TECHNISCHE MECHANIK, Band 20, Heft,4, (2000),305-310
Manuskripteingang: 7. Februar 2000

The Collinearity Principle and Minimum Energy Orbits

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This paper presents two important aspects for application of the Collinearity Principle, viz.: the orbital energy of a point satellite of constant angular momentum as a function of its orbit eccentricity, and satellite collision.

1 The Collinearity Principle

The Collinearity Principle (Rimrott, 1998) states that *in a multibody system of constant angular momentum, the individual angular momenta tend to align themselves into collinearity in the course of time.* This process (Figure 1) may be brought about by a loss of mechanical energy through interaction of bodies within the system such as bearing friction.

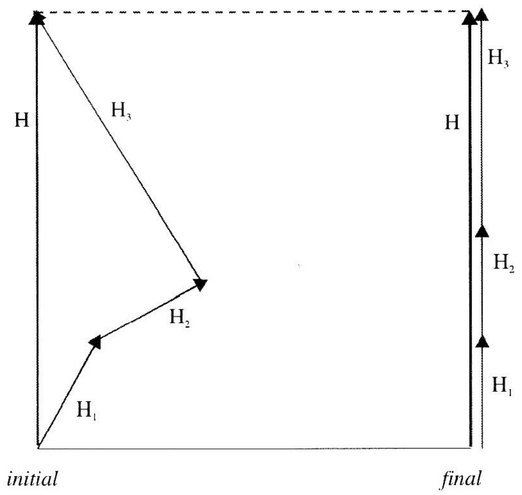


Figure 1: Collinearization of Angular Momenta

Loss of mechanical energy may also be caused by interbody gravitational action (e.g. tides), collisions, atmospheric resistance, and solar wind. Collinearity of the angular momenta alone does not, however, guarantee a minimum energy state. In the present paper, it is shown that only a satellite on a *circular* orbit has minimum energy. The paper also investigates the energy loss associated with a particular *collision*.

2 Master-Satellite Systems

In order to introduce a uniform terminology, we shall use the terms *master* for all sun-like bodies, and *satellite* for all planet-like bodies. A master body thus has far greater mass around which satellites (Latin for bodyguards) orbit. In the Earth-Moon system, Earth is the master, while the Moon is the satellite. Man-made, unpowered spacecraft are satellites of a master such as Earth, the Moon, or the Sun.

3 Orbit Eccentricity

Let us consider a small point satellite of mass m on an elliptical orbit of eccentricity, ε, about a large point master with gravitational parameter μ. The angular momentum can be shown (Rimrott, 1989) to be

$$H = m\sqrt{\mu\, p} \tag{1}$$

where p is the semi-parameter of the orbit. The Collinearity Principle requires that the angular momentum about the master is constant. That is

$$H = \text{constant} \tag{2}$$

From equations (1) and (2), it follows that (Figure 2)

$$p = \frac{H^2}{\mu m^2} = \text{constant} \tag{3}$$

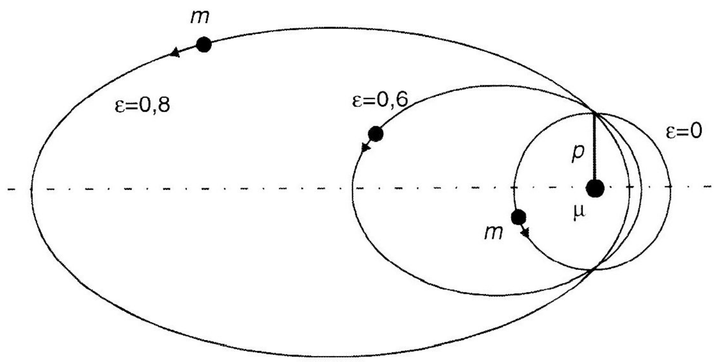


Figure 2: Coplanar Orbits of Equal Angular Momentum

Thus, for a given angular momentum, any elliptical orbit, regardless of its eccentricity, has the same semi-parameter, p.

The orbital (kinetic plus potential) energy of a satellite (Rimrott, 1989) is

$$E = -\frac{\mu m}{2a} \tag{4}$$

where a is the semi-major axis of the elliptical orbit. Since

$$a = \frac{p}{1-\varepsilon^2} \tag{5}$$

one obtains

$$E = -\frac{\mu m}{2p}(1-\varepsilon^2) \tag{6}$$

The orbital energy of a satellite of constant angular momentum on an elliptical orbit is exclusively a function of the orbit's eccentricity provided the satellite mass and the master gravitational parameter remain constant. In addition, for a given angular momentum, the orbital energy is a minimum for a circular orbit $(\varepsilon = 0)$, which will be designated as E_f.

As an example, consider a satellite of mass m = 200 kg, finally assuming a circular orbit, 230 km above the surface of Earth (μ = 3.99 × 10^5 km^3/s^2; radius of Earth = 6,370 km). Its angular momentum is

$$H = m\sqrt{\mu p} = 200\sqrt{3.99\times10^5(6370 + 230)}\ \text{kg km}^2/\text{s} = 1.03\times10^7\ \text{kg km}^2/\text{s} \tag{7}$$

and the final (minimum) orbital energy is

$$E_f = -\frac{\mu m}{2p} = \frac{3.99\times10^5\ 200}{2(6{,}370 + 230)}\text{km}^2\text{kg/s}^2 = -\ 6{,}050\ \text{km}^2\text{kg/s}^2 \tag{8}$$

Let us assume that the satellite initially had an elliptic orbit with $\varepsilon_i = 0.8$. Its energy then was

$$E_i = -\frac{\mu m}{2p}(1 - \varepsilon_i^2) = -\frac{3.99\times10^5 \; 200}{2(6{,}370+230)}(1-0.8^2)\,\text{km}^2\text{kg/s}^2 = -\,2{,}180\,\text{km}^2\text{kg/s}^2 \qquad (9)$$

While going from the elliptic orbit to circular orbit, the change of orbital energy was

$$\Delta E = E_f - E_i = -6{,}050\,\text{km}^2\text{kg/s}^2 - (-2{,}180)\,\text{km}^2\text{kg/s}^2 = -3{,}870\,\text{km}^2\text{kg/s}^2 \qquad (10)$$

The energy change is negative, representing a loss of orbital energy (Figure 3).

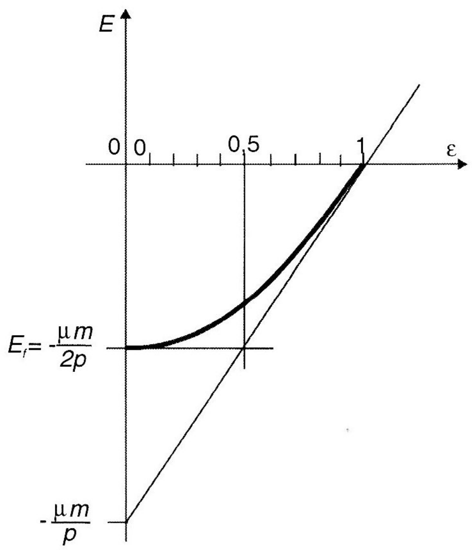


Figure 3: Orbit Energy as Function of Orbit Eccentricity

Differentiating equation (6) with respect to ε, we obtain

$$\frac{dE}{d\varepsilon} = \frac{\mu m}{p}\,\varepsilon \qquad (11)$$

showing that the greater the eccentricity, the more pronounced the slope.

We conclude that a satellite on an elliptic orbit aspires to a circular orbit, and the greater the eccentricity, the greater the aspiration. The aspiration is relatively weak for near-circular orbits.

Rewriting equation (11), we obtain

$$dE = \frac{\mu m}{p}\,\varepsilon\, d\varepsilon \qquad (12)$$

showing that when $d\varepsilon$ is negative, then dE is also negative. Thus, any eccentricity decrease is associated with an energy loss.

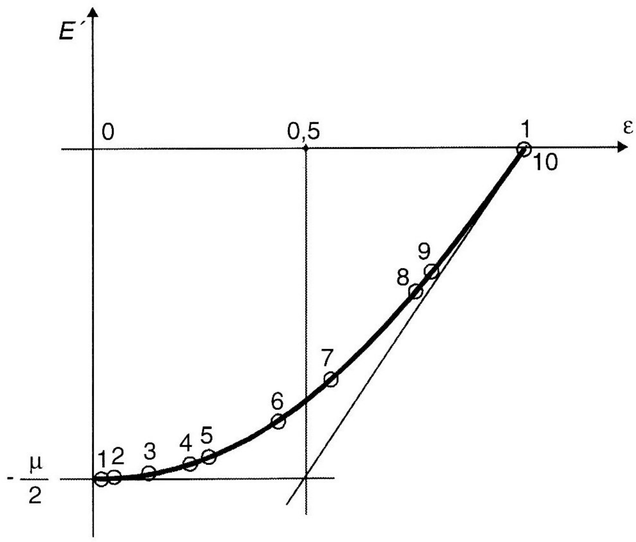
As experimental evidence, we can use the solar system. It has obviously almost reached a minimum energy state. All planets travel on orbits that, in addition to being almost coplanar, are almost circular. There are only a few comets left that travel on highly elliptical orbits about the Sun.
They are in the process of disintegration, and will later collide with larger satellites (Figure 4).

Eccentricities

Planets	
Mercury	0.206
Venus	0.007
Earth	0.017
Moon	0.055
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.05
Neptune	0.01
Pluto	0.25
Comets	
Encke	0.85
Halley	0.97
Kohoutek	0.99
Meteor	
Taurid	0.80
Asteroid	
Apollo	0.56
Meteorite	
Leutkirch	0.40

Specific Orbital Energy

$$E' = \frac{p}{m} E = -\frac{\mu}{2}(1-\varepsilon^2)$$



E = satellite orbital energy

p = satellite orbit semi-parameter

m = satellite mass

μ = master gravitational parameter

1 - Venus, Earth, Neptune
2 - Jupiter, Saturn, Uranus, Moon (about Earth)
3 - Mars
4 - Mercury
5 - Pluto
6 - Leutkirch
7 - Apollo
8 - Taurid
9 - Encke
10 - Halley, Kohoutek

Figure 4: Specific Orbital Energy versus Orbit Eccentricity Diagram for Selected Planets, Comets, Meteors, Asteroids and Meteorites of the Solar System

If we look at the Earth-Moon system as independent from external influences, and make allowances for the finite size of Earth, then the global angular momentum contains Earth's eigenrotation and the Moon's rotation about Earth. The system is not yet near its final stage. The Moon is known to move some 4 cm/year further away from Earth. The Earth, in turn, spins at a rate of 0.02ms/year more slowly.

4 Satellite Collision and Merger

There are, of course, many collisions that actually can and do occur in a master-satellite system such as the solar system. They typically involve a small satellite colliding and merging with a large satellite, having catastrophic consequences for the smaller of the two. Theoretically, one can study any number of collisions. A few special cases of collision have already been investigated. Satellites on coplanar circular orbits, (e.g. rings around Saturn), cannot collide. Their paths do not intersect. In circular orbits, that are not coplanar, collision is possible (Rimrott, 1998). And if coplanar orbits are not circular and intersect, a collision is also possible (Rimrott, 1999). The present study has been chosen to be as simple as possible and emphasizes the fact that typically the final orbit has, in addition to less orbital energy, also less eccentricity than the original orbits before collision.

Let us assume two satellites on coplanar and highly eccentric orbits. Their angular momenta are collinear. They collide in the semi-parameter position P (Figure 5). The velocity of each satellite just before collision (Rimrott, 1989) was

$$v_i = \sqrt{\frac{\mu}{p_i}}\sqrt{1 + \varepsilon_i^2} \tag{13}$$

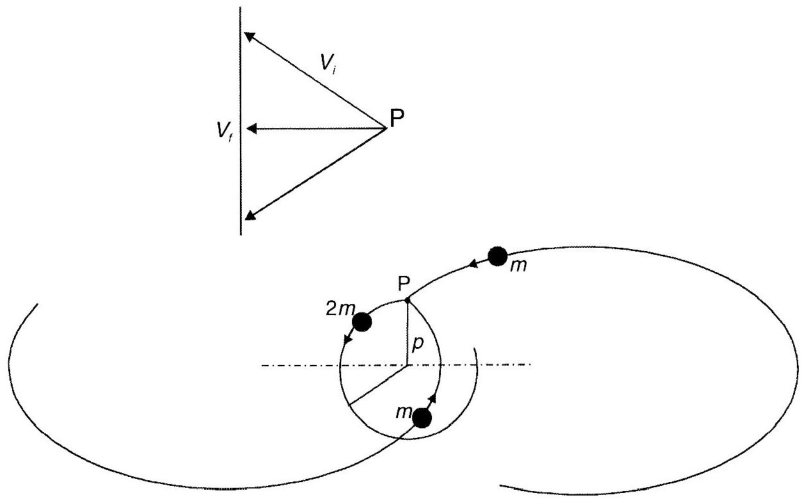


Figure 5: A Satellite Collision and Merger

After collision, only the transverse component of the velocity remains, which is

$$v_f = \sqrt{\frac{\mu}{p_i}} \tag{14}$$

The semiparameter p_i of the old orbits becomes the perigee distance r_{PE} of the new orbit, thus

$$r_{PE} = \frac{p_i}{1+\varepsilon_f} = p_i \tag{15}$$

The angular momentum of the system remains constant, with

$$H = m\sqrt{\mu\, p_i} + m\sqrt{\mu\, p_i} = 2\,m\sqrt{\mu\, p_i} \tag{16}$$

and

$$H = 2\,m\sqrt{\mu\, p_f} \tag{17}$$

Comparing equations (15), (16) and (17), we conclude that

$$p_f = p_i \qquad\qquad \varepsilon_f = 0 \tag{18}$$

and thus the final orbit happens to be circular. Its orbital energy level is

$$E_f = -\frac{\mu\, 2\, m}{2\, r} = \frac{\mu\, m}{p_i} \tag{19}$$

The initial orbital energy level was

$$E_i = E_1 + E_2 = -2\frac{\mu\, m}{2\, p_i}\left(1 - \varepsilon_i^2\right) \tag{20}$$

The energy change is

$$\Delta E = E_f - E_i = -\frac{\mu m}{p_i} \varepsilon_i^2 \tag{21}$$

which is obviously negative, thus it represents a loss of energy. It should equal the change of kinetic energy during the collision. Each satellite has velocities before and after the collision, given in equations (13) and (14). The kinetic energy has changed from

$$T_i = 2\frac{m v_i^2}{2} = m\frac{\mu}{p_i}\left(1+\varepsilon_i^2\right) \tag{22}$$

to

$$T_f = 2\frac{m v_f^2}{2} = m\frac{\mu}{p_i} \tag{23}$$

The difference is

$$\Delta T = T_f - T_i = -\frac{\mu m}{p_i} \varepsilon_i^2 \tag{24}$$

and thus the same as equation (21).

5 Conclusion

In the present paper, the Collinearity Principle has been applied to two aspects of the behavior of master-satellite systems. First, it has been shown that circular orbits have less orbital energy than elliptical ones, and it has been concluded that satellites on high elliptical orbits will try to change to less elliptical orbits, and if possible to circular orbits if they can find the means to lose orbital energy. The solar system supplies ample experimental evidence with nearly all planets on almost circular orbits and only a few comets and meteors left on highly elliptical orbits. Secondly, collision and merger of two satellites have been investigated, and it has been shown that the impact energy loss equals the resultant orbital energy loss.

Literature

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