TECHNISCHE MECHANIK, Band 19, Heft 1, (1999), 45-48
Manuskripteingang: 05. Mai 1998

About a Torsional Problem for the Orthotropic Non-Homogeneous Rod of Rectangular Cross-Section

A. M. Eishinskii, V. J. Adlucky, E. Ts. Tsadikova

The paper deals with an analytical solution of the torsional problem for the orthotropic non-homogeneous rod of rectangular cross-section with two shear moduli which are functions of the cross-sectional height coordinate. Computational results are presented for the torsional rigidity, maximum shear stresses and their locations for various values of the problem parameters.

The solution of a torsional problem for the orthotropic non-homogeneous rod of rectangular cross section (Figure 1) with variable shear moduli under the action of a torsional moment M is considered. It is supposed that the planes of elastic symmetry are parallel to the sides of the rod, and two shear moduli G_1 and G_2 are given in the following form

$$G_1(y) = \beta^{-2} G_2(y) \qquad G_2(y) = g\left(e^{-\alpha y} + Ce^{\alpha y}\right)^{-2} \tag{1}$$

where β, g, α, C are constants. Giving various values to the parameter C it is possible to simulate both hardening and unhardening of various zones of the rod material. When $|C| > e^{2\alpha b}$ and $|C| < e^{-2\alpha b}$ the monotonic variation of moduli takes place.

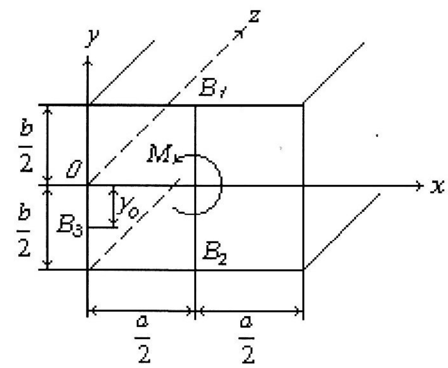

Figure 1. Non-Homogeneous Rod of Rectangular Cross-Section

Hereafter for definiteness we assume that $\alpha = 1/b$. It should be noted that the case $C=1$ was considered by Eishinskii et al. (1973).
Plots of the value $G_2(y)$ depending on C are presented in Figure 2.

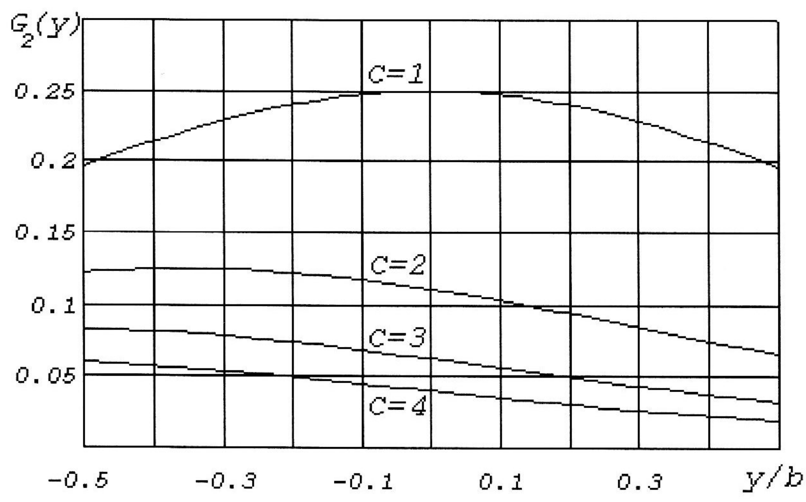

Figure 2. Plots of the Value $G_2(y)$ Depending on C

The equation for the stresses function $\psi(x,y)$ has the form (see Lechnicky, 1972)

$$\frac{\partial}{\partial x}\left[\frac{1}{G_1(y)}\frac{\partial\psi}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{1}{G_2(y)}\frac{\partial\psi}{\partial y}\right]=-2 \qquad (2)$$

Solving equation (2), in view of relations (1), by the method of expansion in Fourier series, we obtain

$$\psi(x,y)=\sum_{k=1,3,5,\ldots}^{\infty}\sin\frac{k\pi x}{a}Y_k(y) \qquad (3)$$

where

$$Y_k(y)=\frac{4gb^2}{\pi k s_k\left(e^{-\frac{y}{b}}+Ce^{\frac{y}{b}}\right)}\left\{\frac{e^{-s_k\frac{y}{b}}}{e^{-s_k}-e^{s_k}}\left[\int_0^{0,5}\frac{e^{-s_k t}dt}{e^{-t}+Ce^{t}}-e^{-s_k}\int_0^{0,5}\frac{e^{s_k t}dt}{e^{-t}+Ce^{t}}-\int_0^{-0,5}\frac{e^{-s_k t}dt}{e^{-t}+Ce^{t}}+\right.\right.$$

$$\left.+e^{s_k}\int_0^{-0,5}\frac{e^{s_k t}dt}{e^{-t}+Ce^{t}}\right]+\frac{e^{s_k\frac{y}{b}}}{e^{-s_k}-e^{s_k}}\left[e^{-s_k}\int_0^{-0,5}\frac{e^{-s_k t}dt}{e^{-t}+Ce^{t}}-\int_0^{-0,5}\frac{e^{s_k t}dt}{e^{-t}+Ce^{t}}-\right.$$

$$\left.\left.-e^{s_k}\int_0^{0,5}\frac{e^{-s_k t}dt}{e^{-t}+Ce^{t}}+\int_0^{0,5}\frac{e^{s_k t}dt}{e^{-t}+Ce^{t}}\right]+e^{-s_k\frac{y}{b}}\int_0^{\frac{y}{b}}\frac{e^{s_k t}dt}{e^{-t}+Ce^{t}}-e^{s_k\frac{y}{b}}\int_0^{\frac{y}{b}}\frac{e^{-s_k t}dt}{e^{-t}+Ce^{t}}\right\}$$

$$s_k=\sqrt{\left(\frac{k\pi}{d}\right)^2+1}$$

$$d=\frac{a}{b\beta}$$

The series (3) is absolutely convergent because the remainder after the k-th term is $O(k^{-3})$. Its first derivatives with respect to x and y are absolutely convergent too. In case of $C=0$ solution (3) coincides with the well-known Lechnicky's solution (Lechnicky, 1971).

The stress state of the rod can be determined as follows

$$\tau_{xz}=\vartheta\frac{\partial\psi}{\partial y} \qquad \tau_{yz}=-\vartheta\frac{\partial\psi}{\partial x}$$

where

$\vartheta=\frac{M}{D}$ is the torsion angle per unit length;

$D=2\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_0^a\psi(x,y)dxdy=gab^3B$ is the torsional rigidity.

The values of the factor B are plotted in Figure 3 for various values of C and d.

Maximum shear stresses are determined using the following formulas:

$$\text{At point } B_1\left(\frac{a}{2},\frac{b}{2}\right): \quad \tau_{max} = \frac{M}{ab^2}k_1(B_1)$$

$$\text{At point } B_2\left(\frac{a}{2},-\frac{b}{2}\right): \quad \tau_{max} = \frac{M}{ab^2}k_1(B_2)$$

$$\text{At point } B_3\left(0,y_0\right): \quad \tau_{max} = \frac{M}{ab^2\beta}k_2(B_3)$$

where

$$k_1(B_1) = \sum_{k=1,3,5,...}^{\infty}(-1)^{\frac{k-1}{2}}Y_k'\left(\frac{b}{2}\right)$$

$$k_1(B_2) = \sum_{k=1,3,5,...}^{\infty}(-1)^{\frac{k-1}{2}}Y_k'\left(-\frac{b}{2}\right)$$

$$k_2(B_3) = \sum_{k=1,3,5,...}^{\infty}kY_k\left(y_0\right) \qquad \sum_{k=1,3,5,...}^{\infty}kY_k'\left(y_0\right) = 0$$

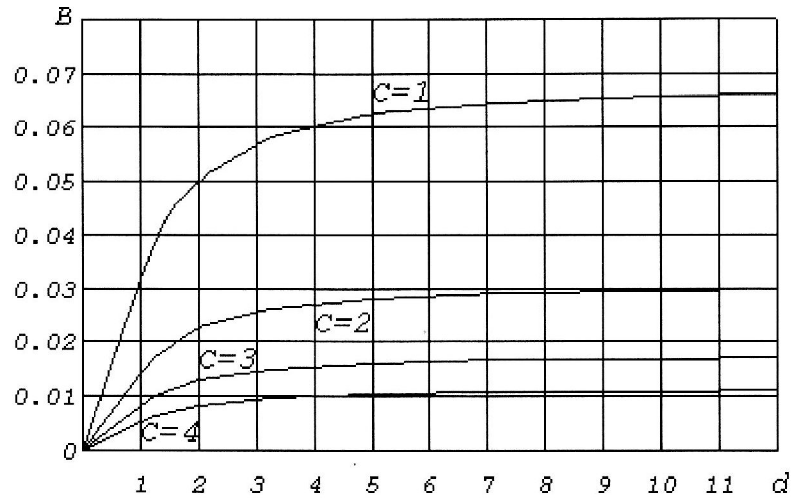

Figure 3. The Values of the Factor B Depending on C and d

The values of $k_1(B_1)$, $k_1(B_2)$ depending on C and d are presented in Figure 4. It should be noted that the quantity $k_2(B_3)$ has only a weak dependency on C, therefore its values are presented not in a graphical but in a tabulated form (see Table 1).

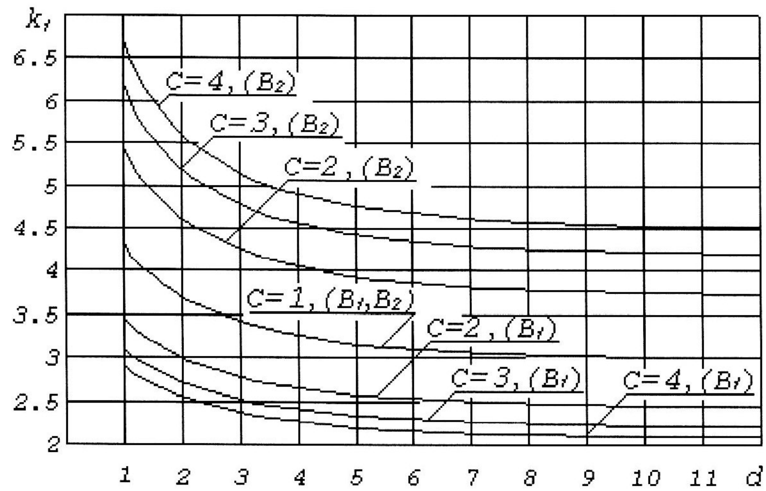

Figure 4. The Values of $k_1(B_1)$, $k_1(B_2)$ Depending on C and d

The values of y_0/b are presented in Figure 5. It is evident that for C=1, y_0=0.

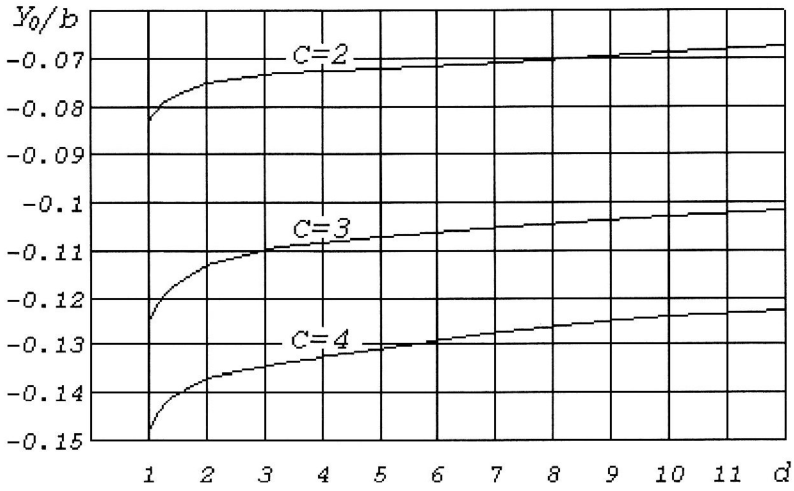

Figure 5. The Values of y_0/b Depending on C and d

d	C=2	C=3	C=4
1.00	4.87	4.95	5.01
1.25	4.17	4.23	4.32
2.00	3.25	3.30	3.33
3.25	2.75	2.78	2.84
5.00	2.51	2.54	2.57
7.25	2.37	2.40	2.43
10.00	2.27	2.30	2.32
13.25	2.19	2.22	2.24

Table 1. The Values of $k_2(B_3)$ Depending on C and d

It follows from the data above that the material non-homogeneity has essential influence on the distribution of the stresses in the rod.

Example:
Let us assume that $\beta = 1, a/b = 2, C = 3$. Then d=2. With the results from Figures 4,5 and Table 1 we can obtain that

$$\tau_{max}(B_1) = 2.70\,\frac{M}{ab^2},\ \tau_{max}(B_2) = 5.30\,\frac{M}{ab^2},\ \tau_{max}(B_3) = 3.00\,\frac{M}{ab^2},\ y_0 = -0.113\ b.$$

Literature

1. Eishinskii, A. M.; Adlucky, V. J.; Tsadikova E. T.; et al.: Torsion of the non-homogeneous rods with cross sections in the form of a rectangular and part of a circular ring (Russian). Modern problems of machinery industry and details of machines. Dnepropetrovsk, (1973), 132-149.

2. Lechnicky, S. G.: Torsion of the non-isotropic and non-homogeneous bars (Russian). Moscow, Nauka, (1971), 240 p.

Addresses: Alexander M. Eishinskii, Research Fellow, Mechanical Department, National Mining Academy of Ukraine, Serova 3, apt 7, UA-320000 Dnepropetrovsk, Ukraine. Ph. D. Victor J. Adlucky, Department of Applied Mathematics, Dnepropetrovsk State University, Zaporozhskoe shosse 4, apt 66, UA-320107 Dnepropetrovsk, Ukraine. Th. D. Elena Ts. Tsadikova. Lochnerstr, 41, D- 90441 Nürnberg, Germany.