TECHNISCHE MECHANIK, Band 19, Heft 2, (1999), 127-140
Manuskripteingang: 20. August 1998

Dynamic Modeling of Multi-Elastic Body Systems using Kane's Method and Congruency Transformations

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In this article, an improved method for deriving elastic generalized coordinates is considered. Then Kane's equations of motion for multibody systems consisting of an arbitrary number of rigid and elastic bodies is presented. The equations are in general form and they are applicable for any desired holonomic system. Flexibility in choosing generalized speeds in terms of generalized coordinate derivatives in Kane's method is used. It is shown that proper choice of a congruency transformation between generalized coordinate derivatives and generalized speeds leads to first order decoupled equations of motion for holonomic multibody systems consisting of an arbitrary number of rigid and elastic bodies. In order to show the use of this method, a simple system consisting of a lumped mass, a spring and a clamped-free elastic beam is modeled. Finally, numerical implementation of decoupling using congruency transformation is discussed and shown via simulation of a two degree of freedom flexible robot.

1 Introduction

Much effort has been focused on formulation of procedures to yield the differential equations describing the motion of multibody systems (Hartog, 1948; Gibbs, 1961; Crandall, 1968; Kane and Levinson, 1985; Roberson and Schwertassek, 1988; Huston, 1990). In most cases the resulting equations are numerically integrated to obtain a trajectory of the system's motion. In addition, the equations of motion are often analyzed directly to determine the nature of the nonlinear behavior. In this case, if the nonlinear differential equations are written in the form $\dot{x} = f(x,t)$, the analysis is simpler. In addition, this form is ideal for numerical integration. Loduha (1994) and Ravani (1995), presented a method, using Kane's equations (Kane and Levinson, 1985), for generating equations of motion which are decoupled in the highest derivative terms. They showed the application for holonomic and nonholonomic multi rigid body systems. Their method is a systematic and efficient one for modeling multi rigid body systems. Other applications of this method are shown by Meghdari (1997).

On the other hand, much investigations have been carried out on the dynamic modeling of elastic systems. Earlier models were based on the assumption that small deformation of the bodies do not affect the rigid body motion (Sunada and Dubowsky, 1981; Naghanatan and Soni, 1986; Meghdari, 1991). These models do not yield accurate results when high speed systems are considered. In later works formulation of equations of motion of each elastic body is done in terms of its rigid body and elastic degrees of freedom. Then the rigid body motion and elastic deformations are solved simultaneously. Both finite element and assumed mode methods were used to model elastic degrees of freedom extensively (Yoo and Haug, 1985; Agraval and Shabana, 1985; Meghdari and Ghasempouri, 1994). Assumed mode methods decrease the dimension of the problem and are superior to finite element methods. Ider and Amirouche (1989) presented a method to obtain assumed modes by prior finite element analysis of each body. In their method one can easily model the coupling effects of longitudinal and transverse deformations and the geometrical nonlinearities. They applied Kane's equations beside the mentioned method to model multi elastic body dynamics.

In this paper we plan to combine the efficiency of Kane's equations, the power of the first order decoupling method and the improvement of assumed mode techniques and present a method to derive a first order decoupled form of differential equations of motion for multibody systems consisting of an arbitrary number of rigid and elastic bodies.

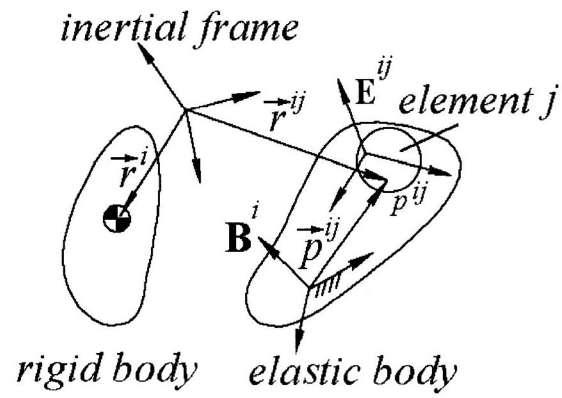

Figure 1. Multi Body System Consisting of Elastic and Rigid Bodies

2 Generalized Coordinates for Multi-Elastic Body Systems

Consider an elastic body B_i as shown in Figure 1. A frame $\mathbf{B}^i$ is attached to a point of it. We divide this body to E^i elements. Position of a point p^{ij} in element j of body i with respect to a frame $\mathbf{E}^{ij}$ attached to the element can be written in terms of nodal coordinates by applying element shape functions. Then with the aid of transformation matrices, one can obtain the position vector of point p^{ij} in the body frame $\mathbf{B}^i$:

$$\vec{p}^{ij} = \Phi^{ij}\vec{e}^{ij} \tag{1}$$

where Φ^{ij} is a transformation matrix containing element shape functions and $\vec{e}^{ij}$ is the nodal coordinates vector of element j of body i which can be expanded as below:

$$\vec{e}^{ij} = \vec{e}_0^{ij} + \vec{\alpha}^{ij} \tag{2}$$

where $\vec{e}_0^{ij}$ is the initial nodal coordinate vector of element j, yielding the undeformed position vector and $\vec{\alpha}^{ij}$ is the nodal displacement vector of element j. After assemblage with the use of boundary conditions, the nodal displacement vector of body i is formed as $\vec{\alpha}^i$. Now we can formulate the eigenvalue problem for the body i.

$$\mathbf{M}^i\ddot{\vec{\alpha}}^i + \mathbf{K}^i\vec{\alpha}^i = \mathbf{0} \tag{3}$$

where $\mathbf{M}^i$ and $\mathbf{K}^i$ are the generalized mass and generalized stiffness matrices of the body i, respectively. An approximate solution to the above problem is obtained:

$$\vec{\alpha}^i = \mathbf{X}^i\vec{\eta}^i \tag{4}$$

where $\mathbf{X}^i$ is the matrix with eigenvectors (mode shapes) as columns and $\vec{\eta}^i$ is the modal coordinate vector of body i. If we extract the $\vec{\alpha}^{ij}$ of each element in terms of the modal coordinate vector $\vec{\eta}^i$ and use equation (2) and equation (1), we can obtain the position vector of a point p^{ij} with respect to the body frame in terms of the modal coordinate vector $\vec{\eta}^i$. Now the components of $\vec{\eta}^i$ can be used as elastic coordinates for the body i.
Finally if the system contains N bodies, the position of the system can be described by $3N$ Euler angles, $3N$ relative translations, and m modal coordinates (selected components of $\vec{\eta}^i$ for each body). So they can form the generalized coordinates vector $\vec{q}$ of the system. For details see Ider and Amirouche (1989).

3 Kane's Equations for Multi Body Systems Consisting of Rigid and Elastic Bodies

The velocity of each point in the element j can be derived in terms of generalized coordinates. If the velocity of point p^{ij} in the element j of body i is shown by $\vec{v}^{ij}$, its partial velocity is obtained in matrix form as:

$$\mathbf{V}^{ij} = \frac{\partial \vec{v}^{ij}}{\partial \vec{u}^T} \tag{5}$$

where $\mathbf{V}^{ij}$ is the partial velocity matrix and its kth column contains the kth partial velocity of point p^{ij}. $\vec{u}$ is the generalized speed vector and it is related to the generalized coordinate vector derivative as follows:

$$\dot{\vec{q}} = \mathbf{T}\vec{u} \tag{6}$$

If there exists any rigid body in the system and $\vec{v}^i$ is its mass center velocity and $\vec{\omega}^i$ is its angular velocity, its partial velocity matrix $\mathbf{V}^i$ and partial angular velocity matrix $\mathbf{\Gamma}^i$ are written:

$$\mathbf{V}^i = \frac{\partial \vec{v}^i}{\partial \vec{u}} \tag{7}$$

$$\mathbf{\Gamma}^i = \frac{\partial \vec{\omega}^i}{\partial \vec{u}} \tag{8}$$

where the kth column of $\mathbf{\Gamma}^i$ contains the kth partial angular velocity of the rigid body i. If the surface traction on the element j of the elastic body i is shown by $\vec{f}^{ij}$ and the body force exerted on it is shown by $\vec{b}^{ij}$, and the resulting force on the rigid body i is $\vec{f}^i$ and the moment exerted on it is $\vec{M}^i$, one can write the generalized active force as:

$$\vec{F} = \sum_{i=1}^{N^E} \sum_{j=1}^{E^i} [\int_{S^{ij}} \mathbf{V}^{ij^T} \vec{f}^{ij} dS + \int_{V^{ij}} \mathbf{V}^{ij^T} \vec{b}^{ij} dV] + \sum_{i=1}^{N^s} [\mathbf{V}^{i^T} \vec{f}^i + \mathbf{\Gamma}^{i^T} \vec{M}^i] \tag{9}$$

where S^{ij} denotes that part of the element surface which lies within the global boundary and V^{ij} is the volume of the element. N^s, N^E and E^i are the number of rigid bodies, the number of elastic bodies and the number of elements in the elastic body i, respectively. The generalized inertial force can be written:

$$\vec{F}^* = -\sum_{i=1}^{N^E} \sum_{j=1}^{E^i} [\int_{V^{ij}} \rho^{ij} \mathbf{V}^{ij^T} \vec{a}^{ij} dV] - \sum_{i=1}^{N^s} [m^i \mathbf{V}^{i^T} \vec{a}^i + \mathbf{\Gamma}^{i^T} \dot{\vec{H}}^i] \tag{10}$$

where $\vec{a}^{ij}$ is the acceleration of point p^{ij} in element j of the elastic body i, and $\vec{a}^i$ is the mass center acceleration of the rigid body i. Now, Kane's equations of motion become:

$$\vec{F} + \vec{F}^* - \vec{F}^s = \vec{0} \tag{11}$$

where $\vec{F}^s$ is the generalized internal force vector due to elasticity of the elastic bodies.

4 The Influence of the Generalized Speed Component Selection

Our goal is to prescribe a choice of generalized speeds components that would yield decoupled equations of motion. So it will be helpful to reveal the influence of generalized speed components on the resulting dynamical equations. Acceleration of point p^{ij} in element j of the elastic body i is derived as:

$$\vec{a}^{ij} = \frac{d}{dt}\vec{v}^{ij} = \frac{d}{dt}(\mathbf{V}^{ij}\vec{u}) = \frac{d}{dt}(\mathbf{V}^{ij})\vec{u} + \mathbf{V}^{ij}\dot{\vec{u}} \tag{12}$$

Acceleration of the mass center of the rigid body i and its angular acceleration are

$$\vec{a}^{i} = \frac{d}{dt}(\mathbf{V}^{i})\vec{u} + \mathbf{V}^{i}\dot{\vec{u}} \tag{13}$$

$$\dot{\vec{\omega}}^{i} = \frac{d}{dt}(\mathbf{\Gamma}^{i})\vec{u} + \mathbf{\Gamma}^{i}\dot{\vec{u}} \tag{14}$$

The rate of angular momentum of the rigid body i is written as:

$$\dot{\vec{H}} = \mathbf{I}^{i}\mathbf{\Gamma}^{i}\dot{\vec{u}} + \mathbf{I}^{i}\frac{d}{dt}(\mathbf{\Gamma}^{i})\vec{u} + \vec{\omega}^{i} \times \mathbf{I}^{i}\vec{\omega}^{i} \tag{15}$$

where $\mathbf{I}^{i}$ is the moment of inertia matrix for the rigid body i. Now the generalized inertial force can be written.

$$\begin{aligned} \vec{F}^{*} = & -\sum_{i=1}^{N^E}\sum_{j=1}^{E^i}\Big[\int_{V^{ij}} \rho^{ij}\Big(\mathbf{V}^{ij^T}\mathbf{V}^{ij}\dot{\vec{u}} + \mathbf{V}^{ij^T}\frac{d}{dt}(\mathbf{V}^{ij^T})\vec{u}\Big)dV\Big] - \sum_{i=1}^{N^s}\Big[m^{i}\Big(\mathbf{V}^{i^T}\mathbf{V}^{i}\dot{\vec{u}} + \mathbf{V}^{i^T}\frac{d}{dt}(\mathbf{V}^{i})\vec{u}\Big) \\ & + \mathbf{\Gamma}^{i^T}\mathbf{I}^{i}\mathbf{\Gamma}^{i}\dot{\vec{u}} + \mathbf{\Gamma}^{i^T}\mathbf{I}^{i}\frac{d}{dt}(\mathbf{\Gamma}^{i})\vec{u} + \mathbf{\Gamma}^{i^T}\vec{\omega}^{i} \times \mathbf{I}^{i}\vec{\omega}^{i}\Big] \end{aligned} \tag{16}$$

Partial velocities of equations (5) and (7) and the partial angular velocity of equation (8) can be expressed as

$$\mathbf{V}^{ij} = \frac{\partial \vec{v}^{ij}}{\partial \vec{u}^{T}} = \frac{\partial \vec{v}^{ij}}{\partial \dot{\vec{q}}^{T}}\mathbf{T} = \frac{\partial \vec{r}^{ij}}{\partial \vec{q}^{T}}\mathbf{T} = \mathbf{J}^{ij}\mathbf{T} \tag{17}$$

$$\mathbf{V}^{i} = \frac{\partial \vec{v}^{i}}{\partial \vec{u}^{T}} = \frac{\partial \vec{v}^{i}}{\partial \dot{\vec{q}}^{T}}\mathbf{T} = \frac{\partial \vec{r}^{i}}{\partial \vec{q}^{T}}\mathbf{T} = \mathbf{J}^{i}\mathbf{T} \tag{18}$$

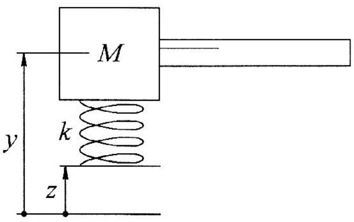

Figure 2. Lumped Mass, Cantilever Beam and Spring

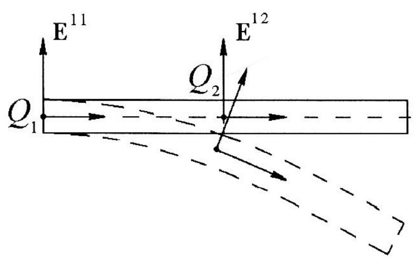

Figure 3. The Beam Divided into Two Elements

$$\mathbf{\Gamma}^{i} = \frac{\partial \vec{\omega}^{i}}{\partial \vec{u}^{T}} = \frac{\partial \vec{\omega}^{i}}{\partial \dot{\vec{q}}^{T}}\mathbf{T} = \mathbf{\Omega}^{i}\mathbf{T} \tag{19}$$

Substitution of equations (17), (18) and (19) in equations (9), (16) and (11) yields the following expression for the equations of motion

$$\mathbf{T}^T\mathbf{A}\mathbf{T}\dot{\vec{u}} = -\left\{\sum_{i=1}^{N^E}\sum_{j=1}^{E^i}\int_{V^{ij}} \rho^{ij}\mathbf{T}^T\mathbf{J}^{ij^T}\frac{d}{dt}(\mathbf{J}^{ij}\mathbf{T})dV + \sum_{i=1}^{N^s}[m^i\mathbf{T}^T\mathbf{J}^{i^T}\frac{d}{dt}(\mathbf{J}^i\mathbf{T}) + \mathbf{T}^T\Omega^{i^T}\mathbf{I}^i\frac{d}{dt}(\Omega^i\mathbf{T})]\right\}\vec{u}$$

$$+\left\{\sum_{i=1}^{N^E}\sum_{j=1}^{E^i}[\int_{V^{ij}}\mathbf{T}^T\mathbf{J}^{ij^T}\vec{b}^{ij}dV + \int_{S^{ij}}\mathbf{T}^T\mathbf{J}^{ij^T}\vec{f}^{ij}dS] + \sum_{i=1}^{N^s}[\mathbf{T}^T\mathbf{J}^{i^T}\vec{f}^i + \mathbf{T}^T\Omega^{i^T}\vec{M}^i]\right\} + \mathbf{T}^T\Omega^{i^T}\vec{\omega}^i\times\mathbf{I}^i\vec{\omega}^i - \mathbf{T}^T\vec{F}^s$$

$$\dot{\vec{q}} = \mathbf{T}\vec{u} \tag{20}$$

where

$$\mathbf{A} = \left\{\sum_{i=1}^{N^E}\sum_{j=1}^{E^i}\int_{V^{ij}}\rho^{ij}\mathbf{J}^{ij^T}\mathbf{J}^{ij}dV + \sum_{i=1}^{N^s}[m^i\mathbf{J}^{i^T}\mathbf{J}^i + \Omega^{i^T}\mathbf{I}^i\Omega^i]\right\} \tag{21}$$

It is now clear that $\mathbf{T}$ manifests itself in the transformation $\mathbf{T}^T\mathbf{A}\mathbf{T}$ in the first term of the left hand side of equation (20), as far as first order generalized speed components are concerned. Now the analyst can choose the rate transformation matrix $\mathbf{T}$ such that the first order generalized speed vector in equation (20) is diagonal. (For details of finding $\mathbf{T}$ see Loduha and Ravani (1994, 1995)).

5 Example of Decoupling with the Congruency Transformation

Consider a system consisting of a lumped mass attached to a spring and a clamped-free beam attached to the lumped mass, as shown in the Figure 2. This model can simply represent a robot clamp which is transporting an electronic board.

First we generate elastic generalized coordinates for the elastic beam, using finite elements and assumed modes. The beam is divided into two elements. These elements have the local frames $\mathbf{E}^{11}$ and $\mathbf{E}^{12}$ located at points Q_1 and Q_2, respectively. To define the deflection of the first element, a shape function satisfying the boundary conditions is considered. Since the first element is connected to the lumped mass at Q_1, its deflection and slope at Q_1 will be zero all the time. If the deflection and slope at the end point of the first element are $y_2(t)$ and $\theta_2(t)$ respectively, one can write the deflection in the element (see Figure 3).

$$w^{11} = (3\zeta^2 - 2\zeta^3)y_2(t) + l(\zeta^3 - \zeta^2)\theta_2(t) \tag{22}$$

where l is the length of the element and $0 \le \zeta \le 1$. If the deflection and the slope at the end point of the second element are $y_3(t)$ and $\theta_3(t)$ respectively, the shape function for the second element will be of the form:

$$\begin{aligned} w^{12} &= (1 - 3\zeta^2 + 2\zeta^3)y_2(t) + l(\zeta - 2\zeta^2 + \zeta^3)\theta_2(t) \\ &+ (3\zeta^2 - 2\zeta^3)y_3(t) + l(-\zeta^2 + \zeta^3)\theta_3(t) \end{aligned} \tag{23}$$

Thus the deflection of an arbitrary point in the first and second element, w^{11} and w^{12}, respectively, is written as:

$$w^{11} = \begin{bmatrix} 3\zeta^2 - 2\zeta^3 \\ l(\zeta^3 - \zeta^2) \\ 0 \\ 0 \end{bmatrix}^T \cdot \begin{bmatrix} y_2(t) \\ \theta_2(t) \\ y_3(t) \\ \theta_3(t) \end{bmatrix} \qquad w^{12} = \begin{bmatrix} 1 - 3\zeta^2 + 2\zeta^3 \\ l(\zeta - 2\zeta^2 + \zeta^3) \\ 3\zeta^2 - 2\zeta^3 \\ l(-\zeta^2 + \zeta^3) \end{bmatrix}^T \cdot \begin{bmatrix} y_2(t) \\ \theta_2(t) \\ y_3(t) \\ \theta_3(t) \end{bmatrix} \tag{24}$$

At this stage, four coordinates $y_2(t)$, $\theta_2(t)$, $y_3(t)$ and $\theta_3(t)$ are necessary to define the deflection of the body. To reduce the number of elastic coordinates, the standard component mode technique will be utilized. First the boundary conditions are imposed. That is, the bending moment is continuous at Q_2 and zero at point 3 (see Figure 3.). Thus, the following constraint equation can be written:

$$\begin{bmatrix} y_2(t) \\ \theta_2(t) \\ y_3(t) \\ \theta_3(t) \end{bmatrix} = \begin{bmatrix} 3/4 & -7l/12 \\ 3/4l & -1/4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_3(t) \\ \theta_3(t) \end{bmatrix} \tag{25}$$

Using the shape functions (22) and (23), the generalized mass and generalized stiffness matrix of the beam are computed.

$$\mathbf{M} = \begin{bmatrix} \frac{291}{280}\rho l & \frac{-827}{1680}\rho l^2 \\ \frac{-827}{1680}\rho l^2 & \frac{191}{630}\rho l^3 \end{bmatrix} \qquad \mathbf{K} = \frac{EI}{l^3}\begin{bmatrix} 3 & \frac{-7l}{2} \\ \frac{-7l}{2} & \frac{14l^2}{3} \end{bmatrix} \tag{26}$$

In order to find the corresponding mode shapes, the following eigenvalue problem is solved.

$$\mathbf{M}^{-1}\mathbf{K} - \mathbf{I}\omega^2 = \mathbf{0} \tag{27}$$

Therewith the mode shape matrix $\mathbf{X}^1$ is found. The nodal coordinates are written in terms of the mode amplitudes.

$$\mathbf{X}^1 = \begin{bmatrix} m_1 & m_3 \\ m_2 & m_4 \end{bmatrix} \qquad \begin{bmatrix} y_3(t) \\ \theta_3(t) \end{bmatrix} = \begin{bmatrix} m_1 & m_3 \\ m_2 & m_4 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{28}$$

Once the mode shapes are computed, one may choose the generalized coordinates. For simplicity, only the first mode shape is used here. Thus, the first mode amplitude η, and the position of the lumped mass with respect to a fix point on the ground y are chosen as the generalized coordinates. Now, the deflection of an arbitrary point in each element is written in terms of the generalized coordinate η.

$$w^{11} = \begin{bmatrix} 3\zeta^2 - 2\zeta^3 \\ l(\zeta^3 - \zeta^2) \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 3/4 & -7l/12 \\ 3/4l & -1/4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \eta = q^{11}(\zeta)\eta \tag{29}$$

$$w^{12} = \begin{bmatrix} 1 - 3\zeta^2 + 2\zeta^3 \\ l(\zeta - 2\zeta^2 + \zeta^3) \\ 3\zeta^2 - 2\zeta^3 \\ l(-\zeta^2 + \zeta^3) \end{bmatrix}^T \begin{bmatrix} 3/4 & -7l/12 \\ 3/4l & -1/4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \eta = q^{12}(\zeta)\eta \tag{30}$$

The position vector of an arbitrary point within the elements and the position vector of the lumped mass center may be written as:

$$\vec{r}^{11} = \begin{bmatrix} l\zeta \\ y + q^{11}(\zeta)\eta \end{bmatrix} \qquad \vec{r}^{12} = \begin{bmatrix} l + l\zeta \\ y + q^{12}(\zeta)\eta \end{bmatrix} \qquad \vec{r}^{2} = \begin{bmatrix} 0 \\ y \end{bmatrix} \tag{31}$$

Using the generalized coordinates η and y, we assemble the matrix $\mathbf{A}$ by finding $\mathbf{J}^{11}$, $\mathbf{J}^{12}$ and $\mathbf{J}^2$ from equations (17) and (18). We obtain

$$\mathbf{J}^{11} = \begin{bmatrix} 0 & 0 \\ 1 & q^{11}(\zeta) \end{bmatrix} \qquad \mathbf{J}^{12} = \begin{bmatrix} 0 & 0 \\ 1 & q^{12}(\zeta) \end{bmatrix} \qquad \mathbf{J}^{2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{32}$$

and combine them as in equation (21) to obtain

$$\mathbf{A}=\begin{bmatrix} 2\rho l+M & \frac{\rho l}{12}(15m_1-8m_2) \\ \frac{\rho l}{12}(15m_1-8m_2) & \frac{\rho l}{2520}(764m_2^2l^2+2619m_1^2-2481m_1m_2l) \end{bmatrix} \tag{33}$$

We form the congruency transformation $\mathbf{T}$ for $\mathbf{A}$ as

$$\mathbf{T}=\begin{bmatrix} 1 & \dfrac{\rho l(8m_2l-15m_1)}{2(2\rho l+M)} \\ 0 & 1 \end{bmatrix} \tag{34}$$

The following surface traction and body forces are acting on the elements and the force of the spring is

$$\vec{f}^{11}=\vec{f}^{12}=\vec{0} \qquad \vec{b}^{11}=\vec{b}^{12}=\begin{bmatrix} 0 \\ -\dfrac{\rho g}{A} \end{bmatrix} \qquad \vec{f}^{2}=\begin{bmatrix} 0 \\ -k(y-z)-Mg \end{bmatrix} \tag{35}$$

where A is the cross sectional area of the beam. The generalized internal force is obtained using the generalized stiffness of the beam and the mode shape matrix

$$S=\mathbf{m}^T\mathbf{K}\mathbf{m}\eta=\frac{EI}{l^3}(3m_1^2-7m_1m_2l+\frac{14}{3}m_2^2l^2)\eta \tag{36}$$

$$\vec{F}^{s}=\begin{bmatrix} 0 \\ S \end{bmatrix} \tag{37}$$

Now using equation (20), we can write Kane's equations of motion as follows

$$\dot{u}_1=\frac{-k(y-z)-(2\rho l+M)g}{2\rho l+M} \tag{38}$$

$$\dot{u}_2=\frac{420}{z_1}[(8m_2l-15m_1)k(y-z)+\frac{EI}{l^4}(2\rho l+M)(9m_1^2-21m_1m_2+14m_2^2l^2)\eta] \tag{39}$$

$$\dot{y}=u_1+\frac{\rho l(8m_2l-15m_1)}{12(2\rho l+M)}u_2 \tag{40}$$

$$\dot{\eta}=u_2 \tag{41}$$

where

$$z_1=\rho l(2601m_1^2-1524m_1m_2l+816m_2^2l^2)+M(5238m_1^2-4962m_1m_2l+1528m_2^2l^2) \tag{42}$$

It can be seen that the obtained equations of motion are decoupled in the first order terms and are ready for numerical integration.

6 Numerical Simulation

The following numerical values where used for simulating the above model.

$$\rho = 0.078\frac{\text{kg}}{\text{m}}; \qquad l = 0.25\text{ m}; \qquad E = 2.1\times10^{11}\frac{\text{N}}{\text{m}^2}; \qquad I = 8.33\times10^{-13}\text{ m}^4; \qquad M = 0.2\text{ kg}; \qquad k = 1000\frac{\text{N}}{\text{m}}$$

First the eigenvalue problem of equation (27) was solved to find the mode shapes of the beam. The answer was

$$\mathbf{X}^1 = \begin{bmatrix} -0.09626 & -0.14587 \\ -0.27094 & -1.44942 \end{bmatrix} \qquad \begin{matrix} \omega_1 = 20.61 \\ \omega_2 = 130.4 \end{matrix} \quad \text{(rad/s)} \tag{43}$$

The deflection of an arbitrary point in each element due to these mode shapes are drawn in Figure 4 and Figure 5.

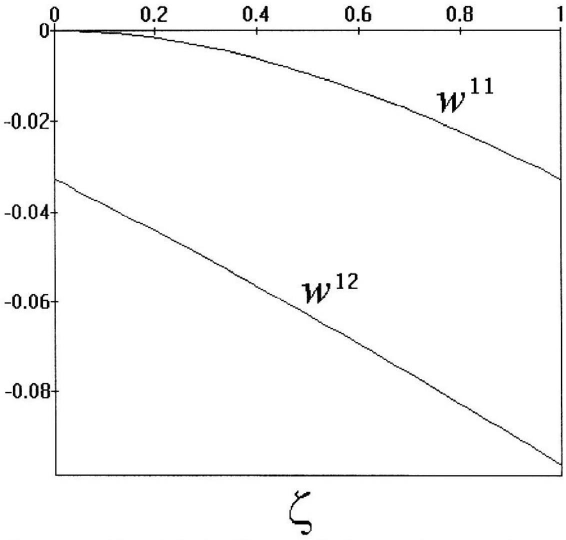

Figure 4. First Mode Shape of First and Second Element ($\eta_1 = 1$)

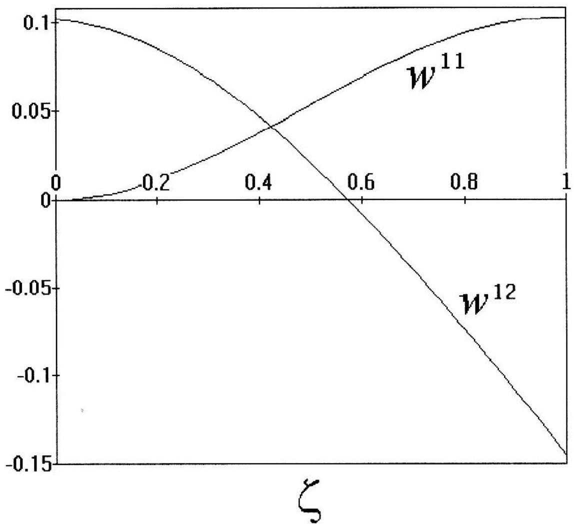

Figure 5. Second Mode Shape of First and Second Element ($\eta_2 = 1$)

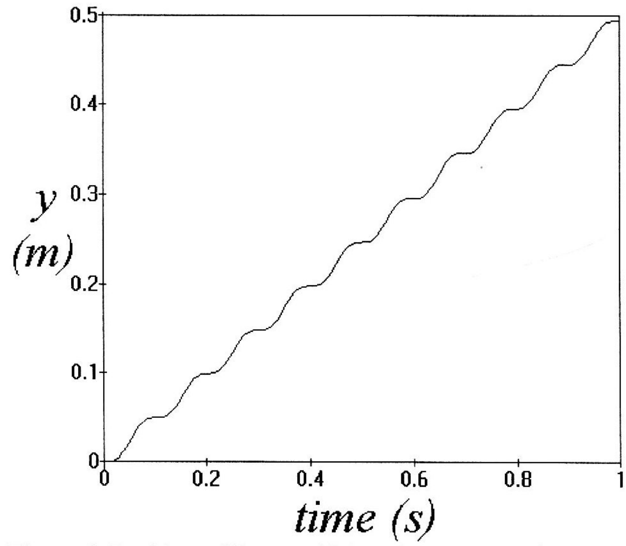

Figure 6. Position of Lumped Mass (m) Versus Time (s)

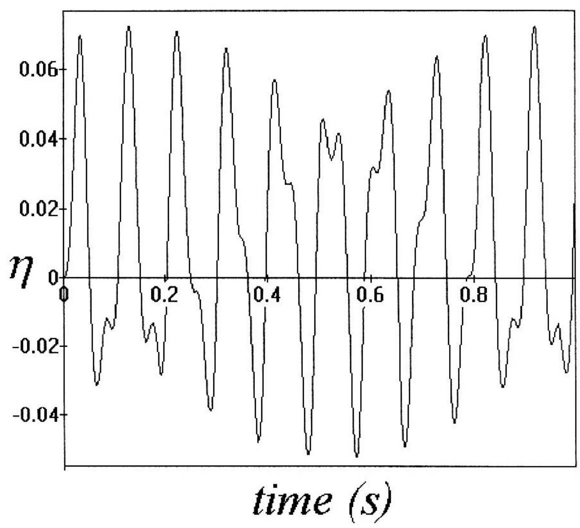

Figure 7. First Mode Amplitude Versus Time (s)

Only the first mode shape was used to derive the equations of motion ($\eta = \eta_1$). It was assumed that the system is at rest initially and the beam has no initial deflection. Then the spring is suddenly given a constant speed of

$0.5\,\text{m/s}$ (i.e. $z = 0.5t$). Time history of the lumped mass position y and first mode shape amplitude η are shown in Figure 6 and Figure 7, respectively.
It can be seen that the maximum of the first mode amplitude is about 0.07 which corresponds to a tip deflection of about 7 mm.

7 Numerical Implementation of First Order Decoupling

We now discuss the numerical implementation of a congruency transformation to be used when a large scale system is encountered and it would be very tedious to symbolically obtain the proper congruency transformation. Again consider the expressions of equations of motion in equations (20) and (21). Here, we do not require the matrix $\mathbf{T}$ to be a diagonalizing congruency transformation. In this case $\mathbf{T}$ may be any matrix that yields a valid vector of generalized velocity components. For example, it is common to chose a trivial set of generalized velocity components by selecting $\mathbf{T}$ as the identity matrix. The matrix product premultiplying $\dot{\mathbf{u}}$ in equation (20) is written as

$$\mathbf{A}_1 = \mathbf{T}^T \left\{ \sum_{i=1}^{N^E} \sum_{j=1}^{E^i} \int_{V^{ij}} \rho^{ij} \mathbf{J}^{ij^T} \mathbf{J}^{ij} dV + \sum_{i=1}^{N^s} [m^i \mathbf{J}^{i^T} \mathbf{J}^i + \mathbf{\Omega}^{i^T} \mathbf{I}^i \mathbf{\Omega}^i] \right\} \mathbf{T} \tag{44}$$

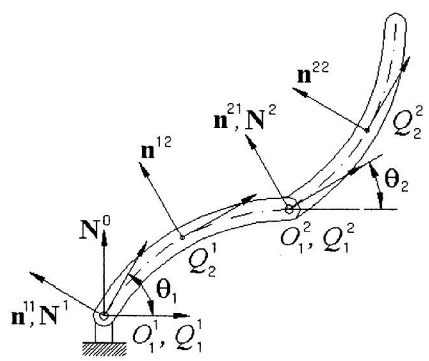

Figure 8. A two-Degree-of-Freedom Flexible Robot

and the right hand side of equation (20) is written as $\mathbf{h}(\mathbf{q},\mathbf{u})$. Thus the equations of motion become:

$$\mathbf{A}_1\dot{\mathbf{u}} = \mathbf{h}(\mathbf{q},\mathbf{u}) \tag{45}$$

The vector $\mathbf{x}$ is introduced in the following expression:

$$\dot{\mathbf{u}} = \mathbf{\Psi}\mathbf{x} \tag{46}$$

where ψ is independent of the original dynamical equations (20). Substituting this into equation (45) yields

$$\mathbf{A}_1\psi\mathbf{x} = \mathbf{h}(\mathbf{q},\mathbf{u}) \tag{47}$$

We can now premultiply both sides of the above equation by $\mathbf{\Psi}^T$ to get

$$\mathbf{\Psi}^T\mathbf{A}_1\mathbf{\Psi}\mathbf{x} = \mathbf{\Psi}^T\mathbf{h}(\mathbf{q},\mathbf{u}) \tag{48}$$

If $\mathbf{\Psi}$ is chosen as a congruency transformation of the symmetric matrix $\mathbf{A}$, $\mathbf{\Psi}^T\mathbf{A}_1\mathbf{\Psi}$ will be diagonal. Let $\mathbf{L} = \mathbf{\Psi}^T\mathbf{A}_1\mathbf{\Psi}$ then equation (48) becomes

$$\mathbf{L}\mathbf{x} = \mathbf{\Psi}^T\mathbf{h}(\mathbf{q},\mathbf{u}) \tag{49}$$

The inverse of $\mathbf{L}$ simply comprises of the reciprocals of its diagonal elements. The vector $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{L}^{-1}\mathbf{\Psi}^{T}\mathbf{h}(\mathbf{q},\mathbf{u}) \tag{50}$$

Finally the first order decoupled form is obtained

$$\dot{\mathbf{u}} = \mathbf{\Psi}\mathbf{L}^{-1}\mathbf{\Psi}^{T}\mathbf{h}(\mathbf{q},\mathbf{u}) \tag{51}$$

which is well structured for direct numerical integration.

8 An Exemple for Numerical Implementation

Consider a two-degree-of-freedom planar robot with flexible links. The length, mass and material of the two links are assumed to be the same. The inertial reference frame $\mathbf{N}^0$ is located at the ground pivot of the first link. The local frame of the first link $\mathbf{N}^1$ is attached to a point O_1^1 of it. This link is divided into two elements with local frames $\mathbf{n}^{11}$ and $\mathbf{n}^{12}$. At point O_1^2, the second link is pivoted to the first link. Its local frame $\mathbf{N}^2$ is located at point O_1^2. This link is divided into two elements with local frames $\mathbf{n}^{21}$ and $\mathbf{n}^{22}$. As described in section 5, the deflection of an arbitrary point in the first and second link can be written:

$$w^{11} = q^{11}(\zeta)\eta_1 \qquad w^{12} = q^{12}(\zeta)\eta_1 \qquad w^{21} = q^{21}(\zeta)\eta_2 \qquad w^{22} = q^{22}(\zeta)\eta_2 \tag{52}$$

where η_1 and η_2 are the first mode shape amplitudes of the first and second link, respectively.
The generalized coordinates vector will be

$$\mathbf{q} = [\theta_1\ \eta_1\ \theta_2\ \eta_2]^T \tag{53}$$

where θ_1 is the angle between frames $\mathbf{N}^1$ and $\mathbf{N}^0$, and θ_2 is the angle between frames $\mathbf{N}^2$ and $\mathbf{N}^0$. Using the deflections obtained in equation (52), one can write the position vector of an arbitrary point in each of the elements of the links.
Position vector of an arbitrary point in the first element of the first link:

$$\mathbf{r}^{11} = \begin{bmatrix} l\zeta\cos\theta_1 - q^{11}(\zeta)\eta_1\sin\theta_1 \\ l\zeta\sin\theta_1 + q^{11}(\zeta)\eta_1\cos\theta_1 \end{bmatrix} \qquad 0 \le \zeta \le 1 \tag{54}$$

Position vector of an arbitrary point in the second element of the first link:

$$\mathbf{r}^{12} = \begin{bmatrix} (l+l\zeta)\cos\theta_1 - q^{12}(\zeta)\eta_1\sin\theta_1 \\ (l+l\zeta)\sin\theta_1 + q^{12}(\zeta)\eta_1\cos\theta_1 \end{bmatrix} \qquad 0 \le \zeta \le 1 \tag{55}$$

Position vector of an arbitrary point in the first element of the second link:

$$\mathbf{r}^{21} = \begin{bmatrix} 2l\cos\theta_1 - q^{12}(1)\eta_1\sin\theta_1 + l\zeta\cos\theta_2 - q^{21}(\zeta)\eta_2\sin\theta_2 \\ 2l\sin\theta_1 + q^{12}(1)\eta_1\cos\theta_1 + l\zeta\sin\theta_2 + q^{21}(\zeta)\eta_2\cos\theta_2 \end{bmatrix} \qquad 0 \le \zeta \le 1 \tag{56}$$

Position vector of an arbitrary point in the second element of the second link:

$$\mathbf{r}^{22} = \begin{bmatrix} 2l\cos\theta_1 - q^{12}(1)\eta_1\sin\theta_1 + (l+l\zeta)\cos\theta_2 - q^{22}(\zeta)\eta_2\sin\theta_2 \\ 2l\sin\theta_1 + q^{12}(1)\eta_1\cos\theta_1 + (l+l\zeta)\sin\theta_2 + q^{22}(\zeta)\eta_2\cos\theta_2 \end{bmatrix} \qquad 0 \le \zeta \le 1 \tag{57}$$

Now, Jacobian matrices are computed from equation (17).

$$\mathbf{J}^{11} = \begin{bmatrix} -l\zeta\sin\theta_1 - q^{11}(\zeta)\eta_1\cos\theta_1 & -q_{11}(\zeta)\sin\theta_1 & 0 & 0 \\ l\zeta\cos\theta_1 - q^{11}(\zeta)\eta_1\sin\theta_1 & q_{11}(\zeta)\cos\theta_1 & 0 & 0 \end{bmatrix} \tag{58}$$

$$\mathbf{J}^{12} = \begin{bmatrix} -(l+l\zeta)\sin\theta_1 - q^{12}(\zeta)\eta_1\cos\theta_1 & -q^{12}(\zeta)\sin\theta_1 & 0 & 0 \\ (l+l\zeta)\cos\theta_1 - q^{12}(\zeta)\eta_1\sin\theta_1 & q^{12}(\zeta)\cos\theta_1 & 0 & 0 \end{bmatrix} \tag{59}$$

$$\mathbf{J}^{21} = \begin{bmatrix} -2l\sin\theta_1 - q^{12}(1)\eta_1\cos\theta_1 & -q^{12}(1)\sin\theta_1 & -l\zeta\sin\theta_2 - q^{21}(\zeta)\eta_2\cos\theta_2 & -q^{21}(\zeta)\sin\theta_2 \\ 2l\cos\theta_1 - q^{12}(1)\eta_1\sin\theta_1 & q^{12}(1)\cos\theta_1 & l\zeta\cos\theta_2 - q^{21}(\zeta)\eta_2\sin\theta_2 & q^{21}(\zeta)\cos\theta_2 \end{bmatrix} \tag{60}$$

$$\mathbf{J}^{22} = \begin{bmatrix} -2l\sin\theta_1 - q^{12}(1)\eta_1\cos\theta_1 & -q^{12}(1)\sin\theta_1 & -(l+l\zeta)\sin\theta_2 - q^{22}(\zeta)\eta_2\cos\theta_2 & -q^{22}(\zeta)\sin\theta_2 \\ 2l\cos\theta_1 - q^{12}(1)\eta_1\sin\theta_1 & q^{12}(1)\cos\theta_1 & (l+l\zeta)\cos\theta_2 - q^{22}(\zeta)\eta_2\sin\theta_2 & q^{22}(\zeta)\cos\theta_2 \end{bmatrix} \tag{61}$$

The matrix $\mathbf{A}$ is computed from equation (21).

$$\mathbf{A} = \rho l \left[\begin{array}{c|c|c|c} \frac{32}{3}l^2 + z_1\eta_1^2 & 4q^{12}(1)l + z_2 l & \begin{array}{c} z_3 q^{12}(1)\eta_1\eta_2 c_{12} + 2z_3 l\eta_2 s_{12} + \\ 4l^2 c_{12} - 2q^{12}(1)\eta_1 l s_{12} \end{array} & 2z_3 l c_{12} - z_3 q^{12}(1)\eta_1 s_{12} \\ \hline & 2q^{12}(1)^2 + z_1 & 2q^{12}(1)^2 + z_1 & z_3 q^{12}(1)\eta_2 c_{12} \\ \hline & & \frac{8}{3}l^2 + z_1\eta_2^2 & z_2 l \\ \hline SYM. & & & z_1 \end{array} \right] \tag{62}$$

where

$$z_1 = \frac{291}{280}m_1^2 + \frac{191}{630}m_2^2 l^2 - \frac{827}{840}m_1 m_2 l \tag{63}$$

$$z_2 = \frac{33}{20}m_1 - \frac{11}{15}m_2 l \tag{64}$$

$$z_3 = \frac{5}{4}m_1 - \frac{2}{3}m_2 l \tag{65}$$

$$c_{12} = \cos(\theta_1 - \theta_2) \qquad s_{12} = \sin(\theta_1 - \theta_2)$$

Now, time derivatives of the Jacobian matrices are computed. In order to complete the equation of motion (20), we compute the following.

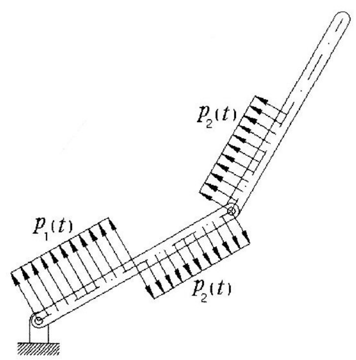

Figure 9. Traction Forces on Elements

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{V^{ij}} \rho^{ij}\mathbf{J}^{ij^T}\frac{d}{dt}(\mathbf{J}^{ij})dV =$$

$$\rho l\left[\begin{array}{c|c|c|c} z_1\eta_1\dot{\eta}_1 & (2q^{12}(1)^2+z_1)\eta_1\dot{\theta}_1 & \begin{array}{l} z_3(q^{12}(1)\eta_1\dot{\eta}_2-2l\eta_2\dot{\theta}_2)c_{12} \\ +z_3(q^{12}(1)\eta_1\eta_2\dot{\theta}_2-2l\dot{\eta}_2)s_{12} \\ +2q^{12}(1)\eta_1\dot{\theta}_2lc_{12}+4l^2\dot{\theta}_2s_{12} \end{array} & z_3(2ls_{12}+q^{12}(1)\eta_1c_{12})\dot{\theta}_2 \\ \hline -(2q^{12}(1)^2+z_1)\eta_1\dot{\theta}_1 & 0 & \begin{array}{l} z_3q^{12}(1)(\dot{\eta}_2s_{12}-\eta_2\dot{\theta}_2c_{12}) \\ +2q^{12}(1)l\dot{\theta}_2s_{12} \end{array} & z_3q^{12}(1)\dot{\theta}c_2s_{12} \\ \hline \begin{array}{l} z_3(q^{12}(1)\dot{\eta}_1\eta_2-2l\eta_2\dot{\theta}_1)c_{12} \\ +z_3q^{12}(1)\eta_1\eta_2\dot{\theta}_1s_{12} \\ -(2lq^{12}(1)\dot{\eta}_1+4l^2\dot{\theta}_1)s_{12} \end{array} & q^{12}(1)\dot{\theta}_1(z_3\eta_2c_{12}+2ls_{12}) & z_1\eta_2\dot{\eta}_2 & z_1\eta_2\dot{\theta}_2 \\ \hline \begin{array}{l} -z_3(2l\dot{\theta}_1+q^{12}(1)\dot{\eta}_1)s_{12} \\ +z_3q^{12}(1)\eta_1\dot{\theta}_1c_{12} \end{array} & -z_3q^{12}(1)\dot{\theta}_1s_{12} & z_1\eta_2\dot{\theta}_2 & 0 \end{array}\right] \qquad (66)$$

We assume gravitation as a body force. Thus the body force term in equation (20) becomes:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}[\int_{V^{ij}}\mathbf{J}^{ij^T}\vec{b}^{ij}dV=\rho l\begin{bmatrix}(z_3+2q^{12}(1))\eta_1gs_1-6gl\cos\theta_1 \\ -(z_3+2q^{12}(1))gc_1 \\ z_3g\eta_2s_1-2glc_2 \\ -z_3gc_2\end{bmatrix} \qquad (67)$$

where

$$s_1=\sin\theta_1 \qquad c_1=\cos\theta_1 \qquad s_2=\sin\theta_2 \qquad c_2=\cos\theta_2$$

Assuming traction forces as shown in Figure 9, the traction force term in equation (20) will be computed.

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{S^{ij}}\mathbf{J}^{ij^T}\vec{f}^{ij}dS=\begin{bmatrix} M_1(t)+(\frac{4c_{12}-3-2q^{12}(1)\eta_1s_{12}}{l})M_2(t) \\ (\frac{5m_1}{8l}-\frac{13m_2}{24})M_1(t)+(\frac{q^{12}(1)c_{12}}{l}+\frac{19m_2}{24}-\frac{15m_1}{8l})M_2(t) \\ M_2(t) \\ (\frac{5m_1}{8l}-\frac{13m_2}{24})M_2(t)\end{bmatrix} \qquad (68)$$

where

$$M_1(t)=\frac{p_1(t)l^2}{2} \qquad M_2(t)=\frac{p_2(t)l^2}{2} \qquad (69)$$

are equivalent joint moments. Now, the equations of motion for the mentioned robot are obtained by substituting equations (62), (66), (67), and (68) into equation (20). A numerical example will be presented in the following.

9 Numerical Simulation

In order to perform numerical integrations, the following numerical values were assumed.

$l=1\,\mathrm{m}$, $\rho=1\frac{\mathrm{kg}}{\mathrm{m}}$, $E=2\times10^{11}\frac{\mathrm{N}}{\mathrm{m}^2}$, $I=1.3\times10^{-9}\,\mathrm{m}^4$, $\mathrm{m}_1=-0.579\,\mathrm{m}$, $\mathrm{m}_2=-0.815\,\mathrm{m}$, $M_1(t)=M_2(t)=0\,\mathrm{Nm}$

The matrix $\mathbf{A}_1$ is computed in each time step. Then, its corresponding congruency transformation $\mathbf{\Psi}$ is derived and $\mathbf{L}$ is obtained. Finally, the first order decoupled form equation (51) is formed and integrated numerically. The following numerical results are obtained.

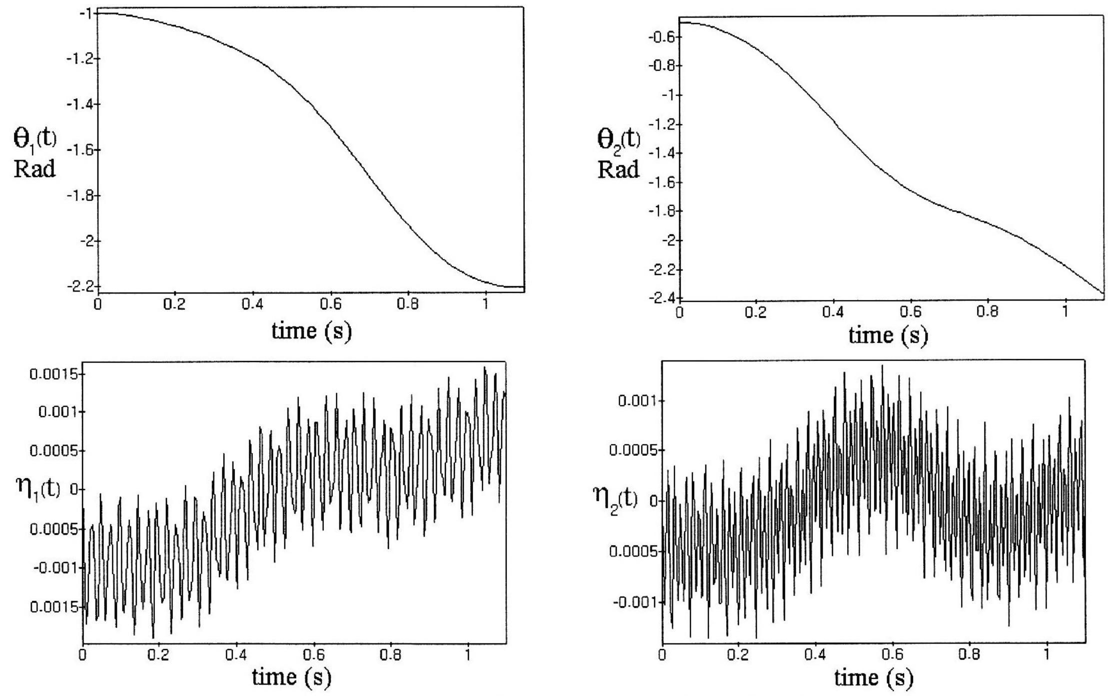

Figure 10. Trajectory of Generalized Coordinates

10 Conclusion

A method of finding elastic generalized coordinates has been considered in which a finite element analysis is done prior to finding component mode shapes. In this way, the nonlinear geometric stiffening can be automatically incorporated through proper finite element formulation. Then, a method for creating first order decoupled equations of motion for multibody systems consisting of rigid and elastic bodies has been proposed and demonstrated through an example. This is achieved using a matrix form of Kane's equations of motion in conjunction with choosing a proper congruency transformation between derivatives of the generalized coordinates and the generalized speeds. The resulting equations are in a form convenient for nonlinear behavior analysis. Moreover, it becomes an easy matter to implement general integration routines for first order differential equations to obtain the generalized coordinate trajectories for simulation purposes. Finally, numerical implementation of decoupling using congruency transformation is discussed and shown via simulation of a two degree of freedom flexible robot.

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