 = 0,\ x_2 \neq L$ or $x_2 = L,\ x_1 \neq 0$).

2 Perturbation Procedure

In the framework of the semi-inextensional theory (Alfutov, 1978; Reissner, 1964) (or the quasimembrane theory, if we use the terminology of (Awrejcewicz et al., 1998)) for each part of the shell one may use the following equations:

$$B_i \frac{\partial^4 W}{\partial x^4} + \frac{D_i}{R^6} \frac{\partial^4}{\partial \varphi^4}\left(\frac{\partial^2}{\partial \varphi^2}+1\right)^2 W + \frac{q}{R^3}\frac{\partial^4}{\partial \varphi^4}\left(\frac{\partial^2}{\partial \varphi^2}+1\right) W = 0 \qquad i = 1,2 \tag{1}$$

Here $B_i = \frac{Eh_i}{1-\nu^2}$; $D_i = \frac{Eh_i^3}{12(1-\nu^2)}$, $i = 1,2; E, \nu$ - modulus of elasticity and Poisson ratio; φ - circumferential coordinate.

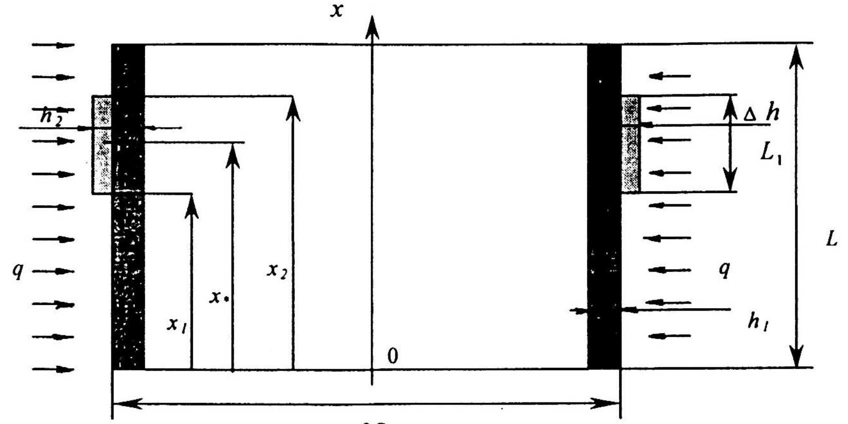

Figure 1. Cylindrical Shell of Variable Thickness

At the inner boundaries of the shell $(x = x_1,\ x = x_2)$ conditions of joining must be posed. It means, that displacements, axial and shear stresses must be equal. Using distributions, one may write the stability equation for the whole shell in the following form

$$B\frac{\partial^4 W}{\partial x^4} + \frac{D}{R^6}\frac{\partial^4}{\partial \varphi^4}\left(\frac{\partial^2}{\partial \varphi^2}+1\right)^2 W + \frac{q}{R^3}\frac{\partial^4}{\partial \varphi^4}\left(\frac{\partial^2}{\partial \varphi^2}+1\right)W = 0 \tag{2}$$

Here $B = \frac{Eh_1}{1-\nu^2}\{1+\varepsilon[H(x-x_1)-H(x-x_2)]\}$ $\quad D = \frac{Eh_1^3}{12(1-\nu^2)}\{1+\varepsilon\alpha[H(x-x_1)-H(x-x_2)]\}$

$$\varepsilon = \frac{h_2-h_1}{h_1} \qquad \varepsilon\alpha = \left(\frac{h_2}{h_1}\right)^3 - 1$$

$H(x)$ is the Heaviside function.

We use the linear theory of stability and don't take into account initial imperfections and moments of the critical state. These factors are not important in the case under consideration. We suppose that the shell is simply supported, then

$$W = \frac{\partial^2 W}{\partial x^2} = 0 \qquad \text{for } x = 0, L \tag{3}$$

Satisfying conditions of periodicity in circumferential direction one may write the unknown eigenvalue function $W(x,\varphi)$ in the following form

$$W = f(x)\sin n\varphi$$

Then the function $f(x)$ may be obtained from the ordinary differential equation with variable coefficients:

$$B\frac{d^4 f}{dx^4} + \frac{n^4(n^2-1)}{R^3}\left[\frac{D}{R^3}(n^2-1) - q\right]f = 0 \tag{4}$$

Now we have the following possibilities for a first approximation of the eigenvalue problem. If $L_1 << 0.5L$, in equation (4) one can suppose $B = B_2$, $D = D_2$. For $L_1 \sim 0.5L$ it will be more suitable to suppose $B = 0.5(B_1+B_2)$, $D = 0.5(D_1+D_2)$. Further we shall deal with the case $0 << L_1 << 0.5L$, then as first approximation one may use equation (4) with $\varepsilon = 0$.

A solution of the eigenvalue problem equations (4) and (3) may be easily obtained:

$$f_0 = \sin\frac{\pi x}{L}$$

$$q_0 = \left(\frac{n\pi}{L}\right)^4 \frac{R^3}{n^4(n^2-1)} + \frac{Eh_1^3}{12R^3(1-\nu^2)}(n^2-1) \tag{5}$$

Minimization of expression (5) with respect to n gives us a known formula (Alfutov, 1978; Grigolyuk and Kabanov, 1978), and for $\nu = 0.3$ and $L > 2R$ one may write

$$q_0 = 0.92E\frac{R}{L}\left(\frac{h_1}{R}\right)^{5/2}$$

It is worth noting that further approximations may change the value of n, so, one must use expression (5) and the minimization procedure with respect to n in each approximation.
The solution of the governing eigenvalue problem we represent in the form of formal expansions:

$$q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \dots \tag{6.1}$$

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \tag{6.2}$$

Substituting series (6) in equation (4) and boundary conditions (3) and splitting it with respect to powers of ε, one obtains the recurrent sequence of eigenvalue problems:

$$M(f_0) = q_0 N(f_0) \tag{7}$$

$$M(f_1) + M_1(f_0) = q_1 N(f_0) + q_0 N(f_1) \tag{8}$$

$$M(f_2) + M_1(f_1) = q_2 N(f_0) + q_1 N(f_1) + q_0 N(f_2) \tag{9}$$

$$M(f_3) + M_1(f_2) = q_3 N(f_0) + q_2 N(f_1) + q_1 N(f_2) + q_0 N(f_3) \tag{10}$$

$$\vdots$$

$$f_i = \frac{d^2 f_i}{dx^2} = 0 \qquad i = 0, 1, 2, \dots \tag{11}$$

Here

$$M(f) = B_1 \frac{d^4 f}{dx^4} + A \cdot f \qquad N(f) = C \cdot f \qquad M_1(f) = \left[B_1 \frac{d^4 f}{dx^4} + A \cdot \alpha \cdot f \right] [H(x - x_1) - H(x - x_2)]$$

$$A = \frac{n^4 (n^2 - 1) D_1}{R^6} \qquad C = \frac{n^4 (n^2 - 1)}{R^3}$$

Since the nonperturbative eigenvalue problem is self-adjoint, one can easily obtain q_1 from equation (8), using scalar product by f_0 (Nayfen, 1973)

$$q_1 = \left[\frac{2E\pi^4 R^3}{(1 - \nu^2) L^5 n^4 (n^2 - 1)} (h_2 - h_1) + \frac{E(n^2 - 1)}{6(1 - \nu^2) R^3 L} (h_2^3 - h_1^3) \right] \cdot \left(\frac{L_1}{2} - \frac{L}{2\pi} \cos \frac{2\pi x_*}{L} \sin \frac{\pi L_1}{L} \right)$$

where $x_* = (x_1 + x_2)/2$.

Minimization with respect to n gives us in the first approximation

$$\omega = \left[\frac{4.3896 R^{3/2} (h_2 - h_1)}{L h_1 (7.29 R^{3/2} - L h_1^{1/2})} + \frac{1}{h_1^{5/2}} \left(\frac{1.458}{L h_1^{1/2}} - \frac{0.2}{R^{3/2}} \right) (h_2^3 - h_1^3) \right] \left(\frac{L_1}{2} - \frac{L}{2\pi} \cos \frac{2\pi x_*}{L} \sin \frac{\pi L_1}{L} \right) \tag{12}$$

Neglecting small terms in formula (12) one obtains:

$$\omega = \left[4.976 \frac{\Delta h}{h_1} + 4.374 \left(\frac{\Delta h}{h_1} \right)^2 + 1.458 \left(\frac{\Delta h}{h_1} \right)^3 \right] \left(\frac{L_1}{2L} - \frac{1}{2\pi} \cos \frac{2\pi x_*}{L} \sin \frac{\pi L_1}{L} \right)$$

Here $\Delta h = h_2 - h_1$. The coefficient ω takes into account the influence of the parameters L_1, Δh and x_* on the critical pressure (Figure 1). The expression in square brackets gives us the dependence of ω on Δh, the expression in parentheses contains parameters L_1 and x_* respectively. One may write

$$\omega = V'(\Delta h / h_1)\, V''(x_*, A)$$

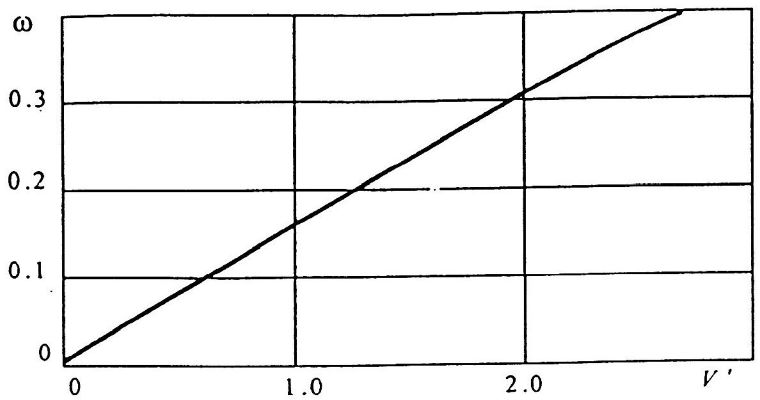

Figure 2. Dependence of Nondimensional Critical Pressure from Variation of Thickness

As one can see from Figure 2, the dependence of ω from $\Delta h / h_1$ is almost linear for small values of $\Delta h / h_1$. Some numerical results are shown in Figure 3, where we use the following geometrical parameters: $L_1 = 1.5$ m, L = 15 m and $R = 10.5$ m. Maximal influence of the inclusion takes place for $x_* = L/2$.

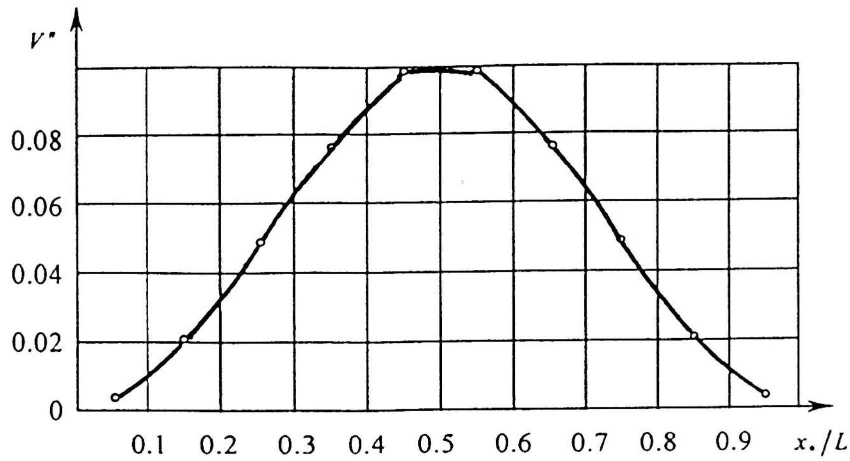

Figure 3. Dependence of Nondimensional Critical Pressure from Coordinate x_*

In the first approximation one obtains the eigenform as follows

$$f_1(x) = \sum_{j=2}^{\infty} A_j \sin nax \tag{13}$$

Here $a = \pi / L$.
Coefficients of expansion (13) may be written in the following form:

$$A_j = \left(a^4 + \alpha\right)\left[\left(\frac{\sin(1-j)ax}{(1-j)2a} - \frac{\sin(1+j)ax}{(1+j)2a}\right)\Bigg|_{x_1}^{x_2}\right]$$

$$A_j = -\frac{\left(B_1 a^4 + D_1 \alpha\right)}{\pi\left[B_1 (ja)^4 + A - Cq_0\right]}\left(\frac{\sin(1-j)ax}{(1-j)} - \frac{\sin(1+j)ax}{(1+j)}\right)$$

Using solution (13), one obtains term q_2 in the expansion (6.1)

$$q_2 = \frac{2}{CL}\left[(q_0 C + A(\alpha - 1))\sum_{j=2}^{\infty} A_j\left(\frac{\sin(1-j)ax}{(1-j)} - \frac{\sin(1+j)ax}{(1+j)}\right) + \right.$$

$$+2\frac{A-q_0C}{a}\sum_{j=2}^{\infty}\frac{A_j}{j}\cos jax \sin ax + \frac{A-q_0C}{a}\sum_{j=2}^{\infty}\frac{A_j}{j^2}(\cos jax\sin ax + \cos ax \sin jax) +$$

$$\left. + \left(2B_1a^3 + \frac{A\alpha}{a} - \frac{4q_1C}{a}\right)\cdot(\cos ax \cdot \sin ax)\right|_{x_1}^{x_2}$$

3 Using Adjoint Equations

For the coefficient q_3 one may write

$$q_3 = \frac{((M_1(f_2) - q_1N(f_2)), f_0) - q_2(N(f_1), f_1)}{(N(f_0), f_0)} \tag{14}$$

Since the function f_2 is unknown as well, we transform the first term in the numerator of the ratio (Marchuk et al., 1996) as follows

$$((M_1(f_2) - q_1N(f_2)), f_0) = ((M_1(f_0) - q_1N(f_0)), f_2) \tag{15}$$

From equation (8) we have:

$$M_1(f_0) - q_1N(f_0) = -M(f_1) + q_0N(f_1) \tag{16}$$

and

$$((M_1(f_0) - q_1N(f_0)), f_2) = -((M(f_1) - q_0N(f_1)), f_2) =$$

$$= -((M(f_2) - q_0N(f_2)), f_1) = -((-M_1(f_1) + q_1N(f_1) + q_2N(f_0) + q_2N(f_0)), f_1) =$$

$$= ((M_1(f_1) - q_1N(f_1)), f_1) = ((q_2N(f_0)), f_1)$$

Finally we obtain

$$q_3 = \frac{((M_1(f_1) - q_1N(f_1)), f_1) - ((2q_2N(f_0)), f_1)}{(N(f_0), f_0)} \tag{17}$$

4 Error Estimation and Padé Approximants

To estimate the accuracy of the obtained solution we also solved the governing eigenvalue problem by the finite difference method. Some numerical results are shown in Figure 4 for $L/R = 4$; $R/h_1 = 500$, $x_* = 0.5L$; $L_1 = 0.3L$.

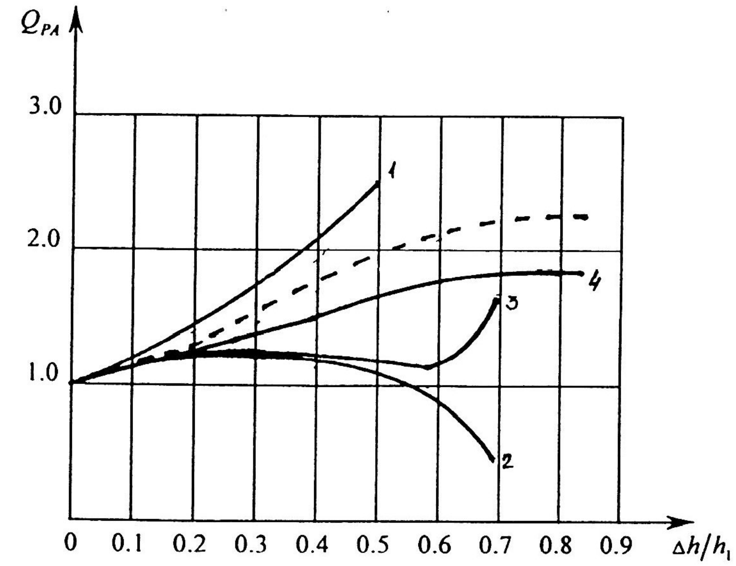

Figure 4. Comparison of Analytical and Numerical Results

Results obtained by the perturbation method are presented by curves 1 $(Q_1 = q_0 + \varepsilon q_1)$, 2 $(Q_2 = Q_1 + \varepsilon^2 q_2)$ and 3 $(Q_3 = Q_2 + \varepsilon^3 q_3)$, the finite difference solution (FDS) is presented by the dotted line.

The perturbation series is divergent. Really, for $\Delta h / h_1 = 0.3$ the discrepancy between FDS and Q_1 is 8.5 %, but between FDS and Q_3 the discrepancy is 38 %.

For overcoming this drawback we use the analytical continuation by Padé approximants (Awrejcewicz, et al. 1998; Baker and Graves-Morris, 1996). The Padé approximant for Q_2 has the following form

$$Q_{PA} \equiv \frac{q_{\kappa p}}{q_0} = \frac{1+a\varepsilon}{1+b\varepsilon} \tag{18}$$

$b = -q_2 / q_1$; $\quad a = q_1 / q_0 + b$.

Numerical values of function Q_{PA} are depicted in Figure 4 by curve 4. Evidently, formula (18) gives good results even for a large change of the thickness.

5 Concluding Remarks

The presented approach allows to obtain the closed analytical formula for the critical pressure of the cylindrical shell of variable thickness.

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