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Theoretical and Numerical Studies of the Shell Equations of Bauer, Reiss and Keller, Part I: Mathematical Theory

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We study the solution field $\mathcal{M}$ of a parameter dependent nonlinear two-point boundary value problem suggested by Bauer et al. (1970). This problem describes the buckling of a thin-walled spherical shell under a uniform axisymmetric external static pressure. The boundary value problem is formulated as an abstract operator equation $T(x, \lambda) = 0$ in appropriate Banach spaces. By exploiting the equivariance of T we obtain detailed information about the structure of $\mathcal{M}$. In Part II of this work, the theoretical results are used to efficiently compute interesting parts of $\mathcal{M}$ with numerical standard techniques.

1 Introduction

Let us consider the following parameter-dependent nonlinear two-point boundary value problem (BVP) which describes the buckling behavior of a thin-walled spherical shell under a uniform axisymmetric external static pressure

$$\begin{aligned} y_1'(t) &= (\nu - 1)\cot(t) y_1(t) + y_2(t) + [k\cot^2(t) - \lambda] y_4(t) + \cot(t) y_2(t) y_4(t) \\ y_2'(t) &= y_3(t) \\ y_3'(t) &= [\cot^2(t) - \nu] y_2(t) - \cot(t) y_3(t) - y_4(t) - 0.5\cot(t) y_4^2(t) \\ y_4'(t) &= \beta\, y_1(t) - \nu\cot(t) y_4(t) \qquad\qquad 0 < t < \pi \end{aligned} \tag{1}$$

$$y_2(0) = y_4(0) = y_2(\pi) = y_4(\pi) = 0 \tag{2}$$

where $y_1(t) = m(t)$, $y_2(t) = q(t)$, $y_3(t) = s(t)$ are proportional to the radial bending moment, the transversal shear and the circumferential membrane stress, respectively. The component $y_4(t)$ is proportional to the angle of rotation of a tangent to a meridian and ν is Poisson's ratio. Let the radii of the inner and outer surface of the spherical shell be given by $r = R \mp h$, where R is the radius of the midsurface of the shell and $2h$ is the uniform thickness. The parameter λ and the constants k, β are defined by

$$\lambda \equiv \frac{pR}{4Eh} \qquad k \equiv \frac{1}{3}\left(\frac{h}{R}\right)^2 \qquad \beta \equiv \frac{1-\nu^2}{k}$$

where E is Young's modulus and p is a uniform compressive load. In the sequel we refer to λ as the load.

The above BVP was at first treated by Bauer, Reiss and Keller (1970). Since in the subsequent years the theory of numerical methods for bifurcation problems was coming on, Hermann, Ullmann and Ullrich (1991) consider equation (1) from the point of view of modern bifurcation theory. However, the work of these authors is restricted to the buckling of a hemisphere, i.e., they use the following boundary conditions instead of equation (2):

$$y_2(0) = y_4(0) = y_2(\pi/2) = y_4(\pi/2) = 0 \tag{3}$$

In the present paper we continue the investigations of Hermann et al. (1991) for the more general problem of a full sphere, equations (1) and (2). In particular, the symmetry properties of the problem and of its solutions are examined. During the presence of such symmetries the amount of numerical work can be decreased considerably. But the concept of a symmetric solution given in the paper of Bauer et al. (1970) must be modified since the only solution of equations (1) and (2) which satisfies this criterion is the trivial one.

For our further considerations it is useful to transform equations (1) and (2) into a second order BVP

$$\begin{aligned} x_1''(t) &= -\cot(t)x_1'(t) + [\cot^2(t) - \nu]\, x_1(t) - x_2(t) - 0.5\cot(t)x_2^2(t) \qquad 0 < t < \pi \\ x_2''(t) &= -\cot(t)x_2'(t) + [\cot^2(t) + \nu]\, x_2(t) + \beta\, [x_1(t) - \lambda x_2(t) + \cot(t)x_1(t)x_2(t)] \end{aligned} \tag{4}$$

$$x_1(0) = x_2(0) = x_1(\pi) = x_2(\pi) = 0 \tag{5}$$

It is easy to show that equations (1) and (2) and equations (4) and (5) are equivalent. The solutions of both problems are correlated by $x_1(t) = y_2(t)$, $x_2(t) = y_4(t)$, $y_1(t) = [y_4'(t) + \nu\cot(t)y_4(t)]/\beta$, $y_3(t) = y_2'(t)$.

To simplify the representation we write equation (4) in vector notation

$$x''(t) = f(t, x(t); \lambda) \qquad 0 < t < \pi \tag{6}$$

Problem (6),(5) contains the (control) parameter λ which may vary over a certain specified interval $I_\lambda \subset \mathbb{R}$. Our goal is to obtain a global bifurcation picture of this BVP. In other words, we want to determine numerically a part of the solution manifold

$$\mathcal{M} \equiv \{(x, \lambda) :\ \lambda \in \mathbb{R},\ x = x(t; \lambda) \text{ solution of equations (6) and (5)}\} \tag{7}$$

In the following $\mathcal{M}$ is also called solution field of the parameter dependent problem. Obviously, the curve of trivial solutions $\mathcal{C}_{triv} \equiv \{(0, \lambda) :\ \lambda \in \mathbb{R}\}$ is a subset of $\mathcal{M}$.

In addition to the dependence of the differential equations on the parameter λ, problem (1),(2) (or (4),(5) respectively) shows a further difficulty. The right hand sides of the differential equations have singularities at $t = 0$ and $t = \pi$ which are caused by the trigonometric function $\cot(t)$. For instance, in the neighbourhood of $t = 0$ we have $\cot(t) = t^{-1} - \frac{1}{3}t - \frac{1}{45}t^3 + \ldots$. Therefore, special techniques must be applied to eliminate the so-called regular singularity.

The organization of the paper is as follows: In Section 2, the shell equations (6),(5) are transformed into an abstract operator equation $T(x, \lambda) = 0$. On the basis of this operator equation fundamental properties of the above model are discussed. In Section 3 we develop an explicit formula for the bifurcation points on $\mathcal{C}_{triv}$, i.e., for those points where curves of nontrivial solutions branch off. We show that there are only simple and double bifurcation points. However, double bifurcation points occur very rarely. In Section 4, the consequences of the equivariance of the operator T for the symmetry properties of the solutions (x, λ) are described. In Part II of this paper we show how these theoretical results and appropriate numerical methods can be applied to compute interesting parts of the solution field $\mathcal{M}$ of the shell equations. Moreover, in the second part some interesting pictures of deformed shells are presented.

For the theoretical studies in Part I of this paper we assume that $\nu \in [0.2, 0.5]$ and $k > 0$. The numerical results in Part II are based on the values $\nu = 0.32$ (steel) and $k = 10^{-5}$ (thin shell).

In a forthcoming paper we will show that the analytical and numerical techniques presented can also be applied to shell models with a higher degree of nonlinearity, e.g. to the shell equations of Troger and Steindl (1991), pp. 356–365.

2 Operator Form of the Shell Equations

Since we want to use the results of modern bifurcation theory (see e.g. Chow and Hale (1982), Golubitsky and Schaeffer (1984), Golubitsky et al. (1988), Kuznetsov (1995)), the BVP (6),(5) has to be reformulated as an equivalent operator equation. In the definitions of the operator and the corresponding abstract function spaces the (regular) singularities in the right hand sides of the differential equations must be taken into account.

Let $(C_0^k, \|\cdot\|)$ be the real Banach space of all k-times continuously differentiable vector functions $x \equiv (x_1, x_2) : (0, \pi) \to \mathbb{R}^2$ for which the limits $\lim_{t \to 0+} x_i^{(j)}(t)$, $\lim_{t \to \pi-} x_i^{(j)}(t)$, $i = 1, 2$, $j = 0, \ldots, k$, exist. The associated norm is $\|x\| \equiv \max_{\substack{i=1,2 \\ j=0,\ldots,k}} \sup_{t \in (0,\pi)} |x_i^{(j)}(t)|$.

We can now define the operator $T : X \times \mathbb{R} \to Y$ by

$$[T(x,\lambda)](t) \equiv x''(t) - f(t,x(t);\lambda) \qquad 0 < t < \pi \tag{8}$$

where $X \equiv \{x \in C_0^2 : \lim_{t\to 0+} x(t) = \lim_{t\to\pi-} x(t) = 0\}$ and $Y \equiv C_0^0$. The problem of solving the BVP (6),(5) is eqiuvalent to solving the operator equation

$$T(x,\lambda) = 0 \tag{9}$$

Looking for symmetries of the operator T we find that T is $\mathbb{Z}_2$-equivariant, i.e.,

$$T(S_X x, \lambda) = S_Y T(x,\lambda) \qquad \forall\,(x,\lambda) \in X \times \mathbb{R} \tag{10}$$

where the linear continuous involutions $S_X : X \to X$, $S_Y : Y \to Y$ are given by

$$(S_X x)(t) \equiv -x(\pi - t) \qquad (S_Y y)(t) \equiv -y(\pi - t) \qquad \forall t \in (0,\pi) \tag{11}$$

With the subspaces of symmetric elements (odd functions) and antisymmetric elements (even functions) of X, namely $X_s \equiv \{x_s \in X : S_X x_s = x_s\}$ and $X_a \equiv \{x_a \in X : S_X x_a = -x_a\}$ respectively, we write X as the direct sum $X = X_s \oplus X_a$. Similarly Y can be decomposed as $Y = Y_s \oplus Y_a$.

Let $x_s, \varphi_s \in X_s$ and $\varphi_a \in X_a$. Then the equivariance of T implies

$$T(x_s,\lambda) \in Y_s \qquad T_x(x_s,\lambda)\varphi_s \in Y_s \qquad T_x(x_s,\lambda)\varphi_a \in Y_a \qquad T_\lambda(x_s,\lambda) \in Y_s$$

We now show that the $\mathbb{Z}_2$-equivariance of T allows us to reduce the amount of computational work when solving equation (9). In the following we call a solution $z = (x,\lambda) \in X \times \mathbb{R}$ of equation (9) symmetric, antisymmetric or nonsymmetric if x is an element of X_s, X_a or $X\backslash(X_s \cup X_a)$ respectively. The symmetric solutions of equation (9) are the solutions of

$$T|_{X_s\times\mathbb{R}}(x_s,\lambda) = 0 \tag{12}$$

This equation represents a BVP which consists of the differential equations (6), where the interval is reduced to $(0,\pi/2]$, and the boundary conditions $x(0) = x(\pi/2) = 0$. As will be demonstrated in Part II of this paper, the restriction on the subspace $X_s \times \mathbb{R}$ reduces by one half the number of the differential equations to be integrated. Moreover, equation (10) implies that each solution (x,λ) of equation (9) satisfies also $T(S_X x, \lambda) = 0$. In other words, the nonsymmetric solutions of equation (9) occur in pairs (x,λ) and $(S_X x,\lambda)$, where $x \neq S_X x$. Thus equation (9) must be solved only once to obtain both solutions. One of the basic themes in the classical bifurcation theory is the study of the so-called singular points $z_0 \equiv (x_0,\lambda_0) \in X \times \mathbb{R}$ which are characterized by $T(z_0) = 0$, $\dim \mathcal{N}(T_x(z_0)) \geq 1$, where $T_x(z) : X \to Y$ is the partial Fréchet derivative of T with respect to x and $\mathcal{N}$ denotes the null space of the operator. Using equation (8) we compute

$$\begin{aligned}
[T_x(z)\varphi]_1(t) &= \varphi_1''(t) + \cot(t)\varphi_1'(t) - \cot^2(t)\varphi_1(t) + \nu\varphi_1(t) + [1 + \cot(t)x_2(t)]\varphi_2(t) \\
[T_x(z)\varphi]_2(t) &= \varphi_2''(t) + \cot(t)\varphi_2'(t) - \cot^2(t)\varphi_2(t) - \beta[1 + \cot(t)x_2(t)]\varphi_1(t) + \\
&\quad + (\beta[\lambda - \cot(t)x_1(t)] - \nu)\varphi_2(t)
\end{aligned} \tag{13}$$

for all $\varphi \equiv (\varphi_1,\varphi_2) \in X$, $z \equiv (x,\lambda)$ and $0 < t < \pi$. With the linear and continuous operator

$$L : X \to Y \qquad (L\varphi)(t) \equiv \varphi''(t) + \cot(t)\varphi'(t) - \frac{1}{\sin^2(t)}\varphi(t) \tag{14}$$

we write equation (13) in the form

$$[T_x(z)\varphi](t) = (L\varphi)(t) + [A(z)](t)\varphi(t) \tag{15}$$

It is straightforward to show that the continuous elements $a_{ij}(z) : (0,\pi) \to \mathbb{R}$ of the matrix function $A(z)$ can be extended to 0 and π. Let the linear and continuous operator $K(z) : X \to Y$ be defined by the formula $[K(z)\varphi](t) \equiv [A(z)](t)\varphi(t)$. Then, equation (15) implies the splitting $T_x(z) = L + K(z)$. We are now in a position to prove the key result of this section.

THEOREM 1 *Let $z \in X \times \mathbb{R}$ be fixed. Then $T_x(z)$ is a Fredholm operator with index* 0.

Proof. We show that L is bijective (in particular a Fredholm operator with index 0) and that $K(z)$ is compact. Since the sum of a Fredholm operator with index 0 and a compact operator is again a Fredholm operator with index 0 the claim follows immediately.

1) The operator L is bijective: Because of $S_Y L = L S_X$ the restrictions $L_s \equiv L|_{X_s} : X_s \to Y_s$ and $L_a \equiv L|_{X_a} : X_a \to Y_a$ are well-defined. Obviously, L is bijective if and only if L_s and L_a are bijective. We now show that L_s is bijective (the bijectivity of L_a can be proved analogously). Since each element $x_s \in X_s$ (which satisfies the relations $x_s(\pi/2) = x_s''(\pi/2) = 0$) is characterized by its values on $(0, \pi/2)$, we define the following function spaces

$$\bar{X}_s \equiv \{\bar{x}_s : \bar{x}_s \equiv x\big|_{(0,\pi/2)},\ x \in X_s\} \quad \text{and} \quad \bar{Y}_s \equiv \{\bar{y}_s : \bar{y}_s \equiv y\big|_{(0,\pi/2)},\ y \in Y_s\} \tag{16}$$

Using the homeomorphisms $I_{X,s} : X_s \to \bar{X}_s$, $I_{X,s}x_s \equiv x_s\big|_{(0,\pi/2)}$ and $I_{Y,s} : Y_s \to \bar{Y}_s$, $I_{Y,s}y_s \equiv y_s\big|_{(0,\pi/2)}$ we set $\bar{L}_s \equiv I_{Y,s} L_s I_{X,s}^{-1} : \bar{X}_s \to \bar{Y}_s$. Obviously, L_s is bijective if and only if $\bar{L}_s$ is bijective. A direct calculation shows that for each $\bar{y}_s \in \bar{Y}_s$ there exists an uniquely determined $\bar{x}_s \in \bar{X}_s$ with

$$\bar{x}_s''(t) + (\cot \cdot \bar{x}_s)'(t) = \bar{x}_s''(t) + \cot(t)\bar{x}_s'(t) - \frac{1}{\sin^2(t)}\bar{x}_s(t) = (\bar{L}_s\bar{x}_s)(t) = \bar{y}_s(t) \qquad 0 < t < \pi/2$$

Thus $\bar{L}_s$ is bijective.

2) The operator $K(z)$ is compact: To prove the claim we want to use the Theorem of Arzelà-Ascoli. This theorem characterizes precompact sets of continuous functions which are defined on a closed interval $[a, b]$. For this purpose let $\hat{h}$ denote the extension of a function $h : (0, \pi) \to \mathbb{R}^{m\times n}$ (or $\mathbb{R}^m$), $m, n \in \mathbb{N}$, to the closed interval $[0, \pi]$. There are linear homeomorphisms

$$I_X : \hat{X} \equiv \{\hat{x} : x \in X\} \to X \qquad I_X\hat{x} \equiv x \qquad \text{and} \qquad I_Y : \hat{Y} \equiv \{\hat{y} : y \in Y\} \to Y \qquad I_Y\hat{y} \equiv y$$

which are norm-preserving if the spaces $\hat{X}$ and $\hat{Y}$ have the same norms as X and Y, respectively. The extension of $K(z)$, namely $\widehat{K(z)} : \hat{X} \to \hat{Y}$, $[\widehat{K(z)}\hat{\varphi}](t) \equiv [\widehat{A(z)}](t)\hat{\varphi}(t)$ for all $\hat{\varphi} \in \hat{X}$, $0 \le t \le \pi$, is related to $K(z)$ by $K(z) = I_Y \widehat{K(z)} I_X^{-1}$. Thus it is sufficient to show that $\widehat{K(z)}$ is compact, i.e., that $\widehat{K(z)}$ maps bounded sets $\hat{U} \subset \hat{X}$ onto precompact sets $\hat{V} \equiv \{\widehat{K(z)}\hat{\varphi} : \hat{\varphi} \in \hat{U}\} \subset \hat{Y}$. The precompactness of $\hat{V}$ follows from the Theorem of Arzelà-Ascoli.

REMARK 1 Here we give some results for the hemisphere problem which was mentioned in Section 1.

- Let equations (1) and (3) be transformed into a second order BVP where the differential equations are restricted on $(0, \pi/2]$. Now this second order problem can be written as $\bar{T}(\bar{x}_s, \lambda) = 0$, where $\bar{T} \equiv I_{Y,s}(T|_{X_s\times\mathbb{R}})(I_{X,s}^{-1}(\cdot), \cdot) : \bar{X}_s \times \mathbb{R} \to \bar{Y}_s$. The solutions of this operator equation correspond to those solutions of equation (9) which belong to $X_s \times \mathbb{R}$.
- We have $\bar{T}_{\bar{x}_s}(\bar{x}_s, \lambda) = \bar{L}_s + \bar{K}(\bar{x}_s, \lambda) \equiv \bar{L}_s + I_{Y,s}\, K(I_{X,s}^{-1}\bar{x}_s, \lambda)\Big|_{X_s} I_{X,s}^{-1} : \bar{X}_s \to \bar{Y}_s$. The operator $\bar{L}_s$ is bijective. The operator $\bar{K}(\bar{x}_s, \lambda)$ is compact because $K(I_{X,s}^{-1}\bar{x}_s, \lambda)$ is compact. Consequently $\bar{T}_{\bar{x}_s}(\bar{x}_s, \lambda)$ is a Fredholm operator with index 0.

As we will see in the next sections, further partial derivatives of T are required to characterize bifurcation phenomena more precisely. Apart from $T_x(z)$, the only nonvanishing derivatives are $T_\lambda(z)$, $T_{xx}(z)$ and $T_{x\lambda}(z) = T_{\lambda x}(z)$. Since these derivatives can be obtained by formal differentiation we do not give them here explicitely.

3 Trivial Solution Curve $\mathcal{C}_{triv}$

We begin the discussion of the solution field of equation (9) by studying the trivial solution curve $\mathcal{C}_{triv}$. In this section let $z \equiv (0, \lambda)$. Our aim is to characterize and to determine all primary bifurcation points $z_0 = (0, \lambda_0)$. These singular points must necessarily fulfil $\dim \mathcal{N}(T_x(z_0)) \ge 1$. We deduce from equation (13) that

$$[T_x(z)\varphi](t) = (L\varphi)(t) + A(\lambda)[\varphi(t)] \quad \forall \varphi \in X,\ 0 < t < \pi \qquad A(\lambda) \equiv \begin{pmatrix} \nu + 1 & 1 \\ -\beta & 1 - \nu + \beta\lambda \end{pmatrix} \tag{17}$$

The study of the singular points z_0 is based on

THEOREM 2 *Let* $\lambda \in \mathbb{R}$ *be a fixed value. The operator equation* $T_x(0,\lambda)\varphi = 0$, $\varphi \in X$, *has only the trivial solution if there do not exist eigenvalues of* $A(\lambda)$ *which are of the form* $\mu_n \equiv n(n+1)$, $n \in \mathbb{N}$. *Otherwise, nontrivial solutions exist and a basis of the null space of* $T_x(0,\lambda)$ *is given in the next table.*

$A(\lambda)$ *is similar to a*	*eigenvalues* η_i	*eigenvectors* v_i	*basis of* $\mathcal{N}(T_x(z))$
diagonal matrix	$\eta_1 = \mu_n, \eta_2 \neq \mu_m \forall m$	v_1, v_2	$(P_n^1 \circ \cos)v_1$
diagonal matrix	$\eta_1 = \mu_n, \eta_2 = \mu_m$	v_1, v_2	$(P_n^1 \circ \cos)v_1, (P_m^1 \circ \cos)v_2$
Jordan block	$\eta_1 = \eta_2 = \mu_n$	v_1	$(P_n^1 \circ \cos)v_1$

Table 1. *A basis of* $\mathcal{N}(T_x(z))$

Here, $P_n^1 \circ \cos : t \mapsto P_n^1(\cos(t))$, $0 < t < \pi$, *denotes the associated Legendre function of first order which is even (odd) if* n *is odd (even).*

Proof. 1) Let $A(\lambda)$ be similar to a diagonal matrix with the eigenvalues η_1, η_2 and the corresponding linearly independent eigenvectors $v_1, v_2 \in \mathbb{R}^2$. Given an element $\varphi \in X$, we define $g_1, g_2 : (0,\pi) \to \mathbb{R}$ implicitly by the decomposition

$$\varphi(t) \equiv g_1(t)v_1 + g_2(t)v_2 \qquad 0 < t < \pi \tag{18}$$

Obviously, $g_i \in X_e \equiv \{g : \exists \hat{g} \in C^2([0,\pi],\mathbb{R}), \hat{g}(0) = \hat{g}(\pi) = 0, g = \hat{g}\big|_{(0,\pi)}\}$. From $T_x(z)\varphi = 0$, we obtain

$$(L_e g_i)(t) + \eta_i g_i(t) \equiv g_i''(t) + \cot(t)g_i'(t) - \frac{1}{\sin^2(t)}g_i(t) + \eta_i g_i(t) = 0 \qquad 0 < t < \pi,\ i = 1,2 \tag{19}$$

The L_2-theory of the associated Legendre differential operator (see e.g. Triebel, 1992) implies that equation (19) is nontrivially solvable if and only if $\eta_i = \mu_n$ for some $n \in \mathbb{N}$. Under this condition the solution space of equation (19) is spanned by $P_n^1 \circ \cos$, where $(P_n^1 \circ \cos)v \in X$ for all $v \in \mathbb{R}^2$. Let us define

$$h_i \equiv \begin{cases} P_n^1 \circ \cos & \eta_i = n(n+1) \text{ for some } n \in \mathbb{N} \\ 0 & \eta_i \neq n(n+1) \text{ for all } n \in \mathbb{N} \end{cases} \qquad i = 1,2 \tag{20}$$

The decomposition (18) of φ implies $\mathcal{N}(T_x(z)) = \operatorname{span}\{h_1 v_1, h_2 v_2\}$.
2) Let $A(\lambda)$ be similar to a Jordan block with the double eigenvalue η_1 and the corresponding eigenvector v_1. Then the vector v_2 defined by $Av_2 = \eta_1 v_2 + v_1$ is linearly independent to v_1. Considering decomposition (18) the equation $T_x(z)\varphi = 0$ is equivalent to

$$L_e g_1 + \eta_1 g_1 + g_2 = 0 \qquad L_e g_2 + \eta_1 g_2 = 0 \tag{21}$$

With $L_\eta \equiv L_e + \eta_1 I$ formula (21) implies $g_2 \in \mathcal{N}(L_\eta) \cap \mathcal{R}(L_\eta)$. It can be shown that there exists an extension $\bar{L}_\eta$ of L_η which is selfadjoint on an appropriately defined Hilbert space. Thus, we have

$$g_2 \in \mathcal{N}(L_\eta) \cap \mathcal{R}(L_\eta) \subset \mathcal{N}(\bar{L}_\eta) \cap \mathcal{R}(\bar{L}_\eta) = \mathcal{N}(\bar{L}_\eta^*) \cap \mathcal{R}(\bar{L}_\eta) = \mathcal{R}(\bar{L}_\eta)^\perp \cap \mathcal{R}(\bar{L}_\eta) = \{0\}$$

Then the solutions of equation (21) are $\{(g_1, g_2) : (g_1, g_2) = (c h_1, 0),\ c \in \mathbb{R}\}$, where h_1 is defined by formula (20). Using decomposition (18) we find $\mathcal{N}(T_x(z)) = \operatorname{span}\{h_1 v_1\}$.

By the last theorem the determination of all singular points $z_0 \in C_{triv}$ of T is equivalent to a simple inverse eigenvalue problem of $A(\lambda)$: Compute all numbers $\lambda \in \mathbb{R}$ such that the 2-by-2 matrix $A(\lambda)$ has at least one eigenvalue μ of the form $\mu = n(n+1)$ for some $n \in \mathbb{N}$. This problem is discussed in

THEOREM 3 *Let* $A(\lambda)$ *be the matrix defined in equation (17). Then we have*

(1) $\mu_n \equiv n(n+1)$ *is an eigenvalue of* $A(\lambda)$ *iff* $\lambda = \lambda_n \equiv (\beta^{-1}(2 - \mu_n)\mu_n - k - 1)/(1 + \nu - \mu_n)$. *Obviously,* λ_n *is positive.*

(2) Let t_0 *be defined by* $t_0 \equiv -0.5 + \sqrt{1.25 + \nu + \sqrt{\beta}}$. *Then the sequence* $\{\lambda_n\}$, $n \in \mathbb{N}$, $n \geq t_0$, *is monotonously increasing.*

(3) μ_m *is the other eigenvalue of* $A(\lambda_n)$ *iff* $P(m,n,\nu,k) \equiv \beta - (\mu_n - 1 - \nu)(\mu_m - 1 - \nu) = 0$. *Then we have* $\lambda_n = \lambda_m$.

Proof. 1) Let μ_n, μ be the eigenvalues of $A(\lambda_n)$. The number λ_n can be computed from the two scalar equations $\det(A(\lambda_n)) = \mu_n \, \mu$ and $\mathrm{tr}\,(A(\lambda_n)) = \mu_n + \mu$.
2) Let the function $\Lambda : [1, \infty) \to \mathbb{R}$ be defined by $\Lambda(t) \equiv \dfrac{\beta^{-1}\,[2 - t(t+1)]t(t+1) - k - 1}{1 + \nu - t(t+1)}$. We see that $\Lambda(n) = \lambda_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{t\to\infty} \Lambda'(t) = \infty$. If $t_0 \geq 1$ then the number t_0 is the largest root of $\Lambda'(t) = 0$. Otherwise, $\Lambda'(t) > 0$ for all $t \geq 1$.
3) The third claim follows from $\mu_m = \mu = \mathrm{tr}\,(A(\lambda_n)) - \mu_n$.

REMARK 2 In practical computations the results of Theorem 3 can be used as follows. Part 1 gives an explicit formula for the singular points on $\mathcal{C}_{triv}$. The singular point $z_0 = (0, \lambda)$ which corresponds to the smallest value of λ can be determined by part 2. Part 3 yields a criterion by which it is possible to decide whether the dimension of $\mathcal{N}(T_x(0, \lambda_n))$ is 1 or 2. Given the numbers k, ν and n, $P(m) \equiv P(m, n, \nu, k)$ is a quadratic polynomial in m. Therefore, the null space $\mathcal{N}(T_x(0, \lambda_n))$ is two-dimensional if and only if $P(m)$ has a root in $\mathbb{N}$. But this situation occurs very rarely in practice.

Let us now consider singular points z_0 which satisfy $\dim \mathcal{N}(T_x(z_0)) = 1$. For this purpose some knowledge about the null space of the adjoint operator $[T_x(x, \lambda)]^* : Y^* \to X^*$ is required. The following theorem yields this information.

THEOREM 4 *Let $F, G : X \to Y$ be two Fredholm operators with index 0 which satisfy the relation $(G\psi, \varphi)_\omega = (\psi, F\varphi)_\omega$ for all $\psi, \varphi \in X$, where*

$$(\cdot,\cdot)_\omega : \mathrm{L}_2((0,\pi), \mathbb{R}^2) \times \mathrm{L}_2((0,\pi), \mathbb{R}^2) \to \mathbb{R} \qquad\qquad (h, g)_\omega \equiv \int_0^\pi \omega(t)\, h(t)^\top g(t)\, dt$$

Here, ω is a given non-negative weight function. Assume that $\psi_1, \ldots, \psi_k \in X$, $k \geq 1$, is a basis of $\mathcal{N}(G)$. Then we have $\dim \mathcal{N}(G) = \dim \mathcal{N}(F^*) = \dim \mathcal{N}(F)$, $\mathcal{N}(F^*) = \mathrm{span}\,\{\psi_1^*, \ldots, \psi_k^*\}$, where $\psi_i^* \equiv (\psi_i, \cdot)_\omega|_Y \in Y^*$, and $Y = \mathcal{R}(F) \oplus \mathrm{span}\,\{\psi_1, \ldots, \psi_k\}$, $\mathcal{R}(F) = \{y \in Y : \psi_i^* y = 0,\ i = 1, \ldots, k\}$.

Proof. The claim follows by elementary calculations and Fredholm's alternative.

Applying Theorem 4 with $\omega(t) \equiv \sin(t)$ and

$$F \equiv F(\tilde{z}) \equiv T_x(\tilde{z}) = L + K(\tilde{z}) \qquad\qquad G \equiv G(\tilde{z}) = L + \tilde{K}(\tilde{z}) \tag{22}$$

where $[\tilde{K}(\tilde{z})\psi](t) \equiv [A(\tilde{z})(t)]^\top \psi(t)$ for all $\tilde{z} \in X \times \mathbb{R}$, we find $[G(z)\psi](t) = (L\psi)(t) + A(\lambda)^\top \psi(t)$. The equations $T_x(z)\varphi = 0$ and $G(z)\psi = 0$ look similar. Therefore, replacing the matrix $A(\lambda)$ by $A(\lambda)^\top$ in Theorem 2 a basis of $\mathcal{N}(G(z))$ can be obtained.

We now assume that $\mathcal{N}(T_x(z_0)) = \mathrm{span}\,\{\varphi_0\}$ with $\varphi_0 \neq 0$. By Theorem 4 there exists a $\psi_0^* \neq 0$ such that $\mathcal{N}([T_x(z_0)]^*) = \mathrm{span}\,\{\psi_0^*\}$. The singular point z_0 is a simple bifurcation point if the first bifurcation coefficient $a_1 \equiv \psi_0^* T_{x\lambda}(z_0)\varphi_0$ does not vanish. Then, in the neighbourhood of z_0 the solutions of equation (9) form two curves which intersect transversally in z_0 (see e.g. Wallisch and Hermann, 1987). The following theorem states that all points $z_0 \in \mathcal{C}_{triv}$ for which $\dim \mathcal{N}(T_x(z_0)) = 1$ holds are simple bifurcation points.

THEOREM 5 *Let $z_0 \equiv (0, \lambda_0)$ and elements $\varphi_0 \in X \setminus \{0\}$, $\psi_0^* \in Y^* \setminus \{0\}$ be given such that $\mathcal{N}(T_x(z_0)) = \mathrm{span}\,\{\varphi_0\}$ and $\mathcal{N}([T_x(z_0)]^*) = \mathrm{span}\,\{\psi_0^*\}$. Then $a_1 \neq 0$.*

Proof. By Theorem 2 the matrix $A(\lambda_0)$ has a geometrically simple eigenvalue μ_n for some $n \in \mathbb{N}$. Let the corresponding eigenvector be denoted by $v = (v_1, v_2)^\top \in \mathbb{R}^2$. With the abbreviation $q_n \equiv P_n^1 \circ \cos$, we have $\varphi_0 = c_1 q_n v$ for some $c_1 \in \mathbb{R} \setminus \{0\}$. The null space of $G(z_0)$ is spanned by $\psi_0 \equiv q_n w$, where $w \equiv (w_1, w_2)^\top \in \mathbb{R}^2 \setminus \{0\}$ is an eigenvector of $A(\lambda_0)^\top$ which belongs to the eigenvalue μ_n. Using Theorem 4 we obtain $\psi_0^* = c_2\,(\psi_0, \cdot)_\omega|_Y$ for some $c_2 \in \mathbb{R}\setminus\{0\}$ and $\omega(t) = \sin(t)$. Our assumptions $k > 0$, $\nu \in [0.2, 0.5]$ which we have posed in Section 1 and the equations $A(\lambda_0)v = \mu_n v$, $A(\lambda_0)^\top w = \mu_n w$ give $v_1, v_2, w_1, w_2 \neq 0$. We now compute $a_1 = \beta\, c_1 c_2 v_2 w_2 \int_0^\pi \sin(t) q_n^2(t)\, dt \neq 0$.

4 Utilizing the Symmetry Properties of the Problem

Let us now consider the symmetry properties of the solutions of equation (9). It is useful to distinguish between nonsymmetric, antisymmetric and symmetric solutions. As we have seen in Section 2 the nonsymmetric solutions appear in pairs. The following theorem states that there do not exist nontrivial antisymmetric solutions.

THEOREM 6 *Suppose that* $z_a \equiv (x_a, \lambda)$ *is an antisymmetric solution of equation (9). Then* $x_a = 0$.

Proof. Let $x_a \equiv (x_{a,1}, x_{a,2})$. The equations (10) and $T(x_a, \lambda) = 0$ imply $T(-x_a, \lambda) = 0$. From $[T(x_a, \lambda)]_1 + [T(-x_a, \lambda)]_1 = 0$ we obtain $x_{a,2} = 0$. Now $[T(x_a, \lambda)]_2 = 0$ yields $x_{a,1} = 0$.

In the discussion of symmetric solutions of equation (9) we use the following general result.

THEOREM 7 *Let* $z \equiv (x, \lambda) \in X \times \mathbb{R}$ *be given. Then* $\dim \mathcal{N}(T_x(z)) \leq 2$.

Proof. Assume that $\varphi \in X$ is a solution of $T_x(z)\varphi = 0$. Then the extension $\hat{\varphi} \equiv (\hat{\varphi}_1, \hat{\varphi}_2) \in \hat{X}$ of φ to the interval $[0, \pi]$ satisfies (see equations (14) and (15))

$$\hat{\varphi}''(t) = -\cot(t)\hat{\varphi}'(t) + \frac{1}{\sin^2(t)}\hat{\varphi}(t) - A(t)\hat{\varphi}(t),\ 0 < t < \pi \qquad \hat{\varphi}(0) = 0 \qquad A(t) \equiv [A(z)](t) \tag{23}$$

To obtain statements about the dimension of the solution space of equation (23) we transform this problem into a (singular) system of first order differential equations which has been studied in a series of papers (see e.g. de Hoog and Weiss, 1976,1977,1985, Keller, 1976, Lentini, 1980). Let

$$\Phi_1(t) \equiv \hat{\varphi}_1(t)/t \qquad \Phi_2(t) \equiv \hat{\varphi}_1'(t) \qquad \Phi_3(t) \equiv \hat{\varphi}_2(t)/t \qquad \Phi_4(t) \equiv \hat{\varphi}_2'(t) \tag{24}$$

We compute

$$\Phi_1(0) = \lim_{t\to 0+} \frac{\hat{\varphi}_1(t)}{t} = \lim_{t\to 0+} \frac{\hat{\varphi}_1(t) - \hat{\varphi}_1(0)}{t} = \hat{\varphi}_1'(0) = \Phi_2(0) \quad \text{and} \quad \Phi_3(0) = \Phi_4(0) \tag{25}$$

Using the splittings

$$\cot(t) = \frac{1}{t} - \widehat{\cot}(t) \qquad \frac{1}{\sin(t)} = \frac{1}{t} + \widehat{\sin}(t) \qquad \text{where} \lim_{t\to 0+} \widehat{\cot}(t) = \lim_{t\to 0+} \widehat{\sin}(t) = 0 \tag{26}$$

the system of two scalar second order differential equations in equation (23) can be written as the first order system

$$\Phi'(t) = \frac{1}{t} M\, \Phi(t) + g(t, \Phi(t)) \qquad M_1 \equiv \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M \equiv \operatorname{diag}(M_1, M_1) \tag{27}$$

The components of the function $g : (0, \pi) \times \mathbb{R}^4 \to \mathbb{R}^4$,

$$g_1(t, v) \equiv 0 \qquad g_2(t, v) \equiv [2\widehat{\sin}(t) + t\,\widehat{\sin}^2(t) - t a_{11}(t)]v_1 + \widehat{\cot}(t)v_2 - t a_{12}(t)v_3$$
$$g_3(t, v) \equiv 0 \qquad g_4(t, v) \equiv -t a_{21}(t)v_1 + [2\widehat{\sin}(t) + t\,\widehat{\sin}^2(t) - t a_{22}(t)]v_3 + \widehat{\cot}(t)v_4$$

are continuous on $(0, \pi)$. But g_2 and g_4 can be extended to $t = 0$ and are bounded for $t \to \pi-$. Formula (25) implies $\Phi(0) \in \mathcal{N}(M)$. Let $v^0 \in \mathcal{N}(M)$. By Theorem 2.1 in the paper of de Hoog and Weiss (1985), p. 94, there is a $t_0 > 0$ such that the initial value problem consisting of equation (27) and $\Phi(0) = v^0$ has a unique solution $\bar{\Phi}$ on $[0, t_0]$. Assume that $\bar{\Phi}$ can be extended to $\Phi \in C^1([0, \pi), \mathbb{R}^4)$ which solves this initial value problem. Note that $\tilde{g} : (0, \pi) \times \mathbb{R}^4 \to \mathbb{R}^4$, $\tilde{g}(t, v) \equiv \frac{1}{t}Mv + g(t, v)$, is continuously differentiable with respect to the second argument. In particular $\tilde{g}$ is locally Lipschitz continuous. Thus Φ is uniquely determined (see e.g. Demailly, 1994). Since $\dim \mathcal{N}(M) = 2$, there are at most two linear independent solutions of the linear equation (27) which satisfy $\Phi_1(0) = \Phi_2(0)$ and $\Phi_3(0) = \Phi_4(0)$. The claim follows by retransforming Φ according to equation (24).

Let us now consider elements $z_s \equiv (x_s, \lambda) \in X_s \times \mathbb{R}$.

THEOREM 8 *Suppose that* $\dim \mathcal{N}(T_x(z_s)) \geq 1$. *Then there is a finite number of elements in* $X_s \cup X_a$ *forming a basis of* $\mathcal{N}(T_x(z_s))$.

Proof. Let $\mathcal{B}$ be a basis of $\mathcal{N}(T_x(z_s))$. Note $\dim \mathcal{N}(T_x(z_s)) \leq 2$. For each $\varphi \in \mathcal{B}$ decomposed as $\varphi = \varphi_s + \varphi_a \in X_s \oplus X_a$ the equivariance of T implies $0 = T_x(z_s)\varphi_s + T_x(z_s)\varphi_a \in Y_s \oplus Y_a$. Hence, $T_x(z_s)\varphi_s = 0$ and $T_x(z_s)\varphi_a = 0$.

Theorem 7 and Theorem 8 give rise to classify the elements $z_s \in X_s \times \mathbb{R}$ with $\dim \mathcal{N}(T_x(z_s)) \geq 1$ as follows (see Table 2(a)):

type	basis of $\mathcal{N}(T_x(z_s))$	$T_\lambda(z_s) \in \mathcal{R}(T_x(z_s))$?
1	φ_s	yes
2	φ_s	no
3	φ_a	yes
(4)	φ_a	no
5	$\varphi_{s1}, \varphi_{s2}$	yes
6	$\varphi_{s1}, \varphi_{s2}$	no
7	$\varphi_{a1}, \varphi_{a2}$	yes
(8)	$\varphi_{a1}, \varphi_{a2}$	no
9	φ_s, φ_a	yes
10	φ_s, φ_a	no

(a)

$\Longrightarrow$

basis of $\mathcal{N}(G(z_s))$
ψ_s
ψ_s
ψ_a
ψ_a
ψ_{s1}, ψ_{s2}
ψ_{s1}, ψ_{s2}
ψ_{a1}, ψ_{a2}
ψ_{a1}, ψ_{a2}
ψ_s, ψ_a
ψ_s, ψ_a

(b)

Table 2. (a) Classification of $z_s \in X_s \times \mathbb{R}$, $\dim \mathcal{N}(T_x(z_s)) \geq 1$, (b) Null space of $G(z_s)$

Part (b) of Table 2 is a conclusion from part (a). Let us verify the entries in Table 2(b). Theorem 4 and formula (22) imply $\dim \mathcal{N}(G(z_s)) = \dim \mathcal{N}(T_x(z_s))$. Because of $(G(z_s)\psi, \varphi)_\omega = (\psi, T_x(z_s)\varphi)_\omega$ for all $\psi, \varphi \in X$, we have $(S_Y G(z_s)\psi - G(z_s)S_X\psi, \varphi)_\omega = 0$ for all $\psi, \varphi \in X$. Moreover, X is a dense subset of $\mathrm{L}_2((0,\pi), \mathbb{R}^2)$ with regard to $\|\cdot\|_\omega \equiv \sqrt{(\cdot,\cdot)_\omega}$. Thus $S_Y G(z_s) = G(z_s) S_X$. It follows that there exists a basis $\mathcal{B}$ of $\mathcal{N}(G(z_s))$ with elements in $X_s \cup X_a$ (compare Theorem 8). By Theorem 4, $\psi \notin \mathcal{R}(T_x(z_s))$ for all $\psi \in \operatorname{span} \mathcal{B}$. Because of equation (10), the assumption $T_x(z_s)u = v_s \in Y_s$ and the splitting $u = u_s + u_a \in X_s \oplus X_a$ give $0 = [T_x(z_s)u_s - v_s] + [T_x(x_s)u_a] \in Y_s \oplus Y_a$, i.e., $T_x(z_s)u_a = 0$. Hence, each $v_s \in \mathcal{R}(T_x(z_s)) \cap Y_s$ has a symmetric pre-image u_s. Therefore, $\mathcal{R}(T_x(z_s)|_{X_s}) = \mathcal{R}(T_x(z_s)) \cap Y_s$ and $Y_s = \mathcal{R}(T_x(z_s)|_{X_s}) \oplus (\operatorname{span} \mathcal{B} \cap Y_s)$. Obviously we have $\mathcal{N}(T_x(z_s)|_{X_s}) = \mathcal{N}(T_x(z_s)) \cap X_s$. Since $T_x(z_s)|_{X_s} = L_s + K(z_s)|_{X_s}$ is a Fredholm operator with index 0 (L_s is linear, continuous and bijective, $K(z_s)|_{X_s}$ is compact) we find $\dim(\mathcal{N}(T_x(z_s)) \cap X_s) = \dim(\operatorname{span} \mathcal{B} \cap Y_s)$.

In Table 2(a) the numbers 4 and 8 are put in brackets since these cases cannot occur. This can be shown by the following arguments. Let us first consider elements z_s of the type 4. Looking at Table 2(b) we see that there exists a $\psi_a \in X_a \setminus \{0\}$ such that $\mathcal{N}(G(z_s)) = \operatorname{span}\{\psi_a\}$. Now Theorem 4 implies $\psi_a^* T_\lambda(z_s) = (\psi_a, T_\lambda(z_s))_\omega = \int_0^\pi \sin(t)\psi_a(t)^\top [T_\lambda(z_s)](t)\,dt$. Since $T_\lambda(z_s) \in Y_s$, the integrant is odd and the integral vanishes. From Theorem 4 we obtain $T_\lambda(z_s) \in \mathcal{R}(T_x(z_s))$. This result contradicts the corresponding entry in the third column of Table 2(a). Thus our claim is proved. The existence of elements z_s of type 8 can be excluded by the same arguments.

To prepare the next theorem we give two formulas which can be easily verified:

$$(T|_{X_s \times \mathbb{R}})_{x_s}(z_s) = T_x(z_s)|_{X_s} \quad \text{and} \quad (T|_{X_s \times \mathbb{R}})_\lambda(z_s) = T_\lambda(z_s) \tag{28}$$

THEOREM 9 *Let $z_{0s} \equiv (x_{0s}, \lambda_0) \in X_s \times \mathbb{R}$ be a singular point of the operator T. Using the classification given in Table 2(a), the following statements describe the solution field of equation (9) in a sufficiently small neighbourhood U of z_{0s}. The scalar ε_0 is assumed to be an appropriate positive real number.*

(1) Assume z_{0s} is of type 1, 2, 5 or 6. Then, all solutions $z \in U$ are symmetric.

(2) Assume z_{0s} is of type 2. Then, z_{0s} is a limit point of T. The solutions $z \in U$ form a curve $\mathcal{C} = \{(x(\varepsilon), \lambda(\varepsilon)) \equiv (x_{0s} + \varepsilon\varphi_{0s} + w(\varepsilon), \lambda_0 + \tau(\varepsilon)) : |\varepsilon| < \varepsilon_0\}$, where $w(\varepsilon)$ belongs to the complement of $\mathcal{N}(T_x(z_{0s}))$ in X, $x(0) = x_{0s}$, $\lambda(0) = \lambda_0$ and $\mathcal{C} \subset X_s \times \mathbb{R}$.

(3) Assume z_{0s} is of type 3 or of type 7. Then, the symmetric solutions $z_s \in U$ represent a curve $\mathcal{C} = \{(x_s(\lambda), \lambda) : |\lambda_0 - \lambda| < \varepsilon_0\}$ with $x_s(\lambda_0) = x_{0s}$. In U all other solutions are nonsymmetric.

(4) Assume z_{0s} is of type 10. Then z_{0s} is a limit point of $T|_{X_s \times \mathbb{R}}$. The symmetric solutions $z_s \in U$ form a curve $\mathcal{C} = \{(x_s(\varepsilon), \lambda(\varepsilon)) \equiv (x_{0s} + \varepsilon\varphi_{0s} + w(\varepsilon), \lambda_0 + \tau(\varepsilon)) : |\varepsilon| < \varepsilon_0\}$, where $w(\varepsilon)$ belongs to the complement of $\mathcal{N}((T|_{X_s \times \mathbb{R}})_{x_s}(z_{0s}))$ in X_s, $x_s(0) = x_{0s}$ and $\lambda(0) = \lambda_0$. In U, all other solutions are nonsymmetric.

Proof. 1) For the types 1, 2, 5 or 6 the claim can be proved by the same technique. We demonstrate the basic idea for the case $\mathcal{N}(T_x(z_{0s})) = \operatorname{span}\{\varphi_{0s}\}$, $\varphi_{0s} \in X_s \setminus \{0\}$, only. Table 2(b) and Theorem 4 state that there is a $\psi_{0s} \in X_s$, $\psi_{0s} \notin \mathcal{R}(T_x(z_{0s}))$. Then $X = \mathcal{N}(T_x(z_{0s})) \oplus X_1$, $X_1 \equiv \{w \in X : \varphi_{0s}^* w = 0\}$, and

$Y = \mathcal{R}(T_x(z_{0s})) \oplus \operatorname{span}\{\psi_{0s}\}$, where $\varphi_{0s}^* \in X^*$ satisfies $\varphi_{0s}^*\varphi_{0s} = 1$. Using formulas (28) we conclude $X_s = \mathcal{N}(T_x(z_{0s})) \oplus X_{1s}$, $X_{1s} \equiv \{w_s \in X_s : \varphi_{0s}^* w_s = 0\}$, and $Y_s = \mathcal{R}(T_x(z_{0s})|_{X_s}) \oplus \operatorname{span}\{\psi_{0s}\}$. We define $G : \mathbb{R} \times \mathbb{R} \times X_1 \times \mathbb{R} \to Y$, where $G(\varepsilon, \lambda, w, g) \equiv T(x_{0s} + \varepsilon\varphi_{0s} + w, \lambda_0 + \lambda) + g\psi_{0s}$, and $\tilde{G} : \mathbb{R} \times \mathbb{R} \times X_{1s} \times \mathbb{R} \to Y_s$, where $\tilde{G}(\varepsilon, \lambda, w_s, g) \equiv T|_{X_s \times \mathbb{R}}(x_{0s} + \varepsilon\varphi_{0s} + w_s, \lambda_0 + \lambda) + g\psi_{0s}$. The solutions $(\varepsilon, \lambda, w)$ of $T(x_{0s} + \varepsilon\varphi_{0s} + w, \lambda_0 + \lambda) = 0$ correspond to the solutions $(\varepsilon, \lambda, w, g)$ of $G(\varepsilon, \lambda, w, g) = 0$ with $g = 0$. Furthermore, $x_{0s} + \varepsilon\varphi_{0s} + w \in X_s$ if and only if $w \in X_s$. Thus we consider the equations $G(u, v) = 0$ and $\tilde{G}(u, \tilde{v}) = 0$, where $u \equiv (\varepsilon, \lambda)$, $v \equiv (w, g)$ and $\tilde{v} \equiv (w_s, g)$. Obviously, $G(0,0) = 0$ and $\tilde{G}(0,0) = 0$. We now show that there exists a neighbourhood $\mathcal{U}$ of $(u_0, v_0) = (0,0)$ such that the solutions of $G(u,v) = 0$ contained in $\mathcal{U}$ have symmetric X_1-components, i.e., $w \in X_1 \cap X_s$. The partial derivatives $G_{(w,g)}(0,0)$ and $\tilde{G}_{(w_s,g)}(0,0)$ are linear homeomorphisms. Thus the implicit function theorem guarantees the existence of neighbourhoods $U \subset \mathbb{R} \times \mathbb{R}$, $V \subset X_1 \times \mathbb{R}$, $\tilde{U} \subset \mathbb{R} \times \mathbb{R}$ and $\tilde{V} \subset X_{1s} \times \mathbb{R}$ of the corresponding origins as well as the existence of continuous functions $\omega : U \to V$, $\omega(0) = 0$, and $\tilde{\omega} : \tilde{U} \to \tilde{V}$, $\tilde{\omega}(0) = 0$, such that $G(u,v) = 0$, $(u,v) \in U \times V$, if and only if $v = \omega(u)$, and $\tilde{G}(\tilde{u}, \tilde{v}) = 0$, $(\tilde{u}, \tilde{v}) \in \tilde{U} \times \tilde{V}$, if and only if $\tilde{v} = \tilde{\omega}(\tilde{u})$. Without loss of generality we can assume that $\tilde{U} \subset U$ and $\tilde{\omega}(\tilde{u}) \in V$ for all $\tilde{u} \in \tilde{U}$. Let $(u,v) \in \tilde{U} \times V$ with $G(u,v) = 0$. Then, $v = \omega(u)$. Because of $X_{1s} \subset X_1$ we have $G(u, \tilde{\omega}(u)) = \tilde{G}(u, \tilde{\omega}(u)) = 0$, where $(u, \tilde{\omega}(u)) \in U \times V$. Thus $v = \omega(u) = \tilde{\omega}(u)$, i.e., the X_1-component of v is symmetric.
2) The solution field in the neighbourhood of limit points is studied in the book of Wallisch and Hermann (1987). The statement $\mathcal{C} \subset X_s \times \mathbb{R}$ follows from part (1).
3) The symmetric solutions of equation (9) satisfy equation (12). Clearly, $T|_{X_s \times \mathbb{R}}(x_{0s}, \lambda_0) = 0$. Since $\mathcal{N}(T_x(z_{0s})) \subset X_a$ and $T_x(z_{0s})|_{X_s}$ is a Fredholm operator with index 0, $(T|_{X_s \times \mathbb{R}})_{x_s}(z_{0s}) = T_x(z_{0s})|_{X_s}$ is a homeomorphism. The statement about symmetric solutions follows from the implicit function theorem.
4) The claim can be proved by arguments which we have used before.

REMARK 3 Let z_{0s} be a solution of type 1 (see Table 2). If $\tau \equiv \alpha\gamma - \beta^2 < 0$ (see e.g. Wallisch and Hermann, 1987) then z_{0s} is a hyperbolic point and the solutions in the neighbourhood of z_{0s} are symmetric and form two curves which intersect transversally at z_{0s}. If $\tau > 0$ then z_{0s} is an isola center (see e.g. Keller, 1981, Seoane, 1994).
Let z_{0s} be a solution of type 9. Then, z_{0s} is a singularity of type 1 of $T|_{X_s \times \mathbb{R}}$.

We now consider those solutions $z_{0s} \in X_s \times \mathbb{R}$ of equation (9) which cannot be assigned to one of the ten classes discussed above. These are the so-called isolated solutions which are defined by $T(z_{0s}) = 0$, $T_x(z_{0s})$ is a linear homeomorphism.

THEOREM 10 *Let $(\lambda_-, \lambda_+) \subset \mathbb{R}$, $\mathcal{C}_{iso} \equiv \{(x(\lambda), \lambda) \in X \times \mathbb{R} : \lambda \in (\lambda_-, \lambda_+)\}$ be a curve of isolated solutions of equation (9) and $\lambda_0 \in (\lambda_-, \lambda_+)$ with $x(\lambda_0) \in X_s$. Then $\mathcal{C}_{iso} \subset X_s \times \mathbb{R}$.*

Proof. Let $(x_*, \lambda_*) \equiv (x(\lambda_*), \lambda_*)$ be an arbitrary element of $\mathcal{C}_{iso} \cap (X_s \times \mathbb{R})$. Using equation (28) it is evident that (x_*, λ_*) is also an isolated solution of equation (12). Now the implicit function theorem states that there exist neighbourhoods $W \equiv U \times V \subset X \times \mathbb{R}$ and $W_s \equiv U_s \times V_s \subset X_s \times \mathbb{R}$ of (x_*, λ_*) as well as continuous functions $\omega : V \to U$, $\omega(\lambda_*) = x_*$, and $\omega_s : V_s \to U_s$, $\omega_s(\lambda_*) = x_*$, such that

$$T(x,\lambda) = 0, (x,\lambda) \in W \Leftrightarrow x = \omega(\lambda) \qquad T|_{X_s \times \mathbb{R}}(x_s, \lambda) = 0, (x_s, \lambda) \in W_s \Leftrightarrow x_s = \omega_s(\lambda) \quad (29\,\mathrm{a,b})$$

Without loss of generality let V_s satisfy $V_s \subset V$ and $\omega_s(\lambda) \in U$ for all $\lambda \in V_s$. Suppose $(x, \lambda) \in U \times V_s$ is a solution of equation (9). We see from equation (29 a) that $x = \omega(\lambda)$. By equation (29 b), $(\omega_s(\lambda), \lambda)$ solves equation (12). Hence, (x, λ) and $(\omega_s(\lambda), \lambda) \in U \times V_s$ are two solutions of equation (9) in $U \times V$. Now equation (29 a) implies $x = \omega(\lambda) = \omega_s(\lambda) \in X_s$. Thus, all solutions $(x(\lambda), \lambda)$ of equation (9) are symmetric for λ in a sufficiently small neighbourhood of λ_*.
Applying the arguments discussed above to $(x_*, \lambda_*) \equiv (x(\lambda_0), \lambda_0)$, we find $x(\lambda) \in X_s$ for all $\lambda \in (\gamma_-, \gamma_+)$, where $\lambda_- \leq \gamma_- < \lambda_0 < \gamma_+ \leq \lambda_+$. Let γ_- and γ_+ be the smallest possible value and the largest possible value, respectively. Assume $\gamma_- > \lambda_-$. Then $\lim\limits_{\lambda \to (\gamma_-)+} (x(\lambda), \lambda) = (x(\gamma_-), \gamma_-) \in \mathcal{C}_{iso}$, where $x(\gamma_-) \in X_s$. We set $(x_*, \lambda_*) \equiv (x(\gamma_-), \gamma_-)$ and obtain $x(\lambda) \in X_s$ for all λ in a sufficiently small neighbourhood of γ_-. Therefore, $\lambda_- = \gamma_-$. This argument can now be repeated almost verbatim to show $\lambda_+ = \gamma_+$.

The following result can be used to detect secondary bifurcation points (see Part II of this paper).

THEOREM 11 *Let $\Gamma_1 \equiv \{(x(\varepsilon), \lambda(\varepsilon)) : |\varepsilon| < \varepsilon_0\}$ and $\Gamma_2 \equiv \{(S_X x(\varepsilon), \lambda(\varepsilon)) : (x(\varepsilon), \lambda(\varepsilon)) \in \Gamma_1\}$ be two branches of nonsymmetric solutions which can be extended to $\varepsilon = \varepsilon_0$. Let $z_0 \equiv (x(\varepsilon_0), \lambda(\varepsilon_0)) = (S_X x(\varepsilon_0), \lambda(\varepsilon_0))$. Then $x(\varepsilon_0) \in X_s$, z_0 is a singular point of T and $\mathcal{N}(T_x(z_0)) \cap X_a \neq \{0\}$.*

Proof. Since T, $\varepsilon \mapsto x(\varepsilon)$ and $\varepsilon \mapsto \lambda(\varepsilon)$ are continuous, $T(z_0) = 0$ holds. From $x(\varepsilon_0) = S_X x(\varepsilon_0)$

it follows $x(\varepsilon_0) \in X_s$. We assume $\dim \mathcal{N}(T_x(z_0)) = 0$. The implicit function theorem says that in a neighbourhood of z_0 all solutions of equation (9) form a uniquely determined curve $\mathcal{C}$. Applying the first part of the proof of Theorem 10 to $(x_*, \lambda_*) \equiv z_0$, we obtain a neighbourhood U of z_0 such that $U \cap \mathcal{C} \subset X_s \times \mathbb{R}$. Then we deduce the contradiction $(\Gamma_1 \cup \Gamma_2) \cap U = (\mathcal{C} \setminus \{z_0\}) \cap U \subset X_s \times \mathbb{R}$. Now, let $\mathcal{N}(T_x(z_0)) \subset X_s$. By Theorem 9(1) there is a neighboorhoud U of z_0 such that all solutions of equation (9) in U are symmetric which contradicts $(\Gamma_1 \cup \Gamma_2) \cap U \subset (X \setminus X_s) \times \mathbb{R}$.

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