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Torsion of a Non-Homogeneous Bar with Periodic Parallelepiped Inclusions: Analytical Expressions for Effective Shear Modulus

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An analytical solution, describing the effective shear modulus for composite materials with periodical cylindrical inclusions of square cross section, has been obtained by asymptotic methods and Padé approximants for any values of inclusions concentration and rigidity.

1 Introduction

One of the main tasks of the theory of composite materials is a theoretical prediction of the effective properties. The subject we discuss in this paper has a long history (Christensen, 1979). Here we determine the effective rigidity q of an infinite simple lattice of identical square section cylindrical inclusions immersed in a matrix.

The calculation of q for a general type of composite was originally discussed by Maxwell, who considered each particle of the composites as an isolated dipole. The second-order approximation was due to Lord Rayleigh, who took into account particle moments up to the octupole and calculated the effective transport coefficients from a truncated system of linear algebraic equations.

Composite materials with square or rectangular fibers were studied by Kohanenko (1993), Bourgat (1979) and Bakhvalov and Panasenko (1989). In Kohanenko (1993) and Bakhvalov and Panasenko (1989) net and finite element methods were used respectively. In Bakhvalov and Panasenko (1989) the limiting case for large (close to the maximal value) rectangular cross section cylindrical cavities was studied as an asymptotic procedure, and simple analytical expressions for effective parameters were produced. It is worth noting that numerical methods in some cases give satisfying solutions, but in the general case its use is not simple.

In many papers the so-called three-phase model (TPM) is used (Christensen, 1979; Kerner, 1956; Van der Pol, 1958). Due to this approach the whole periodic structure, with the exception of one cell, is replaced by a homogenized medium with unknown characteristics. From the mathematical point of view it leads to the replacement of the periodicity conditions to the conditions of junction of the cell with the homogenized medium. Then one comes to the problem of a two-phase inclusion in the infinite domain. That allows to use the method of boundary form perturbations, replacing, as a first approximation, the contour of any inclusion by a spherical circle one.

Neither of the above mentioned methods yields accurate results for a system with nearly touching inclusions of high rigidity. In order to describe such a system an asymptotic formula has been derived in McPhedran et al. (1988). However, the validity range of this formula is not known. Moreover, there still remains a certain parameter range, which is covered neither by the asymptotic formula nor by the solutions based on the assumption of small inclusions concentration.

As have been shown in Andrianov et al. (1998) and Tokarzewski et al. (1994), two-point Padé approximants (TPPA) can be effectively used for the study of the effective properties of composite materials. This paper aims to predict the effective shear rigidity of a two-phase composite material, consisting of an infinite simple square array of identical cylindrical inclusions with square cross section, immersed in an isotropic matrix. In section 2 we describe the homogenization procedure. TPM is used for solving the so-called local problem for the case of small inclusions in section 3. A description of the asymptotic procedure for the case of large inclusions is given in section 4. In section 5 we use TPPA for obtaining analytical expressions for the effective rigidity, valid for any values of inclusion concentration and rigidity. In section 6 we present numerical results, and in section 7 - advantages and limitations of our method is briefly discussed in the light of the results from the previous sections.

2 Governing Relations and Homogenization Procedure

We study the effective shear rigidity q of an infinite simple square array of cylinders with square cross section embedded in a matrix material of unit rigidity (Figure 1).

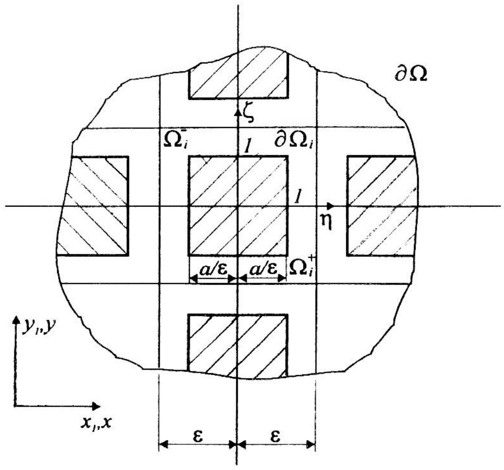

Figure 1. The Composite Material under Consideration with Distinguished Unit Cell $\Omega_i = \Omega_i^+ + \Omega_i^-$. Here a is the characteristic inclusion size (inclusions concentration $c = a^2$).

The governing relations may be written as follows (Muskhelishvili, 1966)

$$\begin{aligned} &\frac{\partial^2 U^+}{\partial x_1^2} + \frac{\partial^2 U^+}{\partial y_1^2} = f^+ \quad \text{in } \Omega^+ \\ &\lambda\left(\frac{\partial^2 U^-}{\partial x_1^2} + \frac{\partial^2 U^-}{\partial y_1^2}\right) = f^- \quad \text{in } \Omega^- \end{aligned} \tag{1}$$

$$\left[U^+ = U^-\right]\Big|_{\partial\Omega_i} \qquad \left[\frac{\partial U^+}{\partial \mathbf{n}_1} = \lambda \frac{\partial U^-}{\partial \mathbf{n}_1}\right]\Bigg|_{\partial\Omega_i} \qquad U\big|_{\partial\Omega} = F(x_1, y_1) \tag{2}$$

Here U is the function of displacements in the axial direction of the rod (U^+, U^- respectively, in the matrix (Ω^+) and in the inclusions (Ω^-)); f is the density of the mass forces, f^+, f^- respectively in the matrix and in the inclusion (we deal with the general case, naturally, our results will be correct for the case $f = 0$); λ is the inclusions rigidity; $\partial / \partial \mathbf{n}_1$ is the derivative normal to the boundary $\partial\Omega_i$ between the phases; $\partial\Omega$ is the outer boundary of the composite material. The study of such problems is important from a theoretical as well as numerical point of view. Because of the complicated structure of the multiply connected domain, any kind of calculation is difficult to perform. If we would treat the boundary value problem, then we would have to impose conditions at the boundary of the inclusions, which are large in number. Therefore, the intention is to approximate the given problem by a homogenized problem on the domain without inclusions. By the method of asymptotic development, a problem on a periodically inhomogeneous domain is reduced to solving problems in the "unit cell" and in the domain without inclusions.

The theory of homogenization has been developed by many authors (Bourgat, 1979; Bakhvalov and Panasenko, 1989), we refer to these publications for bibliographical references. The main problem in this field is the solving of the so-called local (or cell) problem. This problem has usually been treated by numerical methods. In this paper we succeed in solving the local problem by means of some approximate analytical procedures. Let us consider a unit cell of the studied periodical structure with typical size 2ε ($\varepsilon \ll 1$, *Fig.*1) and denote

$$\zeta = \frac{x_1}{\varepsilon} \qquad \eta = \frac{y_1}{\varepsilon} \tag{3}$$

Here ζ and η are the „fast" variables (Bourgat, 1979; Bakhvalov and Panasenko, 1989). For slow variables we use the following notation: $x = x_1$; $y = y_1$.

The operators $\partial / \partial x_1$ and $\partial / \partial y_1$ applied to the functions U^+, U^- become

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial \zeta} \qquad \frac{\partial}{\partial y_1} = \frac{\partial}{\partial y} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta} \qquad \frac{\partial}{\partial \mathbf{n}_1} = \frac{\partial}{\partial \mathbf{n}} + \frac{1}{\varepsilon}\frac{\partial}{\partial \mathbf{k}} \tag{4}$$

where

$$\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\cos\beta \qquad \frac{\partial}{\partial \mathbf{k}} = \frac{\partial}{\partial \zeta}\cos\alpha + \frac{\partial}{\partial \eta}\cos\beta$$

α, β are the angels between the normal vector $\mathbf{n}$ to the boundary $\partial\Omega_i$ and the co-ordinate axes. Let us represent the solution in the form of a formal expansion

$$U = U_0(x,y) + \varepsilon U_1(x,y,\zeta,\eta) + \varepsilon^2 U_2(x,y,\zeta,\eta) + \ldots \tag{5}$$

where

$$U_i(x,y,\zeta+2,\eta+2) = U_i(x,y,\zeta,\eta) \qquad i = 1,2,3\ldots \tag{6}$$

In accordance with multiscale method (Bakhvalov and Panasenko, 1989) we consider formally functions of two variables $U_i(x,y)$ as functions of four variables $U_i(x,y,\zeta(x,y),\eta(x,y))$, $i = 1,2...$
Substituting series (5) into the boundary value problem (1), (2), taking into account relations (4) and splitting it with respect to the powers of ε, one obtains the recurrent system of boundary value problems

$$\frac{\partial^2 U_1^\pm}{\partial \zeta^2} + \frac{\partial^2 U_1^\pm}{\partial \eta^2} = 0 \quad \text{in} \quad \Omega^\pm \tag{7}$$

$$\left[U_1^+ = U_1^-\right]\Big|_{\partial\Omega_i} \qquad \left[\frac{\partial U_1^+}{\partial \mathrm{k}} - \lambda\frac{\partial U_1^-}{\partial \mathrm{k}} = \frac{\partial U_0}{\partial \mathrm{n}}(\lambda - 1)\right]\Bigg|_{\partial\Omega_i} \tag{8}$$

$$L_1\left(U_0, U_1^+, U_2^+\right) \equiv \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} + 2\left(\frac{\partial^2 U_1^+}{\partial x \partial \zeta} + \frac{\partial^2 U_1^+}{\partial y \partial \eta}\right) + \frac{\partial^2 U_2^+}{\partial \zeta^2} + \frac{\partial^2 U_2^+}{\partial \eta^2} - f^+ = 0 \quad \text{in} \quad \Omega^+$$

$$L_2\left(U_0, U_1^-, U_2^-\right) \equiv \lambda\left(\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} + 2\left(\frac{\partial^2 U_1^-}{\partial x \partial \zeta} + \frac{\partial^2 U_1^-}{\partial y \partial \eta}\right) + \frac{\partial^2 U_2^-}{\partial \zeta^2} + \frac{\partial^2 U_2^-}{\partial \eta^2}\right) - f^- = 0 \quad \text{in} \quad \Omega^- \tag{9}$$

$$\left[U_2^+ = U_2^-\right]\Big|_{\partial\Omega_i} \qquad \left[\frac{\partial U_1^+}{\partial \mathbf{n}} + \frac{\partial U_2^+}{\partial \mathbf{k}} = \lambda\frac{\partial U_1^-}{\partial \mathbf{n}} + \lambda\frac{\partial U_2^-}{\partial \mathbf{k}}\right]\Bigg|_{\partial\Omega_i} \qquad U_0\big|_{\partial\Omega} = F(x,y) \tag{10}$$

$$\vdots$$

The equations (7) with the corresponding boundary conditions (8) represent the local boundary value problem. For solving it in the case of small inclusions (inclusions typical size a tends to zero) we use TPM, and in the case of large inclusions (a tends to the unity) we use a singular perturbation approach.

3 Solution of the Local Problem for Small Inclusions

For solving the local problem in the case of small inclusions $(a \to 0)$ we use TPM (Christensen, (1979); Kerner, (1956); Van der Pol, (1958). We replace all periodic structures with the exception of one cell by a homogenized medium $\tilde{\Omega}$ with unknown rigidity $\tilde{\lambda} = q$. That leads to the problem of a two-phase inclusion in an infinite domain (Figure 2)

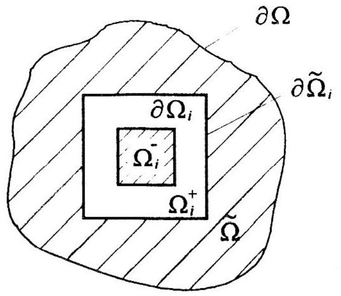

Figure 2. Three-phase Model as Applied to the Composite Material under Consideration in Case of Small Inclusions ($a \to 0$).

Using the method of boundary form perturbations, in the first approximation we replace square contours by circle ones, and we introduce polar coordinates for the cell problem $(\zeta, \eta \to r, \theta)$. Then the functions $U_i(x, y, \zeta, \eta)$ transform into $U_i(x, y, r, \theta)$, $i = 1,2,...$. So in polar coordinates the local boundary value problem can be written as follows:

$$\frac{\partial^2 U_1^{\pm}}{\partial r^2} + \frac{1}{r}\frac{\partial U_1^{\pm}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U_1^{\pm}}{\partial \theta^2} = 0 \quad \text{in} \quad \Omega^{\pm}$$
$$\frac{\partial^2 \tilde{U}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{U}_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{U}_1}{\partial \theta^2} = 0 \quad \text{in} \quad \tilde{\Omega} \tag{11}$$

$$\left[U_1^-\right]\Big|_{r=\frac{2a}{\sqrt{\pi}}} \qquad \left[\frac{\partial U_1^+}{\partial r} - \lambda\frac{\partial U_1^-}{\partial r} = (\lambda - 1)\left(\frac{\partial U_0}{\partial x}\cos\theta + \frac{\partial U_0}{\partial y}\sin\theta\right)\right]\Bigg|_{r=\frac{2a}{\sqrt{\pi}}}$$
$$\left[U_1^+\right]\Big|_{r=\frac{2a}{\sqrt{\pi}}} \qquad \left[\frac{\partial U_1^+}{\partial r} - \tilde{\lambda}\frac{\partial U_1^-}{\partial r} = (\tilde{\lambda} - 1)\left(\frac{\partial U_0}{\partial x}\cos\theta + \frac{\partial U_0}{\partial y}\sin\theta\right)\right]\Bigg|_{r=\frac{2a}{\sqrt{\pi}}} \tag{12}$$
$$\tilde{U}_1\Big|_{r\to\infty} \to 0 \qquad \frac{\partial \tilde{U}_1}{\partial r}\Big|_{r\to\infty} \to 0$$

Solution of equations (11) and (12) can be written in the form:

$$U_1^- = A_1 r\cos\theta + A_2 r\sin\theta$$
$$U_1^+ = (B_1 r + C_1 / r)\cos\theta + (B_2 r + C_2 / r)\sin\theta \tag{13}$$
$$\tilde{U}_1 = D_1 / r\cos\theta + D_2 / r\sin\theta$$

where

$$A_1 = -\left(1+4\tilde{\lambda}T_1\right)\frac{\partial U_0}{\partial x} \qquad B_1 = \left(1+2(\lambda+1)\tilde{\lambda}T_1\right)\frac{\partial U_0}{\partial x}$$

$$C_1 = \frac{8}{\pi}a^2(\lambda-1)\tilde{\lambda}T_1\frac{\partial U_0}{\partial x} \qquad D_1 = \frac{4}{\pi}\left(1+2\left(a^2(\lambda-1)+\lambda+1\right)T_1\right)\frac{\partial U_0}{\partial x}$$

$$T_1\left(\left(\tilde{\lambda}-1\right)(\lambda-1)a^2-\left(\tilde{\lambda}+1\right)(\lambda+1)\right)^{-1} \qquad A_2 = A_1 \qquad B_2 = B_1 \quad C_2 = C_1 \quad D_2 = D_1$$

$$\frac{\partial U_0}{\partial x} \leftrightarrow \frac{\partial U_0}{\partial y}$$

For determination of the effective rigidity $\tilde{\lambda}$ of the homogenized medium we substitute the derived expressions (13) into the boundary value problem (9)

$$\iint_{\Omega_i^+} L_1\left(U_0, U_1^+, U_2^+\right)d\zeta\, d\eta + \iint_{\Omega_i^-} L_1\left(U_0, U_1^-, U_2^-\right)d\zeta\, d\eta = 0 \quad \text{in} \quad \Omega$$

Then the unknown parameter $\tilde{\lambda}$ may be obtained from the linear algebraic equation as follows:

$$q = \tilde{\lambda} = \frac{\lambda\left(1+a^2\right)+1-a^2}{\lambda(1-a)+1+a^2} \qquad (14)$$

Expression (13) has been obtained under the assumption of small inclusions. However, it qualitatively represents the behaviour of the effective rigidity in the case of large inclusions too.

4 Solution of the Cell Problem for Large Inclusions

When inclusions are large $(a \to 1)$, one can not use the approach of TPM, but smallness of the parameter thickness of the wall between two neighbouring inclusions may be taking into account. Then we could construct an asymptotic solution using a singular perturbation technique.

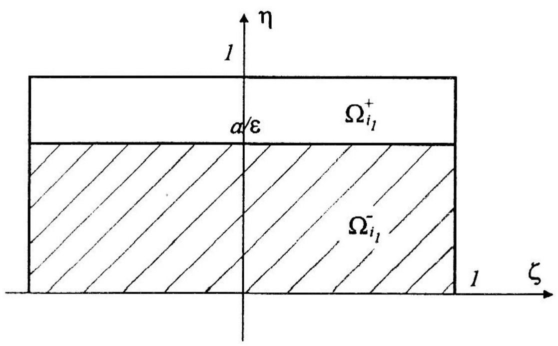

Figure 3. Model of the Composite Material under Consideration in the Case of Large Inclusions ($a \to 1$).

Due to symmetry we can consider each strip (see Figure 3) separately and obtain solution only for that one. For instance for Ω_i^+ it can easily be shown that

$$\frac{\partial^2 U_1^+}{\partial \zeta^2} << \frac{\partial^2 U_1^+}{\partial \eta^2} \qquad (15)$$

and the local boundary value problem can be written in the following form:

$$\frac{\partial^2 U_1^+}{\partial \eta^2} = 0 \tag{16}$$

$$\frac{\partial^2 U_1^-}{\partial \zeta^2} + \frac{\partial^2 U_1^-}{\partial \eta^2} = 0 \tag{17}$$

$$\left[U_1^+ = U_1^-\right]\Big|_{\eta = a/\varepsilon} \qquad \left[\frac{\partial U_1^+}{\partial \eta} - \lambda \frac{\partial U_1^-}{\partial \eta} = \frac{\partial U_0}{\partial y}(\lambda - 1)\right]\Bigg|_{\eta = a/\varepsilon} \tag{18}$$

The condition of periodic continuity is:

$$U_1^+\Big|_{\eta = a/\varepsilon} = 0 \tag{19}$$

Solution of the boundary value problem (16) to (19) is represented as follows:

$$\begin{aligned} U_1^+ &= E_1 + F_1 \eta \\ U_1^- &= G_1 \eta \end{aligned} \tag{20}$$

where

$$E_1 = (1 - \lambda T_2)\frac{\partial U_0}{\partial y} \qquad F_1 = -(1 - \lambda T_2)\frac{\partial U_0}{\partial y} \qquad G_1 = -(1 - T_2)\frac{\partial U_0}{\partial y} \qquad T_2 = (a + \lambda(1 - a))^{-1}$$

After substituting the derived expressions (20) into the boundary value problem (9) and doing homogenisation (see above) we obtain:

$$q = \frac{\lambda(1 - a^2 + a^3) + a^2(1 - a)}{\lambda(1 - a) + a} \tag{21}$$

Formula (21) has been obtained under assumption of large inclusions. However, it qualitatively represents the behaviour of the effective rigidity in the case of small inclusions too.

5 Evaluating of a Two-point Padé Approximant

The notion of TPPA is defined by Baker and Graves-Morris (1995). Let be

$$\varphi(\nu) \cong \sum_{i=0}^{\infty} a_i \nu^i \qquad \text{when} \quad \nu \to 0 \tag{22}$$

$$\varphi(\nu) \cong \sum_{i=0}^{\infty} b_i \nu^{-i} \qquad \text{when} \quad \nu \to \infty \tag{23}$$

Then TPPA are represented by the function

$$\varphi(\nu) = \frac{\sum_{i=0}^{m} c_i \nu^i}{\sum_{j=0}^{n} d_j \nu^j} \tag{24}$$

in which the k_1 and k_2 $(k_1 + k_2 = m + n + 1)$ coefficients of expansions in the Taylor series when $\nu \to 0$ and Laurent series for $\nu \to \infty$ coincide with the corresponding coefficients of the series (22) and (23) respectively.

Here TPPA allows us to obtain an approximate analytical expression of q, valid for all values of inclusion concentrations $c \in [0;1] \left(c = a^2\right)$ and rigidities $\lambda \in [0;\infty)$. For $a \to 0$ we use three first coefficients $\left(k_1 = 3\right)$ of an expansion of the formula (14) in terms of $a/(1-a)$, and for $a \to \infty$ we use two first $\left(k_2 = 2\right)$ coefficients of an expansion of the formula (21) in terms of $(1-a)/a$. The derived two-point Padé approximant (for $m = n = 2$) is:

$$q = \frac{\lambda + 1 + c(\lambda - 1)}{\lambda + 1 - c(\lambda - 1)} \qquad (25)$$

The obtained approximate analytical solution (25) is in good agreement with known numerical data (see section 6). Let us point out that the analytical result of our method (25) coincides with the solution for small inclusions $(a \to 0)$ with expression (14), obtained by means of TPM. This fact shows that TPM allows to achieve a very accurate approximation of q in the case under consideration. However, it would not be so in other problems, which involve calculating different effective coefficients or different geometry of composite materials.

6 Numerical Results

The dependence of the effective rigidity q on the inclusions concentration c and rigidity λ, accordingly to the obtained analytical solution (25), is shown in Figure 4.

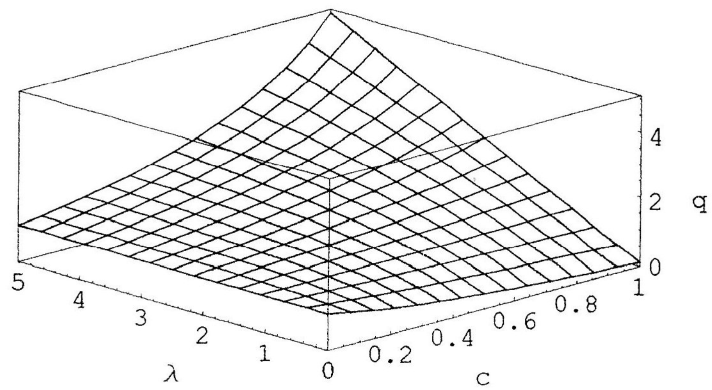

Figure 4. Effective Rigidity q as a Function of Inclusions Concentration $c = a^2$ and Rigidity λ, Accordingy to the Analytical Solution (25)

In Figure 5 formula (25) is compared with Bourgat's numerical results for c = 1/9 (Bourgat, 1979).

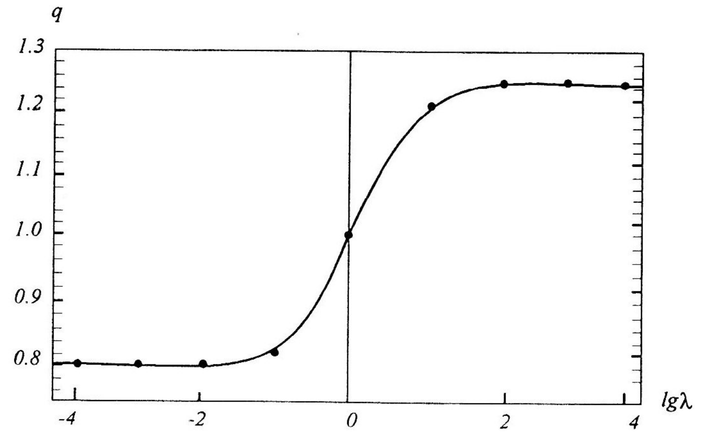

Figure 5. The Analytical Solution (25) (Solid Curve) is Compared with Bourgat's Numerical Results for c = 1/9 (Bourgat, 1979), (Black Points).

7 Concluding Remarks

In this paper we have developed an approximate analytical representation of the effective rigidity q for a periodic array of cylindrical inclusions with square cross section, immersed in a matrix material. Homogenization procedure, three-phase model and some asymptotic approximations have allowed us to obtain solutions for small and large inclusions separately. Two-point Padé approximants have been used to derive the uniform analytical expression of q, valid for all values of inclusions concentration and rigidity. The obtained results are in good agreement with known numerical data.

Our method may be effectively used for calculating other effective coefficients of composite materials with periodic structures. On the other hand, one of the important problems of our procedure is to control the accuracy of the realised approximation. In some cases numerical methods or experimental results can be used for that purpose.

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