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On the Existence of Periodic Solutions of a Gyrostat Similar to Lagrange's Gyroscope

A. I. Ismail, L. Sperling, T. S. Amer

In this paper, the problem of the existence of periodic solutions of motion of a gyrostat fixed at one point under the action of a central Newtonian force field, and a gyrostatic momentum ℓ_i $(i = 1,2,3;\ \ \ell_1 = \ell_2 = 0,\ \ \ell_3 \neq 0)$ *similar to a Lagrange gyroscope is investigated. We assume that the center of mass* G *of this gyrostat is displaced by a small quantity relative to the axis of symmetry, and that quantity is used to obtain the small parameter* ε *(Elfimov, 1978). The equations of motion will be studied under certain initial conditions of motion. The Poincaré small parameter method (Malkin, 1959; Nayfeh, 1973) is applied to obtain the periodic solutions of motion. The periodic solutions for the case of irrational frequencies ratio are given. The periodic solutions are geometrically interpreted to give the forms of Euler angles.*

1 Statement of the Problem and Equations of Motion

Consider the motion of a dynamically symmetrical gyrostat relative to a fixed point O, in response to a Newtonian attraction k of another point O_1 and constant gyrostatic momentum $\ell_i\,(i = 1, 2, 3)$ in which $\ell_1 = \ell_2 = 0$ and ℓ_3 is different from zero. At the fixed point O, two coordinate systems are considered; a fixed one $OXYZ$, in such a way that the point O_1 lies in the negative part of the Z-axis, at a constant distance $R = OO_1$, and another moving one $Oxyz$, fixed to the body, whose axes are directed along the principal axes of inertia of the gyrostat at O see (Figure 1).

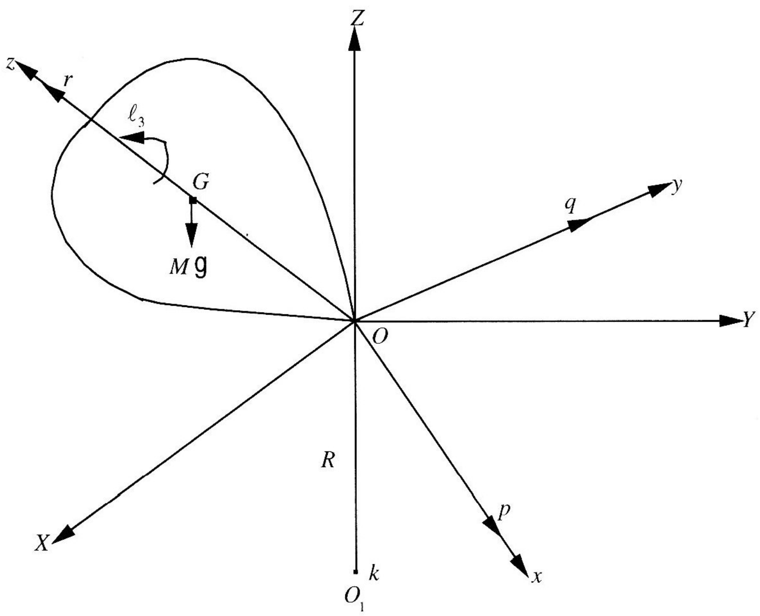

Figure 1. The Force Component

The equations of motion and their three first integrals are

$$
\begin{aligned}
&\frac{dp_1}{d\tau} = a q_1 r_1 - q_1 \ell_3^* - \gamma_1' - k \gamma_1' \gamma_1'' \\
&\frac{dq_1}{d\tau} = -a p_1 r_1 + p_1 \ell_3^* + (\gamma_1 - \gamma_1'') + k \gamma_1 \gamma_1'' \\
&\frac{dr_1}{d\tau} = \frac{\varepsilon}{b} \gamma_1' \\
&\frac{d\gamma_1}{d\tau} = r_1 \gamma_1' - q_1 \gamma_1'' \qquad \frac{d\gamma_1'}{d\tau} = p_1 \gamma_1'' - r_1 \gamma_1 \qquad \frac{d\gamma_1''}{d\tau} = \varepsilon (q_1 \gamma_1 - p_1 \gamma_1')
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
&\varepsilon (p_1^2 + q_1^2 - 2\gamma_1) + b r_1^2 - 2\gamma_1'' - k \gamma_1''^2 = \varepsilon (p_{10}^2 + q_{10}^2 - 2\gamma_{10}) + b r_{10}^2 - 2\gamma_{10}'' - k \gamma_{10}''^2 \\
&\varepsilon (p_1 \gamma_1 + q_1 \gamma_1') + (b r_1 + \ell_3^*) \gamma_1'' = \varepsilon (p_{10} \gamma_{10} + q_{10} \gamma_{10}') + (b r_{10} + \ell_3^*) \gamma_{10}'' \\
&\varepsilon (\gamma_1^2 + \gamma_1'^2) + \gamma_1''^2 = 1
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
&p = \sqrt{\varepsilon}\, \frac{p_1}{n} \qquad q = \sqrt{\varepsilon}\, \frac{q_1}{n} \qquad r = \frac{r_1}{n} \qquad \gamma = \sqrt{\varepsilon}\, \gamma_1 \qquad \gamma' = \sqrt{\varepsilon}\, \gamma_1' \qquad \gamma'' = \gamma_1'' \\
&t = n\tau \qquad n = \sqrt{A / M g z_G} \qquad \ell_3^* = (n/A) \ell_3 \qquad N = 3\lambda / R^3 \qquad k = N a n^2 \\
&a = \frac{A - C}{A} \qquad b = \frac{C}{A} \qquad A = B \neq C \qquad x_G = \sqrt{\varepsilon}\, z_G \qquad y_G = 0
\end{aligned}
\tag{3}
$$

Here A, B, and C are the principal moments of inertia; x_G, y_G, and z_G are the coordinates of the center of mass; p, q, and r are the angular velocity components; γ, γ', and γ'' are the directional cosines of the vertical in the coordinate system attached to the body; ε is a small parameter; M is the mass of the gyrostat; g is the acceleration due to gravity; λ is the constant of gravity of the attracting center; and $p_0, q_0, r_0, \gamma_0, \gamma_0'$, and γ_0'' are the initial values of the corresponding variables.

One of the particular solutions of this problem is the Lagrangian case $(A = B \neq C,\ x_G = y_G = 0)$ $p = q = 0,\quad r = r_0,\quad \gamma = \gamma' = 0,\quad \gamma'' = 1$.

2 The Proposed Method

In this section, Poincaré's small parameter method is applied to obtain the periodic solutions for the equations of motion of the considered problem. From the first and the third equation of (2), we can write

$$
\gamma_1'' = 1 - \varepsilon f_1 \qquad r_1 = r_{10} - \varepsilon f_2 \tag{4}
$$

where

$$
\begin{aligned}
&f_1 = \frac{1}{2} F_1 + \frac{1}{8} \varepsilon F_1^2 + \cdots \\
&f_2 = \frac{1}{2 b r_{10}} (F_2 - F_{20}) + \frac{\varepsilon}{8 b r_{10}} (F_1^2 - F_{10}^2) + \frac{\varepsilon^2}{8 b^2 r_{10}^3} (F_2 - F_{20})^2 + \cdots \\
&F_1 = \gamma_1^2 + \gamma_1'^2 \qquad F_2 = p_1^2 + q_1^2 - 2\gamma_1 + F_1 (k+1)
\end{aligned}
\tag{5}
$$

Thus, F_{10} and F_{20} are the initial values of F_1 and F_2, respectively, and the dots indicate terms of higher order of smallness with respect to ε. Eliminating γ_1'' and r_1 in equations (1) one can obtain the following system

$$\frac{dp_2}{d\tau} = \lambda_1 q_2 + \varepsilon G_1 \qquad \frac{dq_2}{d\tau} = -\lambda_1 p_2 + \varepsilon G_2$$
$$\frac{d\gamma_2}{d\tau} = \lambda_2 \gamma_2' + \varepsilon G_3 \qquad \frac{d\gamma_2'}{d\tau} = -\lambda_2 \gamma_2 + \varepsilon G_4 \tag{6}$$

where

$$\begin{aligned}
&p_1 = p_2 + h\gamma_2 + C_1 \qquad q_1 = q_2 + h\gamma_2' \\
&\gamma_1 = (1+\beta h)\gamma_2 + \beta p_2 + C_2 \qquad \gamma_1' = (1+\beta h)\gamma_2' + \beta q_2 \\
&h = -(1+k)/\pm\sqrt{(br_{10}+\ell_3^*)^2 + 4(1+k)} \\
&\beta = [-(br_{10}+\ell_3^*) \pm \sqrt{(br_{10}+\ell_3^*)^2 + 4(1+k)}\ ]/2(1+k) \\
&C_1 = r_{10}/(1+k+r_{10}\ell_3^* - a r_{10}^2) \qquad C_2 = 1/(1+k+r_{10}\ell_3^* - a r_{10}^2) \\
&\lambda_1 = h - h r_{10}\beta + a r_{10}(1+\beta h) - \beta(1+\beta h)(1+k) - (1+\beta h)\ell_3^* \\
&\lambda_2 = -h - h r_{10}\beta a + r_{10}(1+\beta h) + \beta(1+\beta h)(1+k) + \beta h \ell_3^* \\
&G_1 = h(f_2\gamma_1' - q_1 f_1) - a(1+\beta h) f_2 q_1 + k(1+\beta h) f_1 \gamma_1' \\
&G_2 = h(f_1 p_1 - f_2\gamma_1) + a(1+\beta h) f_2 p_1 - k(1+\beta h) f_1 \gamma_1 + f_1(1+\beta h) \\
&G_3 = \beta(a f_2 q_1 - k f_1 \gamma_1') - (f_2\gamma_1' - q_1 f_1) \\
&G_4 = f_2\gamma_1 - p_1(f_1 + a\beta f_2) + \beta f_1(k\gamma_1 - 1)
\end{aligned} \tag{7}$$

Let $\lambda_1/\lambda_2 = n_1/n_2$ be a rational number; this can be done by a suitable selection of r_0. The general solution for the generating system of (6) is periodic one with period $T_0 = 2\pi n_1/\lambda_1 = 2\pi n_2/\lambda_2$. Let us formulate the problem of determining the $\tau_0(\varepsilon)$-periodic solutions of system (6) with a fairly small value for ε which for $\varepsilon = 0$ would reduce to a solution of period T_0 of the generating system. Consider the following substitution

$$\tau = (1+\varepsilon\alpha)T \tag{8}$$

where α is a function of the small parameter ε, which is to be determined. The problem now reduces to the determination of periodic solutions of period T_0 of the new system of equations (Malkin, 1959)

$$\frac{dp_2}{dT} = \lambda_1 q_2 + \varepsilon H_1 \qquad \frac{dq_2}{dT} = -\lambda_1 p_2 + \varepsilon H_2$$
$$\frac{d\gamma_2}{dT} = \lambda_2 \gamma_2' + \varepsilon H_3 \qquad \frac{d\gamma_2'}{dT} = -\lambda_2 \gamma_2 + \varepsilon H_4 \tag{9}$$

where

$$\begin{aligned}
&H_1 = \lambda_1 \alpha q_2 + (1+\varepsilon\alpha) G_1 \qquad H_2 = -\lambda_1 \alpha p_2 + (1+\varepsilon\alpha) G_2 \\
&H_3 = \lambda_2 \alpha \gamma_2' + (1+\varepsilon\alpha) G_3 \qquad H_4 = -\lambda_2 \alpha \gamma_2 + (1+\varepsilon\alpha) G_4 \\
&H_i = H_i^{(1)} + \varepsilon H_i^{(2)} + \varepsilon^2 H_i^{(3)} + \cdots
\end{aligned} \tag{10}$$

3 Construction of Periodic Solutions in the Case that $\lambda_1\lambda_2^{-1}$ is Rational

We seek the periodic solutions of system (9) in the form

$$\begin{aligned}
&p_2(T,\varepsilon) = M_1 \cos\lambda_1 T + M_2 \sin\lambda_1 T + \textstyle\sum_1 \\
&q_2(T,\varepsilon) = -M_1 \sin\lambda_1 T + M_2 \cos\lambda_1 T + \textstyle\sum_2 \\
&\gamma_2(T,\varepsilon) = M_3 \cos\lambda_2 T + \textstyle\sum_3 \qquad \gamma_2'(T,\varepsilon) = -M_3 \sin\lambda_2 T + \textstyle\sum_4
\end{aligned} \tag{11}$$

$$\left(\textstyle\sum_i = \displaystyle\sum_{n=1}^{\infty} \varepsilon^n C_i^{(n)}(T), \quad i = 1, 2, 3, 4 \right)$$

with the initial conditions

$$p_2(0,\varepsilon)=M_1=M_1^{(0)}+m_1 \qquad q_2(0,\varepsilon)=M_2=M_2^{(0)}+m_2$$
$$\gamma_2(0,\varepsilon)=M_3=M_3^{(0)}+m_3 \qquad \gamma_2'(0,\varepsilon)=0 \tag{12}$$

The periodic solutions of system (6), which correspond to the T_0-periodic solutions of system (9), are of period $\tau_0=(1+\varepsilon\alpha)T_0$. We represent α as $\alpha=\alpha_0+m_4$. In accordance with Poincaré's method we vary the initial conditions, which in this case coincide with the arbitrary constants of solutions of the generating system. We also vary α so as to have solutions (11) of periodic forms, and seek $m_i(\ i=1,\ 2,\ 3,\ 4\ )$ in the form of functions of ε which vanish as $\varepsilon=0$. The substitution of the first approximation for p_1, q_1, γ_1, and γ_1' from (7) into (5) gives

$$\begin{aligned}F_1^{(0)}&=C_2^2+\beta^2(M_1^2+M_2^2)+(1+\beta h)^2M_3^2+2\beta C_2(M_1\cos\lambda_1T+M_2\sin\lambda_1T)\\&+2C_2(1+\beta h)M_3\cos\lambda_2T+2\beta(1+\beta h)M_3[M_1\cos(\lambda_1-\lambda_2)T\\&+M_2\sin(\lambda_1-\lambda_2)T]\end{aligned}$$

$$\begin{aligned}F_2^{(0)}&=C_1^2-2C_2+(M_1^2+M_2^2)+h^2M_3^2+2(C_1-\beta)(M_1\cos\lambda_1T\\&+M_2\sin\lambda_1T)+2(C_1h-1-\beta h)M_3\cos\lambda_2T+2hM_3[M_1\cos(\lambda_1-\lambda_2)T\\&+M_2\sin(\lambda_1-\lambda_2)T]+(1+k)F_1^{(0)}\end{aligned} \tag{13}$$

From (7) we can rewrite the functions $G_i(\ i=1,2,3,4)$ in the following short form

$$G_1=L_1q_2+L_2\gamma_2' \qquad G_2=-(L_1p_2+L_2\gamma_2)+L_3$$
$$G_3=L_4q_2+L_5\gamma_2' \qquad G_4=-(L_4p_2+L_5\gamma_2)+L_6 \tag{14}$$

where

$$\begin{aligned}L_1&=-a(1+\beta h)f_2+h(\beta f_2-f_1)+k\beta(1+\beta h)f_1\\L_2&=-ha(1+\beta h)f_2-h[hf_1-(1+\beta h)f_2]+k(1+\beta h)^2f_1\\L_3&=(1+\beta h)[aC_1f_2+f_1(1-kC_2)]+h(C_1f_1-C_2f_2)\\L_4&=\beta f_2(a-1)+f_1(1-k\beta^2)\\L_5&=h(\beta af_2+f_1)-(1+\beta h)f_2-k\beta(1+\beta h)f_1\\L_6&=-\beta[aC_1f_2+f_1(1-kC_2)]-(C_1f_1-C_2f_2)\end{aligned} \tag{15}$$

If we eliminate terms in the previous formulas that are independent of ε and determined by the generating solutions, then they obtain the following form

$$\begin{aligned}L_i^{(0)}&=[V_{i1}+V_{i2}(M_1^2+M_2^2)+V_{i3}M_3^2]+V_{i4}(M_1\cos\lambda_1T+M_2\sin\lambda_1T)\\&+V_{i5}M_3\cos\lambda_2T+V_{i6}M_3[M_1\cos(\lambda_1-\lambda_2)T+M_2\sin(\lambda_1-\lambda_2)T]\end{aligned} \tag{16}$$

where $V_{ij}(i,j=1,2,\ldots,6)$ are functions of parameters a and r_0 which can be obtained by formulas (4), (13), and (15).

The coefficients $C_i^{(n)}(T)$ in the equations (11) are determined by equations

$$\frac{dC_1^{(n)}(T)}{dT}=\lambda_1C_2^{(n)}(T)+H_1^{(n)}(T) \qquad \frac{dC_2^{(n)}(T)}{dT}=-\lambda_1C_1^{(n)}(T)+H_2^{(n)}(T)$$

$$\frac{dC_3^{(n)}(T)}{dT} = \lambda_2 C_4^{(n)}(T) + H_3^{(n)}(T) \qquad \frac{dC_4^{(n)}(T)}{dT} = -\lambda_2 C_3^{(n)}(T) + H_4^{(n)}(T) \tag{17}$$

with the initial conditions

$$C_i^{(n)}(0) = 0 \qquad (i = 1, 2, 3, 4) \tag{18}$$

For $n = 1$, we get the solutions of (17) in the form

$$\begin{aligned}
C_1^{(1)}(T) &= \lambda_1^{-1} \int_0^T \varphi_{11}(u) \sin \lambda_1 (T-u)\, du \\
C_2^{(1)}(T) &= \lambda_1^{-1} \int_0^T \varphi_{21}(u) \sin \lambda_1 (T-u)\, du \\
C_3^{(1)}(T) &= \lambda_2^{-1} \int_0^T \varphi_{31}(u) \sin \lambda_2 (T-u)\, du \\
C_4^{(1)}(T) &= \lambda_2^{-1} \int_0^T \varphi_{41}(u) \sin \lambda_2 (T-u)\, du
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\lambda_1^{-1}\varphi_{11}(T) = {} & S_{10} + S_{11}(M_1 \cos\lambda_1 T + M_2 \sin\lambda_1 T) + S_{12}\cos 2\lambda_1 T + S_{13}\sin 2\lambda_1 T \\
& + S_{14} M_3 \cos\lambda_2 T + S_{15} M_3 [M_1 \cos(\lambda_1 - \lambda_2)T + M_2 \sin(\lambda_1 - \lambda_2)T] \\
& + S_{16} M_3 [M_1 \cos(\lambda_1 + \lambda_2)T + M_2 \sin(\lambda_1 + \lambda_2)T] + S_{17}\cos(2\lambda_1 - \lambda_2)T \\
& + S_{18}\sin(2\lambda_1 - \lambda_2)T + S_{19} M_3^2 [M_1 \cos(\lambda_1 - 2\lambda_2)T + M_2 \sin(\lambda_1 - 2\lambda_2)T] \\
& + S_{110}\cos 2\lambda_2 T
\end{aligned}$$

$$\begin{aligned}
\lambda_1^{-1}\varphi_{21}(T) = {} & -S_{21}(M_1 \sin\lambda_1 T - M_2 \cos\lambda_1 T) - S_{22}\sin 2\lambda_1 T + S_{23}\cos 2\lambda_1 T \\
& - S_{24} M_3 \sin\lambda_2 T - S_{25} M_3 [M_1 \sin(\lambda_1 - \lambda_2)T - M_2 \cos(\lambda_1 - \lambda_2)T] \\
& - S_{26} M_3 [M_1 \sin(\lambda_1 + \lambda_2)T - M_2 \cos(\lambda_1 + \lambda_2)T] + S_{27}\sin(2\lambda_1 - \lambda_2)T \\
& + S_{28}\cos(2\lambda_1 - \lambda_2)T - S_{29} M_3^2 [M_1 \sin(\lambda_1 - 2\lambda_2)T - M_2 \cos(\lambda_1 - 2\lambda_2)T] \\
& - S_{210}\sin 2\lambda_2 T
\end{aligned}$$

$$\lambda_2^{-1}\varphi_{31}(T) = V_{61} + (V_{63} - \frac{1}{2}V_{55}) M_3^{(0)2} - \frac{3}{2} V_{55} M_3^{(0)2} \cos 2\lambda_2 T + \cdots$$

$$\begin{aligned}
\lambda_2^{-1}\varphi_{41}(T) = {} & -S_{41}(M_1 \sin\lambda_1 T - M_2 \cos\lambda_1 T) - S_{42}\sin 2\lambda_1 T + S_{43}\cos 2\lambda_1 T \\
& - S_{44} M_3 \sin\lambda_2 T - S_{45} M_3 [M_1 \sin(\lambda_1 - \lambda_2)T - M_2 \cos(\lambda_1 - \lambda_2)T] \\
& - S_{46} M_3 [M_1 \sin(\lambda_1 + \lambda_2)T - M_2 \cos(\lambda_1 + \lambda_2)T] \\
& + S_{47}\sin(2\lambda_1 - \lambda_2)T + S_{48}\cos(2\lambda_1 - \lambda_2)T - S_{49} M_3^2 [M_1 \sin(\lambda_1 - 2\lambda_2)T \\
& - M_3 \cos(\lambda_1 - 2\lambda_2)] - S_{410}\sin(2\lambda_2 T)
\end{aligned} \tag{20}$$

and $S_{1i}, S_{2i}, S_{3i}, S_{4i}$ $(i = 1, 2, \cdots, 10)$ are constants that can be obtained easily.

Substituting (20) into (19), one gets

$$\begin{aligned}
C_1^{(1)}(T_0) &= (M_2 E_{11} + R_{11}) T_0 \\
C_2^{(1)}(T_0) &= (M_1 E_{11} + R_{21}) T_0 \\
C_4^{(1)}(T_0) &= -(M_3 E_{31} + R_{41}) T_0
\end{aligned} \tag{21}$$

where

$$E_{11} = -\frac{1}{2} S_{11} \qquad E_{31} = -\frac{1}{2} S_{44} \tag{22}$$

It is shown in (Malkin, 1959) that for the solution (11) to be T_0−periodic, it is necessary and sufficient that

$$\psi_1 = p_2(T_0, \varepsilon) - p_2(0, \varepsilon) = 0 \qquad \psi_2 = q_2(T_0, \varepsilon) - q_2(0, \varepsilon) = 0$$
$$\psi_3 = \gamma_2(T_0, \varepsilon) - \gamma_2(0, \varepsilon) = 0 \qquad \psi_4 = \gamma_2'(T_0, \varepsilon) - \gamma_2'(0, \varepsilon) = 0 \tag{23}$$

where ψ_i $(i = 1, 2, 3, 4)$ are functions of M_i $(i = 1, 2, 3)$, α, and ε. The equalities (23), which determine $M_j^{(0)}$, α_0, and m_i $(j = 1, 2, 3;\ i = 1, 2, 3, 4)$, are not independent due to the existence of the first integral in system (9), which corresponds to the second formula in (2) (Elfimov, 1978). It can be shown that the third condition is a corollary of the remaining if $M_3 \neq 0$. By analogy with the statement in (Arkhangel'skii, 1963), it is possible to consider one of the quantities $M_j^{(0)}$ $(j = 1, 2, 3)$ or α_0 as an arbitrary constant, and one of the m_i $(i = 1, 2, 3, 4)$ as an arbitrary function of ε, which vanishes as $\varepsilon \to 0$.

Reducing equalities (23) by ε and equating to zero the terms at zero powers of ε, yields the following necessary periodicity conditions

$$C_i^{(1)}(T_0) = C_i^{(1)}(M_1, M_2, M_3, \alpha) = 0 \quad (i = 1, 2, 4) \tag{24}$$

which are in accordance with (21); one thus has

$$M_2 E_{11} + R_{11} = 0 \qquad M_1 E_{11} + R_{21} = 0 \qquad M_3 E_{31} + R_{41} = 0 \tag{25}$$

The expressions of R_{11}, R_{21} and R_{41} are nonzero only if $\lambda_1 \lambda_2^{-1}$ is equal to $2, 1/2, 1$ or -1 and are of the forms

i) *for* $\lambda_1 \lambda_2^{-1} = 2$:

$$R_{11} = 0 \qquad R_{21} = \frac{1}{2} V_{25} M_3^2 \qquad R_{41} = \frac{1}{2} M_1 M_3 (V_{45} - V_{66})$$

ii) *for* $\lambda_1 \lambda_2^{-1} = 1/2$:

$$R_{11} = \frac{1}{2} M_2 M_3 (V_{36} - V_{24}) \qquad R_{21} = -\frac{1}{2} M_1 M_3 (V_{36} - V_{24})$$
$$R_{41} = \frac{1}{2} V_{44} (M_1^2 - M_2^2)$$

iii) *for* $\lambda_1 \lambda_2^{-1} = 1$:

$$R_{11} = M_2 M_3 (M_1 V_{16} - \frac{1}{2} M_3 V_{26})$$
$$R_{21} = M_3 [V_{21} + V_{22} (M_1^2 + M_2^2) + V_{23} M_3^2 + V_{16} M_1^2] + \frac{1}{2} V_{26} M_1 M_3^2 - \frac{1}{2} V_{35} M_3$$
$$R_{41} = M_1 [V_{41} + V_{42} (M_1^2 + M_2^2) + V_{43} M_3^2] + \frac{1}{2} V_{46} M_3 (M_1^2 - M_2^2) + V_{56} M_1 M_3^2 - \frac{1}{2} V_{64} M_1$$

iv) *for* $\lambda_1 \lambda_2^{-1} = -1$:

$$R_{11} = 0 \qquad R_{21} = -\frac{1}{2} V_{35} M_3 \qquad R_{41} = -\frac{1}{2} V_{64} M_1 \tag{26}$$

Let $M_1^{(0)}, M_2^{(0)}, M_3^{(0)}$, and α_0 satisfy equations (25). Let us consider Jacobi's matrices of $C_1(T_0)$, $C_2(T_0)$, and $C_4(T_0)$ in terms of M_1, M_2, M_3, and α calculated for $M_j = M_j^{(0)}$ ($j = 1, 2, 3$), $\alpha = \alpha_0$, and also of ψ_1, ψ_2, and ψ_4 in terms of m_i with $m_i = \varepsilon = 0$ ($i = 1, 2, 3, 4$). The calculation of the second matrix does not involve differentiation with respect to ε. Hence it is possible to set $\varepsilon = 0$. And since M_j ($j = 1, 2, 3$), α, and m_i ($i = 1, 2, 3, 4$) appear in solutions in the form of related sums, the considered matrices are the same. We denote them by J.

The solution of equations (23) comprises the following case of the existence of periodic solutions.

4 Formal Construction of Periodic Solutions in the Case that $\lambda_1 \lambda_2^{-1}$ is Irrational

If $M_1^{(0)} = M_2^{(0)} = E_{31} = 0$, $E_{11} \neq 0$, $M_3^{(0)} V_{33} \neq 0$ (M_3 is an arbitrary quantity) (Elfimov, 1978), matrix J is of the third rank and

$$\alpha_0 = \lambda_2^{-1} \left(\frac{1}{2} V_{65} - V_{51} - V_{53} M_3^{(0)^2} \right) \tag{27}$$

Equations (23) have solutions in the form of series of integral powers of ε for m_1, m_2, and m_4 that depend on the arbitrary quantity M_3 and vanish as $\varepsilon \to 0$ (m_3 is to be taken as equal to zero). Under the conditions (27), the independent periodicity conditions are

$$\begin{aligned} &m_1(\cos\lambda_1 T_0 - 1) + m_2 \sin\lambda_1 T_0 + \varepsilon C_1^{(1)}(T_0) + \cdots = 0 \\ &-m_1 \sin\lambda_1 T_0 + m_2(\cos\lambda_1 T_0 - 1) + \varepsilon C_2^{(1)}(T_0) + \cdots = 0 \\ &C_4^{(1)}(T_0) + \cdots = 0 \end{aligned} \tag{28}$$

Using (27), the periodic solutions $p_2(T,\varepsilon)$, $q_2(T,\varepsilon)$, $\gamma_2(T,\varepsilon)$, and $\gamma_2'(T,\varepsilon)$; as $\varepsilon \to 0$; take the forms

$$\begin{aligned} &p_2(T,0) = 0 && q_2(T,0) = 0 \\ &\gamma_2(T,0) = M_3^{(0)} \cos\lambda_2 T && \gamma_2'(T,0) = -M_3^{(0)} \sin\lambda_2 T \end{aligned} \tag{29}$$

with frequency λ_2.

Using the second integral of (2) and the initial conditions (12), we obtain

$$M_3^{(0)} = \frac{1}{2} N_1^{-1} [-N_2 \pm (N_2^2 + 4N_1N_3)^{1/2}] \qquad N_1 \neq 0 \qquad (N_2^2 + 4N_1N_3) > 0 \tag{30}$$

where N_i ($i = 1, 2, 3$) are constants. Making use of (27) and (28), one gets

$$\begin{aligned} m_1 = -\frac{1}{2} \varepsilon \lambda_1^{-1} \{ &V_{31} - V_{33} M_3^{(0)^2} + \lambda_1 V_{25} M_3^{(0)^2} (\lambda_1 - 2\lambda_2)^{-1} + M_3^{(0)} (\lambda_2 - \lambda_1)^{-1} [V_{35}(\lambda_1^2 \\ &+ \lambda_2^2)(\lambda_1 + \lambda_2)^{-1} - (\lambda_1 + \lambda_2)(V_{21} + V_{23} M_3^{(0)^2})] + [V_{31} + (V_{35} - V_{21}) M_3^{(0)} + (V_{33} \\ &- V_{25} - V_{23} M_3^{(0)}) M_3^{(0)^2}] \cos\lambda_1 T_0 \} + \cdots \\ m_2 = -\frac{1}{2} \varepsilon \lambda_1^{-1} [&V_{31} + (V_{35} - V_{21}) M_3^{(0)} + (V_{33} - V_{25} - V_{23} M_3^{(0)}) M_3^{(0)^2}] \sin\lambda_1 T_0 + \cdots \end{aligned} \tag{31}$$

and the calculations for m_4 show that it is of the order $O(\varepsilon^2)$. Making use of (3), (4), (7), (11), (20), (27), and (31), one obtains

$$p=\varepsilon^{\frac{1}{2}} n^{-1}[C_1+N_{11}\cos(\lambda_2 n^{-1}t)]+\varepsilon^{\frac{3}{2}} n^{-1}[N_{12}+N_{13}\cos(\lambda_2 n^{-1}t) + N_{14}\cos(2\lambda_2 n^{-1}t)-N_{15}\sin(\lambda_1 n^{-1}t)-N_{16}\cos(\lambda_1 n^{-1}t)]+\cdots$$

$$q=-\varepsilon^{\frac{1}{2}} n^{-1} N_{11}\sin(\lambda_2 n^{-1}t)+\varepsilon^{\frac{3}{2}} n^{-1}[N_{21}\sin(\lambda_2 n^{-1}t)-N_{22}\sin(2\lambda_2 n^{-1}t) - N_{23}\cos(\lambda_2 n^{-1}t)+N_{23}\cos((\lambda_1-2\lambda_2)n^{-1}t)+N_{24}\sin(\lambda_1 n^{-1}t) - N_{25}\cos(\lambda_1 n^{-1}t)]+\cdots$$

$$r=r_0+\varepsilon N_{31}[1-\cos(\lambda_2 n^{-1}t)]+\varepsilon^{\frac{3}{2}}[0]+\cdots$$

$$\gamma=\varepsilon^{\frac{1}{2}}[C_2+N_{41}\cos(\lambda_2 n^{-1}t)]+\varepsilon^{\frac{3}{2}}[N_{42}+N_{43}\cos(\lambda_2 n^{-1}t)+N_{44}\cos(2\lambda_2 n^{-1}t) + N_{45}\cos(\lambda_1 n^{-1}t)+N_{46}\sin(\lambda_1 n^{-1}t)]+\cdots \tag{32}$$

$$\gamma'=-\varepsilon^{\frac{1}{2}} N_{41}\sin(\lambda_2 n^{-1}t)+\varepsilon^{\frac{3}{2}}[N_{51}\cos(\lambda_2 n^{-1}t)+N_{52}\sin(\lambda_2 n^{-1}t) - N_{53}\sin(2\lambda_2 n^{-1}t)+N_{54}\cos((\lambda_1-2\lambda_2)n^{-1}t)+N_{55}\sin(\lambda_1 n^{-1}t) + N_{56}\cos(\lambda_1 n^{-1}t)]+\cdots$$

$$\gamma''=1-\frac{1}{2}\varepsilon[C_2^2+N_{41}^2+2C_2N_{41}\cos(\lambda_2 n^{-1}t)]+\varepsilon^{\frac{3}{2}}[0]+\cdots$$

$$\alpha(\varepsilon)=\lambda_2^{-1}(\frac{1}{2}V_{65}-V_{51}-V_{53}M_3^{(0)^2})+\varepsilon[0]+\cdots$$

where N_{ij} are constants and $T_0=2\pi\lambda_2^{-1}$. The stability of the solutions will be shown in future in another research.

5 Geometric Interpretation of Motion

We shall investigate the expressions for the Eulerian angles in the form of power series expansions of the small parameter ε, so that we can determine the orientation of the gyrostat at any instant of time. For this case, the Eulerian angles θ, ψ and φ can be written in the following forms (Wittenburg, 1977)

$$\theta=\cos^{-1}\gamma'' \qquad \dot\psi=\frac{p\gamma+q\gamma'}{1-\gamma''^2}$$

$$\dot\varphi=r-\dot\psi\cos\theta \qquad \varphi_0=\tan^{-1}\frac{\gamma_0}{\gamma'_0} \qquad (\dot{\ })\equiv\frac{d()}{dt} \tag{33}$$

Substituting (32) into (33), one has

$$\varphi_0=\frac{\pi}{2}-\lambda_1 n^{-1}h$$
$$\theta=\theta^{(1)}+\frac{3}{2}\varepsilon(1+\beta h)C_2M_3^{(0)}\cos(\lambda_2 n^{-1}t)+\cdots$$
$$\psi=\psi_0+2\varepsilon n^{-1}\{[C_1C_2+h(1+\beta h)M_3^{(0)^2}]t+M_3^{(0)}[(1+\beta h)C_1 + C_2h]n\lambda_2^{-1}\sin(\lambda_2 n^{-1}t)\}+\cdots \tag{34}$$
$$\varphi=\varphi_0+r_0t-\varepsilon M_3^{(0)}(n^2b\,r_0)^{-1}[W_1t+n\lambda_2^{-1}W_2\sin(\lambda_2 n^{-1}t)]+\cdots$$

where

$$\theta^{(1)} = \cos^{-1}\{1-\frac{1}{2}\,\varepsilon[C_2^2+(1+\beta h)^2 M_3^{(0)^2}]\}$$
$$W_1 = (1+\beta h)[1-C_2(1+k)]-C_1 h+2nb\,r_0\, M_3^{(0)^{-1}}[C_1 C_2+h(1+\beta h)M_3^{(0)^2}] \qquad (35)$$
$$W_2 = -(1+\beta h)[1-C_2(1+k)]+C_1 h+2nb\,r_0\,[h\,C_2+(1+\beta h)\,C_1]$$

Formulas (34) show that the expressions of Eulerian angles depend on four arbitrary constants $\theta^{(1)}$, ψ_0, φ_0, and r_0. These formulas describe the orientation of the body at any instant t.

6 Numerical Discussions

In this section we investigate the numerical results by computer codes. Let us consider the following parameters by which the motions of the body are determined

$$A=17.36\,\text{kgmm}^2,\quad C=29.64\,\text{kgmm}^2,\quad R=1000\,\text{mm},\quad r_{10}=100\,\text{mm},$$
$$\varepsilon=0.01E-02,\quad z_G=2\,\text{mm},\quad \gamma_{20}=79.07E-03,\quad \gamma'_{20}=5.44E-03,$$
$$p_{20}=1.67E-05\,\text{s}^{-1},\quad q_{20}=5.23E-08\,\text{s}^{-1},\quad \ell_3=0,350\,\text{kgmm}^2\text{s}^{-1},$$
$$M=300\,\text{kg}$$

Figures (2.a) and (2.b) represent the variation of p_2 and q_2 respectively with time, but figures (2.c) and (2.d) represent the variation of γ_2 and γ'_2 respectively with time.

We note that, when $\ell_3 = 0, 350,$ the solutions are periodic. We conclude that, if ℓ_3 increases, the amplitude of the waves and the number of oscillations also increase.

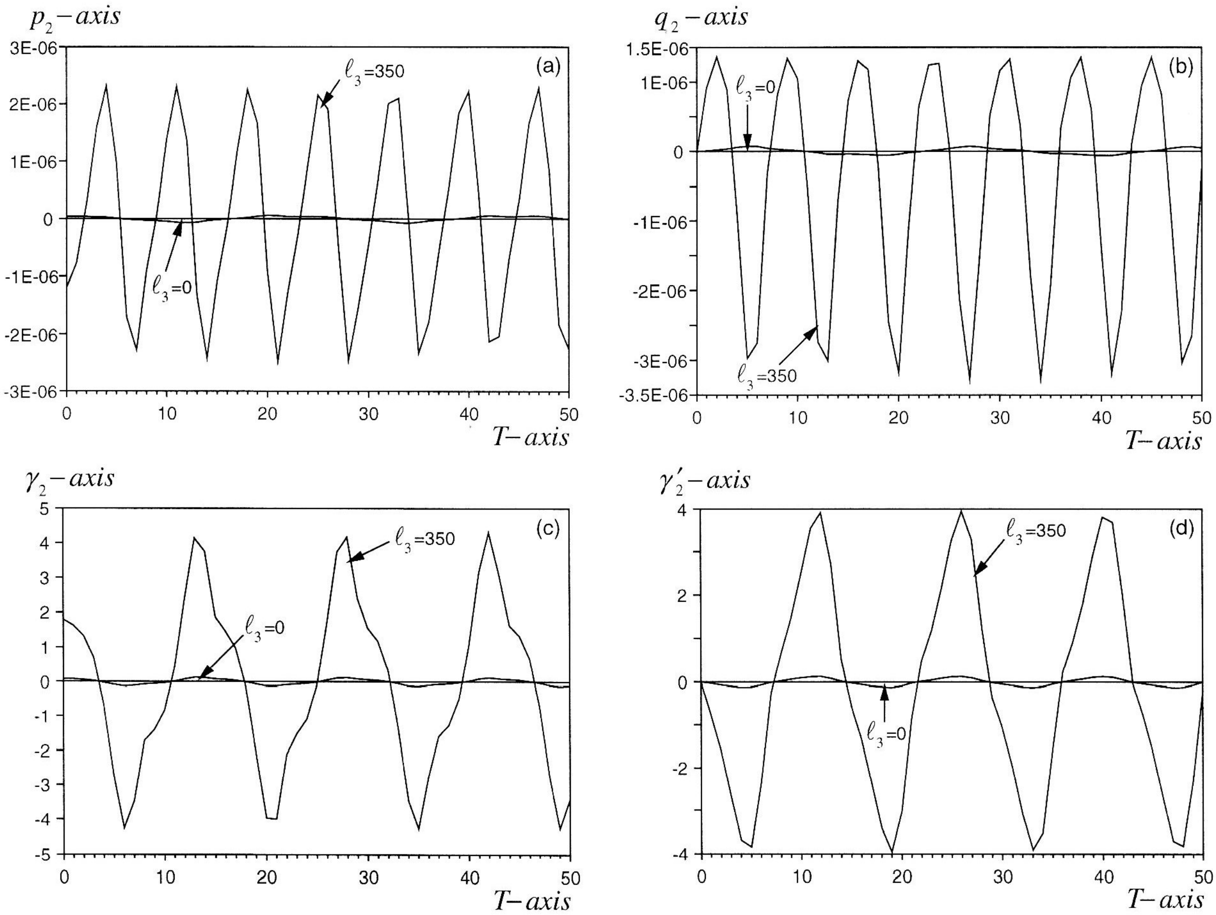

Figure 2. The Effect of ℓ_3 on the Gyroscopic Motion

7 Conclusion

The three-dimensional motion of a rigid body in the Newtonian force field with the third gyrostatic momentum, about one of the principal axes of the ellipsoid of inertia, can be investigated by reducing the six first-order non-linear differential equations of motion and their first three integrals into a quasilinear autonomous system with two degrees of freedom and one first integral. The Poincaré's small parameter method is used to investigate the periodic solutions of our problem up to the first order approximation in terms of the small parameter ε. The obtained periodic solutions are considered as a generalization of those which were obtained by Elfimov (in the case of the uniform force field). The solutions are worked out by computer codes to get their graphical representations. The good effect of the third gyrostatic momentum (ℓ_3) for the mentioned problem is obvious from the graphical representation of this problem.

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Addresses: Prof. Dr.-Ing. habil. Lutz Sperling; Institut für Mechanik, Otto-von-Guericke-Universität Magdeburg, PF 4210, D-39016 Magdeburg, Germany. Dr. Abdel Aziz Ibrahim Ismail and M.Sc. Tarek Saleh Amer; Mathematics Department, Faculty of Science, Tanta University, Tanta 31527, Egypt.