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Chaotic Attitude Motion of a Satellite on a Keplerian Elliptic Orbit

J.H. Peng, Y.Z. Liu

In this paper, we discussed an asymmetric satellite moving on a Keplerian elliptic orbit in a gravitational field of a central mass point. Formulating the Hamiltonian of the rigid body in Serret-Andoyer variables, and by an infinitesimal contact transformation, the system is reduced to a rigid body in torque-free motion, but its moments of inertia vary with time, we take this system as an Euler-Poinsot motion perturbed by a small periodic excitation, then we can apply the Melnikov method to determine the intersection of the stable and unstable manifold of the system's hyperbolic point, usually this can be the cause of chaos. We also manifested the chaotic motion in angular momentum space by the Poincaré surface of section.

1 Introduction

In recent years, some fascinating results on deterministic chaos have been achieved in the field of celestial mechanics. According to the traditional view, the planets of the solar system move along their orbits with the regularity of clockwork, the motion of the planets is strictly periodic, but recent progress (Wisdom,1987) shows chaotic motion may be the reason for the transport of meteorites to the Earth. And the Voyager 1 and 2 space missions, analytical, and numerical analysis (Wisdom,1987) also showed that Hyperion, a satellite of Saturn, performs a chaotic tumbling motion in the sense that its angular velocity and orientation of its axis of rotation are subject to strong and erratic changes, and it is believed this chaotic dance must have also occured in the history of other satellites. The causes of this phenomenon is regarded as the consequence of the asymmetry of the rigid body and the eccentricity of the orbit. A variety of other chaotic phenomena in satellite attitude motion is also reported (Seisl and Steindl, 1989; Ashenberg, 1996).

In this paper, we discuss an asymmetric satellite, as Hyperion, moving on a Keplerian elliptic orbit in a gravitational field of a central mass point (see Figure 1). Formulating the Hamiltonian of the rigid body in Serret-Andoyer variables, and by an infinitesimal contact transformation, the system is reduced to a rigid body in torque-free motion, but its moments of inertia vary in time. We take this system as an Euler-Poinsot motion perturbed by a small periodic excitation, then we can apply the Melnikov method to determine the intersection of the stable and unstable manifold of the system's hyperbolic point. Usually this can be the cause of chaos. We also manifest the chaotic motion in angular momentum space by the Poincaré surface of section.

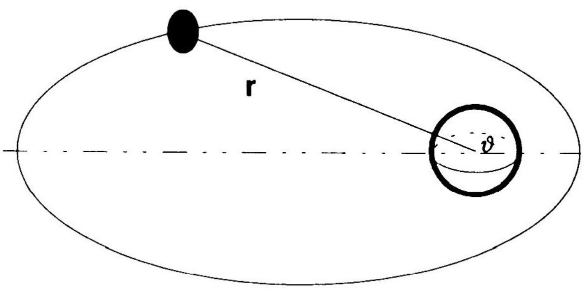


Figure 1. An Asymmetric Satellite Moving on a Keplerian Elliptic Orbit

2 Description of the Model

For a rigid body revolving around a mass point due to gravitational force, its orbital and rotational motion are coupled. Considering the orbital distances are much bigger than the dimensions of the rigid body, it is usual to neglect the gravitational coupling of the attitude to the orbit, and therefore, the orbit is a fixed elliptic orbit. The Hamiltonian is decomposed as the sum (Arribas and Elipe, 1993).

$$H = H_E + \varepsilon H_c + \ldots \tag{1}$$

where H_E stands for the Hamiltonian of a rigid body in torque-free rotation, whereas H_C contains the coupled terms. The small parameter ε is the quotient of the orbital mean motion of the center of mass by a reference value of the rigid body's rotational angular velocity.

The Hamiltonian is formulated in Serret-Andoyer variables (l, g, h, L, G, H) for the attitude motion, and there are defined two angles δ and σ given by $\cos\delta = H/G$, $\cos\sigma = L/G$. The principal moments of inertia of the rigid body (I_1, I_2, I_3) are in the relation $I_1 \le I_2 \le I_3$. The Hamiltonian of the Euler-Poinsot motion of a rigid body in torque-free rotation in Serret-Andoyer variables is

$$H_E = \left(\frac{\sin^2 l}{2I_1} + \frac{\cos^2 l}{2I_2}\right)\left(G^2 - L^2\right) + \frac{1}{2I_3}L^2 \tag{2}$$

This problem is integrable. For the system under the action of gravitational force, we can construct an infinitesimal contact transformation

$$(l, g, h, L, G, H) \to (l', g', h', L', G', H') \tag{3}$$

The new Hamiltonian $H' = H'_E + \varepsilon H'_c$ for the system under the gravitational force can eventually be written as (Arribas and Elipe,1993)

$$H' = \frac{1}{2}\left(\frac{\sin^2 l'}{I_1^*} + \frac{\cos^2 l'}{I_2^*}\right)\left(G'^2 - L'^2\right) + \frac{1}{2I_3^*}L'^2 \tag{4}$$

It has exactly the same form as the Hamiltonian of a rigid body in torque-free motion in equation (2), the angles g', h' being cyclic, their conjugate momenta G', H' are integrals of the motion, then the averaged first order is reduced to one degree of freedom, and therefore it is integrable. The pseudo-moments of inertia I_i^* are

$$\begin{aligned}
\frac{1}{I_1^*} &= \frac{1}{I_1} + \frac{3\varepsilon}{4G'^2 r^3}\left(1 - 3\cos^2\delta'\right)(I_1 - I_2) \\
\frac{1}{I_2^*} &= \frac{1}{I_2} \\
\frac{1}{I_3^*} &= \frac{1}{I_3} + \frac{3\varepsilon}{4G'^2 r^3}\left(1 - 3\cos^2\delta'\right)(I_3 - I_2)
\end{aligned} \tag{5}$$

where $\cos^2\delta' = L'^2/G'^2$

It is worth noting that the pseudo-moments of inertia I_i^* vary in time, because they contain the radial distance r, and time dependence of the moments of inertia is quite common when we consider attitude dynamics of deformable bodies, such as flexible platforms with damping or rotors, and of course the rotation of the Earth.

3 Euler Dynamical Equation and Melnikov Function

In the above section, we reduced the system to a rigid body in torque-free rotation, but its moments of inertia vary in time, so we can discuss the system in angular momentum space, the dynamical equations are the standard Euler equations

$$
\begin{aligned}
\dot{h}_1 &= \frac{I_2^* - I_3^*}{I_2^* I_3^*} h_2 h_3 \\
\dot{h}_2 &= \frac{I_3^* - I_1^*}{I_3^* I_1^*} h_3 h_1 \\
\dot{h}_3 &= \frac{I_1^* - I_2^*}{I_1^* I_2^*} h_1 h_2
\end{aligned}
\tag{6}
$$

where the dot represents the derivative with respect to time. Because the angular momentum L of the motion on the elliptic orbit is constant, we have $\frac{d}{dt} = \frac{L}{mr^2}\frac{d}{d\vartheta}$. The elliptic radial distance is

$$
r = \frac{a\left(1-e^2\right)}{1+e\cos\vartheta}
\tag{7}
$$

where ϑ is the true anomaly, e is the orbital eccentricity, and a is the length of the semi-major axis, respectively. Substituting equation (7) into equations (6) we obtain

$$
\begin{aligned}
\frac{dh_1}{d\vartheta} &= \frac{ma^2\left(1-e^2\right)}{L}\frac{1}{(1+e\cos\vartheta)^2}\frac{I_2 - I_3}{I_2 I_3} h_2 h_3 \\
&\quad + \frac{3\varepsilon}{4G'^2 r^3}\left(1-3\cos^2\delta'\right)(I_1 - I_2)\frac{1}{a\left(1-e^2\right)}(1+e\cos\vartheta)h_2 h_3 + 0\left(\varepsilon^2\right) \\
\frac{dh_2}{d\vartheta} &= \frac{ma^2\left(1-e^2\right)}{L}\frac{1}{(1+e\cos\vartheta)^2}\frac{I_3 - I_1}{I_3 I_1} h_3 h_1 \\
&\quad + \frac{3\varepsilon}{4G'^2}\left(1-3\cos^2\delta'\right)(I_1 - I_3)\frac{1}{a\left(1-e^2\right)}(1+e\cos\vartheta)h_3 h_1 + 0\left(\varepsilon^2\right) \\
\frac{dh_3}{d\vartheta} &= \frac{ma^2\left(1-e^2\right)}{L}\frac{1}{(1+e\cos\vartheta)^2}\frac{I_1 - I_2}{I_1 I_2} h_1 h_2 \\
&\quad + \frac{3\varepsilon}{4G'^2}\left(1-3\cos^2\delta'\right)(I_2 - I_1)\frac{1}{a\left(1-e^2\right)}(1+e\cos\vartheta)h_1 h_2 + 0\left(\varepsilon^2\right)
\end{aligned}
\tag{8}
$$

We define

$$
\begin{aligned}
c &= \frac{ma^2\left(1-e^2\right)}{L} \\
c_1 &= \frac{3}{4G'^2}\left(1-3\cos^2\delta'\right)(I_3 - I_2)\frac{1}{a\left(1-e^2\right)} \\
c_2 &= \frac{3}{4G'^2}\left(1-3\cos^2\delta'\right)(I_1 - I_3)\frac{1}{a\left(1-e^2\right)} \\
c_3 &= \frac{3}{4G'^2}\left(1-3\cos^2\delta'\right)(I_2 - I_1)\frac{1}{a\left(1-e^2\right)}
\end{aligned}
$$

Expanding equations (8) into series and retaining only the first terms of eccentricity e, we obtain

$$\begin{aligned}
\frac{dh_1}{d\vartheta} &= c\frac{I_2-I_3}{I_2I_3}h_2h_3 - 2ec\frac{I_2-I_3}{I_2I_3}\cos\vartheta h_2h_3 + \varepsilon c_1(1+e\cos\vartheta)h_2h_3 + 0(e^2) + 0(\varepsilon^2)\\
\frac{dh_2}{d\vartheta} &= c\frac{I_3-I_1}{I_3I_1}h_3h_1 - 2ec\frac{I_3-I_1}{I_3I_1}\cos\vartheta h_3h_1 + \varepsilon c_2(1+e\cos\vartheta)h_3h_1 + 0(e^2) + 0(\varepsilon^2)\\
\frac{dh_3}{d\vartheta} &= c\frac{I_1-I_2}{I_1I_2}h_1h_2 - 2ec\frac{I_1-I_2}{I_1I_2}\cos\vartheta h_1h_2 + \varepsilon c_3(1+e\cos\vartheta)h_1h_2 + 0(e^2) + 0(\varepsilon^2)
\end{aligned} \tag{9}$$

Redefining $I_i\ (i=1,2,3)$ to be I_i/c, and $a_1 = \frac{I_2-I_3}{I_2I_3}, a_2 = \frac{I_3-I_1}{I_3I_1}, a_3 = \frac{I_1-I_2}{I_1I_2}$, then we can write equations (9) as

$$\begin{aligned}
\frac{dh_1}{d\vartheta} &= a_1h_2h_3 - 2ea_1\cos\vartheta h_2h_3 + \varepsilon c_1(1+e\cos\vartheta)h_2h_3 + 0(e^2) + 0(\varepsilon^2)\\
\frac{dh_2}{d\vartheta} &= a_2h_3h_1 - 2ea_2\cos\vartheta h_3h_1 + \varepsilon c_2(1+e\cos\vartheta)h_3h_1 + 0(e^2) + 0(\varepsilon^2)\\
\frac{dh_3}{d\vartheta} &= a_3h_1h_2 - 2ea_3\cos\vartheta h_1h_2 + \varepsilon c_3(1+e\cos\vartheta)h_1h_2 + 0(e^2) + 0(\varepsilon^2)
\end{aligned} \tag{10}$$

This system can be viewed as a periodically perturbed system. Considering the unperturbed system is a torque-free rigid body motion. $l^2 = h_1^2 + h_2^2 + h_3^2$ is an obvious constant of motion, which defines a sphere in angular momentum space. The flow lines are given by intersecting the ellipsoids H = constant with the sphere. For distinct moments of inertia, the flow on the sphere has saddle points at $(0, \pm l, 0)$ and centers at $(\pm l, 0, 0), (0, 0, \pm l)$. The saddles are connected by four heteroclinic orbits, and they are given by

$$\begin{aligned}
h_1 &= \pm\sqrt{\frac{a_1}{-a_2}}\mathrm{Sech}\left(-\sqrt{a_1a_3}\,l\vartheta\right)\\
h_2 &= \pm l\,\mathrm{Tanh}\left(-\sqrt{a_1a_3}\,l\vartheta\right)\\
h_3 &= \pm\sqrt{\frac{a_3}{-a_2}}\mathrm{Sech}\left(-\sqrt{a_1a_3}\,l\vartheta\right)
\end{aligned} \tag{11}$$

where we chose $I_1 < I_2 < I_3$. The Hamiltonian can now be written as

$$\begin{aligned}
H &= \frac{1}{2}\left(\frac{h_1^2}{I_1}+\frac{h_2^2}{I_2}+\frac{h_3^2}{I_3}\right) = \frac{1}{2}\left(\frac{h_1^2}{I_1}+\frac{h_2^2}{I_2}+\frac{h_3^2}{I_3}\right) + e\left[\left(a_3-\frac{\varepsilon c_3}{2}\right)\cos\vartheta - \frac{\varepsilon}{2e}c_3\right)h_1^2\\
&\quad - e\left[\left(a_1-\frac{\varepsilon c_1}{2}\right)\cos\vartheta - \frac{\varepsilon}{2e}c_1\right)h_3^2\\
&= H_0 + eH_1
\end{aligned} \tag{12}$$

where $\frac{\varepsilon}{e} = 0(1)$, ε and e are of first order infinitesimal.

To show that the perturbed system has transverse heteroclinic orbits for $e \neq 0$ and $\varepsilon \neq 0$, we need only to show that the Melnikov function

$$M(\vartheta_0) = \frac{1}{\Omega_0}\int_{-\infty}^{+\infty}\{\{H_0, H_1\}\}d\vartheta \tag{13}$$

has simple zeros.

According to Holmes (1983), the Melnikov function can be written as

$$M(\vartheta_0)=\frac{1}{\Omega_0}\int_{-\infty}^{+\infty}(-h\nabla_h H_0\times\nabla_h H_1)\,d\vartheta \tag{14}$$

where $\Omega_0=1$ $\quad \nabla_h H_0 = \left(\frac{h_1}{I_1},\frac{h_2}{I_2},\frac{h_3}{I_3}\right)$

$$\nabla_h H_1 = \left(\left[\left(a_3-\frac{\varepsilon}{2}\right)\cos(\vartheta+\vartheta_0)+-\frac{\varepsilon}{2e}c_3\right]h_1,\ 0,-\left[\left(a_1-\frac{\varepsilon c_1}{2}\right)\cos(\vartheta+\vartheta_0)-\frac{\varepsilon}{2e}c_1\right]h_3\right)$$

We obtain

$$M(\vartheta_0) = \left(\frac{(a_3-\varepsilon c_3/2)}{I_3}-\frac{(a_1-\varepsilon c_1/2)}{I_1}\right)\int_{-\infty}^{+\infty}h_1h_2h_3\cos(\vartheta+\vartheta_0)d\vartheta \tag{15}$$

Substituting equations (11) into equation (15), then we obtain

$$M(\vartheta_0) = C\left[\int_{-\infty}^{+\infty}\mathrm{Sech}^2\left(-\sqrt{a_1a_3}l\vartheta\right)\tan\left(-\sqrt{a_1a_3}l\vartheta\right)\sin\vartheta d\vartheta\right]\sin\vartheta_0 \tag{16}$$

where $C = \left(\frac{(a_3-\varepsilon c_3/2)}{I_3}-\frac{(a_1-\varepsilon c_1/2)}{I_1}\right)\frac{l^3\sqrt{a_1a_3}}{a_2}\neq 0$

Integrating equation (16), we obtain

$$M(\vartheta_0) = C\cdot\frac{\pi}{2\left(-\sqrt{a_1a_3}\,l\right)^3}\mathrm{cosech}\left(\frac{\pi}{-2\sqrt{a_1a_3}\,l}\right)\sin\vartheta_0 \tag{17}$$

which has simple zeros. Therefore the system possesses transverse heteroclinic orbits, and this implies that Smale's horseshoe exist, and chaotic motion may occur in this system.

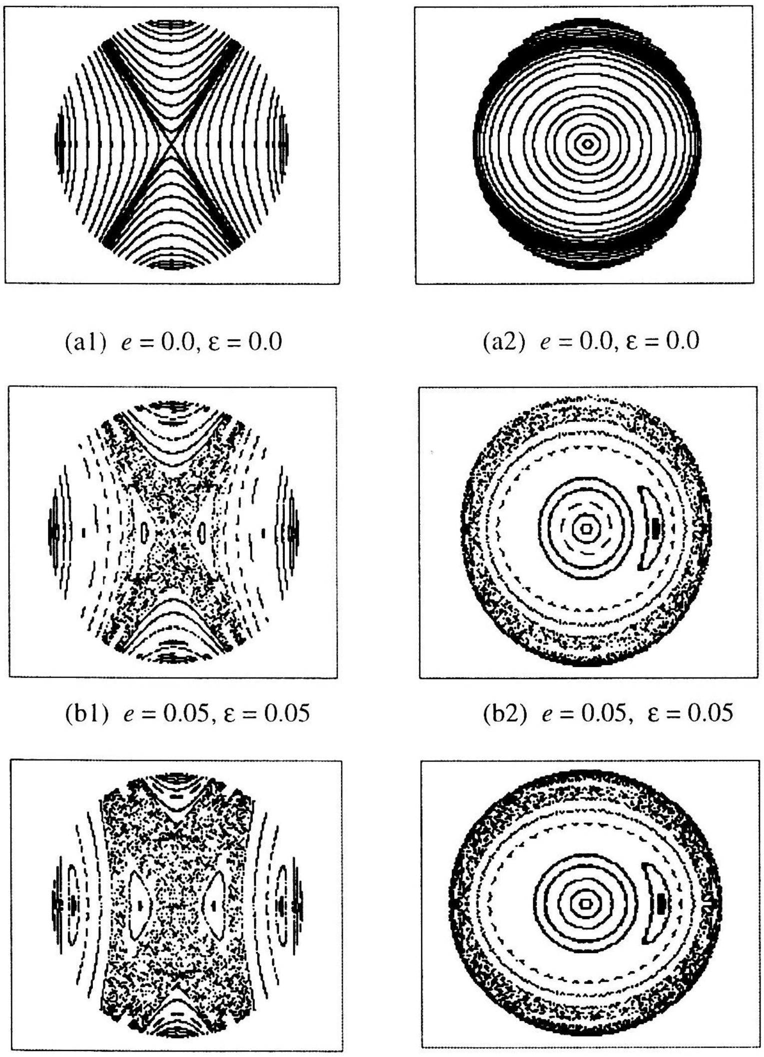
4 Poincaré Surface of Section

Only few nonlinear systems possess closed-form solution, and therefore numerical techniques play a crucial role in the process of analyzing nonlinear phenomena. Especially the Poincaré surface of section has been shown to be well suited for systems with few degrees of freedom. In what follows, equation (10) is numerically integrated for 30 different initial conditions, the Poincaré surfaces of section in the (h_1,h_3) and (h_1,h_2) plane were obtained by plotting points stroboscopically with an orbital period $\vartheta = 2\pi$. Here we let $I_1 = 1$, $I_2=1.5$, $I_3=2$, and $c_1 = c_3 = 1$. The two different types of motion, regular and chaotic, are readily distinguished on Poincaré sections, since for regular motion successive points describe smooth curves or separate points; for chaotic motion the points fill an area in an apparently random manner.

In Figure 2a/b, for fairly small eccentricity e, and small quotient ε, we see that most of the Poincaré maps are fairly well covered by invariant tori, that is, most of the periodic and quasiperiodic motion are preserved.

As we go on increasing e and ε, some tori break into chaotic trajectories in the sense that the successive points on Poincaré maps do not lie on a curve any more, but fill an area densely; other break into island chains along which there is a succession of elliptic and hyperbolic quasiperiodic orbits (see Figure 2c, d). In Figure 2a, b, c, we can also see a hyperbolic point, heteroclinic orbits, and a small region that is close to the separatrix covered by chaotic trajectories. These features corroborate the result obtained previously in the above section by means of the Melnikov theory.

If we increase e and ε somewhat (see Figure 2d), the hyperbolic point disappears, and the topologic structure of the system is changed. As e and ε are further increased, more and more of the regular motion disappears, and finally the points are mixed in a chaotic as shown in Figure 2e, f.

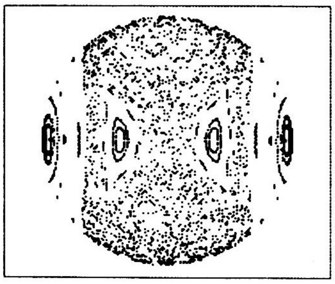


(a1) $e = 0.0, \varepsilon = 0.0$ (a2) $e = 0.0, \varepsilon = 0.0$

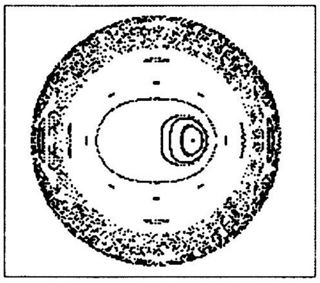
(b1) $e = 0.05, \varepsilon = 0.05$ (b2) $e = 0.05, \varepsilon = 0.05$

(c1) $e = 0.1, \varepsilon = 0.1$ (c2) $e = 0.1, \varepsilon = 0.1$

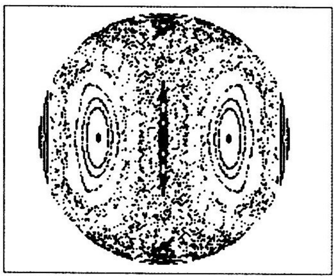
(h_1, h_3) Plane (h_2, h_3) Plane



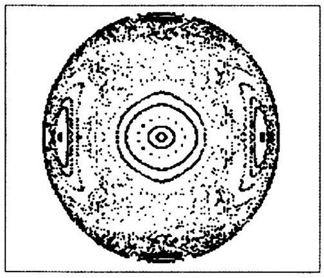
(d1) $e = 0.2, \varepsilon = 0.2$



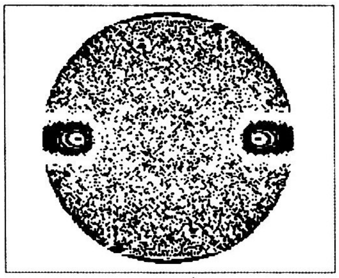
(d2) $e = 0.2, \varepsilon = 0.2$



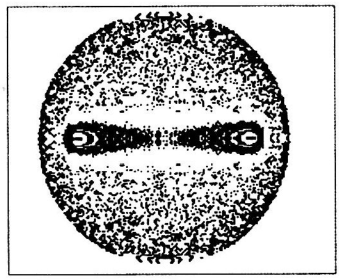
(e1) $e = 0.3, \varepsilon = 0.3$



(e2) $e = 0.3, \varepsilon = 0.3$



(f1) $e = 0.4, \varepsilon = 0.4$



(f2) $e = 0.4, \varepsilon = 0.4$

(h_1, h_3) Plane (h_2, h_3) Plane

Figure 2. Poincaré Surfaces of Section

5 Conclusions

After the system is reduced to a rigid body in torque-free motion and its moments of inertia vary in time, we take this system as an Euler-Poinsot motion perturbed by a small periodic excitation. Then we calculate the Melnikov function. It has simple zeros, this implies that Smale's horseshoe exists. Chaotic motion may occur in this system, and we can see that this chaotic motion is the consequence of the asymmetry of the rigid body and the eccentricity of the orbit. The regular motion and the chaotic motion, as well as the transition from regular motion to chaotic motion is manifested by the Poincaré surface of section.

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Address: Jianhua Peng and Yanzhu Liu, Department of Engineering Mechanics, Jiao Tong University, 200030 Shanghai, P.R. China