Modelling of Elastic Deformation for Initially Anisotropic Materials Sustaining Unilateral Damage

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A continuum unilateral damage mechanics model for elastic deformation of initially anisotropic materials is presented. This model describes simultaneously initial anisotropy, the difference between the damaging processes under compressive and tensile loading types, and damage induced anisotropy. The proposed equation for elastic deformation and proposed evolution equations contain joint invariants of the stress tensor and some material tensors. A scalar cumulative damage parameter is considered and the damage growth equation is formulated. It is shown that under the natural condition that the equivalent stress is always non-negative the second thermodynamic principle is always valid. Specific constitutive equations with a smaller number of material parameters and invariants are obtained. It is shown how these material parameters can be determined from a series of basic experiments.

1 Introduction

In the past two decades, in particular, Continuum Unilateral Damage Mechanics (CUDM) has been applied to creep deformation (Betten et al., 1987, 1998; Zolochevsky, 1988, 1991; Qi and Bertram, 1997), elastic deformation (Lemaitre, 1987; Chaboche, 1992, 1993; Ladeveze et al., 1994; Lubarda et al., 1994; Shan et al., 1994; Chaboche et al., 1995; Halm and Dragon, 1996; Krajcinovic, 1996; Yazdani and Karnavat, 1997), fatigue behaviour (H. Altenbach, J. Altenbach and Zolochevsky, 1995). A systematic consideration of the basic aspects of CUDM can be found in papers by Chaboche (1992, 1993). One of the difficulties in the CUDM is connected with the simultaneous description of the anisotropic nature of damage and the difference between the damaging processes under tensile and compressive loading types (Chaboche, 1992). Even in the elastic case, up to now there are difficulties to reproduce simultaneously the initial anisotropy, the different damage in tension and compression, and the damage induced anisotropy (Chaboche et al., 1995). In most papers under consideration the concept of positive and negative projections of stress and strain tensors was used to account for tensile and compressive loading types. The aim of this paper is to consider a new unilateral damage model for elastic deformation without using the positive and negative stress and strain projection operators. In the following small elastic strains in isothermal processes are considered within the framework of the phenomenological macroscopic approach of CUDM.

2 Formulation of Constitutive Equations

To describe the anisotropic damage we can start from the consideration of elastic deformation in an isotropic medium. For this purpose the mentioned "isotropic concept" (Betten, 1981) will be used by substituting a mapped stress tensor. First let us consider the isotropic case without damage.
The stress-strain relations for elastic behaviour are based on the assumption of the existence of a potential F. For isotropic materials this potential $F = F(\sigma)$ is a scalar-valued tensor function of the Cauchy stress tensor σ. It is evident from the theory of isotropic tensor functions (Betten, 1986) that in an isotropic medium an elastic potential

$$F = F\left[J_1(\sigma), J_2(\sigma), J_3(\sigma)\right] \tag{1}$$

can depend only on the invariants

$$J_1(\sigma) = \sigma \cdot\cdot \mathbf{I} \tag{2a}$$
$$J_2(\sigma) = \sigma \cdot\cdot \sigma \tag{2b}$$
$$J_3(\sigma) = \sigma \cdot\cdot (\sigma \cdot \sigma) \tag{2c}$$

of the Cauchy stress tensor. Here, the symbol (·) denotes the scalar product, **I** represents unit second-order tensor. Taking into consideration the above mentioned assumption, the potential F in equation (1) can be written in the form

$$F = \frac{1}{2}\sigma_e^2 \tag{3}$$

where the equivalent stress σ_e has the following structure (Zolochevskij, 1988)

$$\sigma_e = \sigma_2 + \alpha\sigma_1 + \gamma\sigma_3 \tag{4}$$

Here

$$\sigma_1 = B\, J_1(\sigma) \tag{5a}$$

$$\sigma_2^2 = A\, J_1^2(\sigma) + C\, J_2(\sigma) \tag{5b}$$

$$\sigma_3^3 = D\, J_1^3(\sigma) + K\, J_1(\sigma)\, J_2(\sigma) + L\, J_3(\sigma) \tag{5c}$$

are scalar functions of the stress invariants; A, B, C, D, K, L are material parameters; α, γ are weight coefficients taking into account the influence of linear and cubic polynomials in the expression (4) for σ_e. The representation (4) is a general form. For example, by placing in (4), (5b) $\alpha = \gamma = 0$, $A = -\dfrac{v}{E}$, $C = \dfrac{1+v}{E}$ we arrive (Betten, 1993) at the equivalent stress $\sigma_e = \sqrt{\dfrac{1+v}{E}J_2 - \dfrac{v}{E}J_1^2}$ in the classical linear elastic potential on the base of the initial elastic modulus E and the Poisson's ratio v.

By analogy with the concept of Krajcinovic and Fonseka (1981), let us describe the stress-strain state of the damaged isotropic material on the basis of the relations (1)-(5) for the undamaged material with the parameters A, B, C, D, K, L by replacing the damage scalar functions $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{K}, \widetilde{L}$, respectively.

We then use the assumption (Betten, 1981) that the anisotropy of the material is entirely involved in a fourth rank tensor $\mathbf{M}^{(4)}$, and the anisotropic behaviour is described by the linear transformation

$$\tau = \mathbf{M}^{(4)} \cdot\cdot \sigma \tag{6}$$

where σ is the Cauchy stress tensor in an anisotropic undamaged medium. By analogy with relations (5a,b,c) the basic invariants of the image tensor (6) are given by:

$$\sigma_1(\tau) \equiv \sigma_1(\widetilde{\mathbf{b}}, \sigma) = \widetilde{\mathbf{b}} \cdot\cdot \sigma \tag{7a}$$

$$\sigma_2^2(\tau) \equiv \sigma_2^2(\widetilde{\mathbf{a}}, \tau) = \sigma \cdot\cdot \widetilde{\mathbf{a}} \cdot\cdot \sigma \tag{7b}$$

$$\sigma_3^3(\tau) \equiv \sigma_3^3(\widetilde{\mathbf{c}}, \sigma) = \sigma \cdot\cdot \left(\sigma \cdot\cdot \widetilde{\mathbf{c}} \cdot\cdot \sigma\right) \tag{7c}$$

if we define the material tensors $\widetilde{\mathbf{b}}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{c}}$ as:

$$\widetilde{b}_{ij} \equiv \widetilde{B}\, M_{ijpp} \tag{8a}$$

$$\widetilde{a}_{ijkl} = \widetilde{A}\, M_{ijpp} M_{klqq} + \widetilde{C}\, M_{ijpq} M_{klpq} \tag{8b}$$

$$\widetilde{c}_{ijklmn} = \widetilde{D}\, M_{ijpp} M_{klqq} M_{mnrr} + \widetilde{K}\, M_{ijpp} M_{klqr} M_{mnqr} + \widetilde{L}\, M_{ijrs} M_{klsp} M_{mnpr} \tag{8c}$$

Here we use Einstein's summation convention. The invariants (7a,b,c) are elements of the system of joint invariants, which are the only considered ones, and the material tensors (8a,b,c) are influenced by damage. As known, the elastic infinitesimal strain tensor ε is determined by the rule

$$\varepsilon = \frac{\partial F}{\partial \sigma} \tag{9}$$

Therefore, using equations (3),(4) and the relations

$$\frac{\partial F}{\partial \sigma} = \sigma_e\left(\frac{\partial \sigma_2}{\partial \sigma} + \alpha\frac{\partial \sigma_1}{\partial \sigma} + \gamma\frac{\partial \sigma_3}{\partial \sigma}\right) \tag{10a}$$

$$\frac{\partial \sigma_2}{\partial \sigma} = \frac{\widetilde{\mathbf{a}} \cdot\cdot \sigma}{\sigma_2} \tag{10b}$$

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$$\frac{\partial \sigma_1}{\partial \sigma} = \widetilde{\mathbf{b}} \tag{10c}$$

$$\frac{\partial \sigma_3}{\partial \sigma} = \frac{\sigma \cdot\cdot \widetilde{\mathbf{c}} \cdot\cdot \sigma}{\sigma_3^2} \tag{10d}$$

we arrive at the constitutive equation

$$\varepsilon = \sigma_e \left(\frac{\widetilde{\mathbf{a}} \cdot\cdot \sigma}{\sigma_2} + \alpha \widetilde{\mathbf{b}} + \gamma \frac{\sigma \cdot\cdot \widetilde{\mathbf{c}} \cdot\cdot \sigma}{\sigma_3^2} \right) \tag{11}$$

for the damaged anisotropic materials. This equation can be represented in canonical form (Betten, 1993)

$$\varepsilon = \mathbf{H} + \mathbf{P}^{(4)} \cdot\cdot \sigma + \left(\mathbf{Q}^{(6)} \cdot\cdot \sigma \right) \cdot\cdot \sigma \tag{12}$$

where $\mathbf{H}, \mathbf{P}^{(4)}, \mathbf{Q}^{(6)}$ are tensor functions depending on the stress tensor and the material tensors $\widetilde{\mathbf{b}}, \widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{c}}$. The constitutive equation (11) has a non-linear tensor structure and is a general form. That is why we analyse possible specific relations, resulting from equation (11) and containing a smaller number of parameters. For example, if we assume $\gamma = 0$, we arrive at the linear tensor equation

$$\varepsilon = \sigma_e \left(\frac{\widetilde{\mathbf{a}} \cdot\cdot \sigma}{\sigma_2} + \alpha \widetilde{\mathbf{b}} \right) \tag{13}$$

where

$$\sigma_e = \sigma_e (\sigma, \widetilde{\mathbf{b}}, \widetilde{\mathbf{a}}) \tag{14}$$

In the case of $\alpha = 0$, equation (11) is transformed into the non-linear tensor equation

$$\varepsilon = \sigma_e \left(\frac{\widetilde{\mathbf{a}} \cdot\cdot \sigma}{\sigma_2} + \gamma \frac{\sigma \cdot\cdot \widetilde{\mathbf{c}} \cdot\cdot \sigma}{\sigma_3^2} \right) \tag{15}$$

with the equivalent stress

$$\sigma_e = \sigma_e (\sigma, \widetilde{\mathbf{a}}, \widetilde{\mathbf{c}}) \tag{16}$$

Using the conditions $\alpha = \gamma = 0$, from the relation (11) we obtain Hooke's law

$$\varepsilon = \widetilde{\mathbf{a}} \cdot\cdot \sigma \tag{17}$$

for initially anisotropic damaged material with the same behaviour in tension and compression.

The rule (9) is compatible with the tensor function theory in the isotropic special case provided additional conditions of integrability have been fulfilled. In more complicated cases, i.e., when the potential in (9) is a function not only of the stress tensor but also of the damage tensor or the initial anisotropic material tensor, the rule (9) furnishes only restricted forms of constitutive equations, even if a general potential has been assumed (Betten, 1985).

3 Thermodynamic Consideration

Since the potential (3) is a homogeneous positive definite function of second degree, then, according to Euler's theorem, we find:

$$W \equiv \sigma \cdot\cdot \varepsilon = 2F \tag{18}$$

The potential (3) is the density of the elastic strain energy, which can be interpreted as the thermodynamic potential, in which the thermodynamic parameters of the state are the components of the stress tensor and the material tensors.

Now we obtain the damage energy release rates (see Chaboche, 1992):

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$$\mathbf{y} = \frac{\partial F}{\partial \widetilde{\mathbf{b}}}$$ (19a)

$$\mathbf{Y} = \frac{\partial F}{\partial \widetilde{\mathbf{a}}}$$ (19b)

$$\mathbf{Y}^* = \frac{\partial F}{\partial \widetilde{\mathbf{c}}}$$ (19c)

It follows from equations (3),(4),(7a,b,c) that

$$\frac{\partial F}{\partial \widetilde{\mathbf{b}}} = \alpha \sigma_e \frac{\partial \sigma_1}{\partial \widetilde{\mathbf{b}}}$$ (20a)

$$\frac{\partial F}{\partial \widetilde{\mathbf{a}}} = \sigma_e \frac{\partial \sigma_2}{\partial \widetilde{\mathbf{a}}}$$ (20b)

$$\frac{\partial F}{\partial \widetilde{\mathbf{c}}} = \gamma \sigma_e \frac{\partial \sigma_3}{\partial \widetilde{\mathbf{c}}}$$ (20c)

$$\frac{\partial \sigma_1}{\partial \widetilde{\mathbf{b}}} = \sigma$$ (20d)

$$\frac{\partial \sigma_2}{\partial \widetilde{\mathbf{a}}} = \frac{\sigma \otimes \sigma}{2\sigma_2}$$ (20e)

$$\frac{\partial \sigma_3}{\partial \widetilde{\mathbf{c}}} = \frac{\sigma \otimes \sigma \otimes \sigma}{3\sigma_3^2}$$ (20f)

where the symbol $\otimes$ denotes the dyadic product. Therefore, using equations (19a,b,c), (20a,b, c,d,e,f) we have

$$\mathbf{y} = \alpha \, \sigma_e \, \sigma$$ (21a)

$$\mathbf{Y} = \sigma_e \frac{\sigma \otimes \sigma}{2\sigma_2}$$ (21b)

$$\mathbf{Y}^* = \gamma \, \sigma_e \frac{\sigma \otimes \sigma \otimes \sigma}{3\sigma_3^2}$$ (21c)

Let us define the following damage criterion

$$g(z,r) \equiv z(\mathbf{y}, \mathbf{Y}, \mathbf{Y}^*, \widetilde{\mathbf{b}}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{c}}, \mathbf{b}, \mathbf{a}, \mathbf{c}, \sigma) - r(\omega) = 0$$ (22)

where z is a suitable scalar function, which defines the form of the damage surface, while r is the damage threshold at current time t. If r_0 is the initial damage threshold, it must be $r \geq r_0$. Damage occurs if the value z is equal to the damage threshold at current time. The parameter r, which defines the size of the damage surface, is assumed to be depending on the cumulative damage parameter ω. The case $z < r$ corresponds to the elastic deformation of material without damage.

Let us introduce new constant tensors β, ξ, ζ of second, fourth and sixth rank, respectively, and then new material tensors

$$\mathbf{b} = \beta_{ij} \mathbf{n}_i \otimes \mathbf{n}_j$$ (23a)

$$\mathbf{a} = \xi_{ijkl} \mathbf{n}_i \otimes \mathbf{n}_j \otimes \mathbf{n}_k \otimes \mathbf{n}_l$$ (23b)

$$\mathbf{c} = \zeta_{ijklrs} \mathbf{n}_i \otimes \mathbf{n}_j \otimes \mathbf{n}_k \otimes \mathbf{n}_l \otimes \mathbf{n}_r \otimes \mathbf{n}_s$$ (23c)

where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the three orthogonal principal directions of damage. Different possible definitions of the principal directions of damage are given by Chaboche (1993). In the simplest case we can choose the principal directions of damage as the eigenvectors of the stress tensor.

Let us consider the new polynomials

$$\Sigma_1(\mathbf{b},\sigma) = \mathbf{b}\cdots\sigma \tag{24a}$$

$$\Sigma_2^2(\mathbf{a},\sigma) = \sigma\cdots\mathbf{a}\cdots\sigma \tag{24b}$$

$$\Sigma_3^3(\mathbf{c},\sigma) = \sigma\cdots\left(\sigma\cdots\mathbf{c}\cdots\sigma\right) \tag{24c}$$

and introduce the following structure for the function z in a damage potential

$$z = \sigma_e\left(\alpha_1\Sigma_1 + 2\Sigma_2 + 3\gamma_1\Sigma_3\right) \tag{25}$$

where α_1,γ_1 are numerical coefficients taking into account the influence of linear and cubic polynomials in the expression (25) for z, coefficients „2" and „3" are taken in order to receive the simple formulas in the following. It is not difficult to show on the base of equation (21a,b,c) that

$$z = \alpha_1 \mathrm{t\,r}\,(\mathbf{yb}) + \sqrt{8\sigma_e\sigma_2 \mathrm{t\,r}\,(\mathbf{Ya})} + \gamma_1\sqrt[3]{[9\gamma\sigma_e^2\sigma_2^2 \mathrm{t\,r}\,(\mathbf{Y}^*\mathbf{c})]^2} \tag{26}$$

The damage process is characterized by the following equations of evolution

$$\overset{\cdot}{\tilde{\mathbf{b}}} = \lambda\frac{\partial z}{\partial \mathbf{y}} \tag{27a}$$

$$\overset{\cdot}{\tilde{\mathbf{a}}} = \lambda\frac{\partial z}{\partial \mathbf{Y}} \tag{27b}$$

$$\overset{\cdot}{\tilde{\mathbf{c}}} = \lambda\frac{\partial z}{\partial \mathbf{Y}^*} \tag{27c}$$

which lead to:

$$\overset{\cdot}{\tilde{\mathbf{b}}} = \lambda\alpha_1\mathbf{b} \tag{28a}$$

$$\overset{\cdot}{\tilde{\mathbf{a}}} = 2\,\lambda\sigma_2\,\frac{\mathbf{a}}{\Sigma_2} \tag{28b}$$

$$\overset{\cdot}{\tilde{\mathbf{c}}} = 3\,\lambda\gamma_1\sigma_3^2\,\frac{\mathbf{c}}{\Sigma_3^2} \tag{28c}$$

Using the principle of maximum damage dissipation, one can show that the damage consistency parameter λ satisfies the Kuhn-Tucker relations

$$\dot{\lambda} \geq 0 \tag{29a}$$

$$g(z,r) \leq 0 \tag{29b}$$

$$\dot{\lambda}\,g(z,r) = 0 \tag{29c}$$

The parameter λ can be considered as a measure of the cumulative damage, i.e.

$$\dot{\lambda} = \dot{\omega} \tag{30}$$

Therefore, if the damage in the material is increasing, we have $\dot{\lambda} > 0$. Then, by condition (29c) it follows: $g(z,r) = 0$. If, on the other hand, the damage criterion is not satisfied and $g(z,r) < 0$, the condition (29c) implies that $\dot{\lambda} \equiv \dot{\omega} \equiv 0$.

The second thermodynamic principle has the form:

$$\psi = \mathrm{t\,r}\,(\mathbf{y}\overset{\cdot}{\tilde{\mathbf{b}}}) + \mathrm{t\,r}\,(\mathbf{Y}\overset{\cdot}{\tilde{\mathbf{a}}}) + \mathrm{t\,r}\,(\mathbf{Y}^*\overset{\cdot}{\tilde{\mathbf{c}}}) \tag{31}$$

Substituting the values (21a,b,c), (28a,b,c) into equation (31), we obtain

$$\psi = \dot{\lambda}\,\sigma_e\Sigma_e \tag{32}$$

if we denote

$$\Sigma_e = \Sigma_2 + \alpha\alpha_1\Sigma_1 + \gamma\gamma_1\Sigma_3 \tag{33}$$

Since $\dot{\lambda} \geq 0$ from (29a), we see that under the natural condition that the equivalent stress is always non-negative, i. e. $\sigma_e \geq 0$ and $\Sigma_e \geq 0$, the second thermodynamic principle $\psi \geq 0$ is always valid.

We now turn to the task of defining the rate of the cumulative damage parameter. For this purpose differentiating equations (22), (25) we can write

$$\dot{\sigma}_e\left(\alpha_1\Sigma_1 + 2\Sigma_2 + 3\gamma_1\Sigma_3\right) + \sigma_e\left(\alpha_1\dot{\Sigma}_1 + 2\dot{\Sigma}_2 + 3\gamma_1\dot{\Sigma}_3\right) - \frac{dr}{d\omega}\dot{\omega} = 0 \tag{34}$$

and then

$$\dot{\omega} = \frac{1}{h}[\dot{\sigma}_e\left(\alpha_1\Sigma_1 + 2\Sigma_2 + 3\gamma_1\Sigma_3\right) + \sigma_e\left(\alpha_1\dot{\Sigma}_1 + 2\dot{\Sigma}_2 + 3\gamma_1\dot{\Sigma}_3\right)] \tag{35}$$

where we denote

$$h = \frac{dr}{d\sigma} \tag{36}$$

Since damage growth implies $\dot{\omega} \geq 0$, it follows from equation (35) that during the damage loading we get

$$\text{sign}(h)[\dot{\sigma}_e\left(\alpha_1\Sigma_1 + 2\Sigma_2 + 3\gamma_1\Sigma_3\right) + \sigma_e\left(\alpha_1\dot{\Sigma}_1 + 2\dot{\Sigma}_2 + 3\gamma_1\dot{\Sigma}_3\right)] > 0 \tag{37}$$

The inequality

$$\dot{\sigma}_e\left(\alpha_1\Sigma_1 + 2\Sigma_2 + 3\gamma_1\Sigma_3\right) + \sigma_e\left(\alpha_1\dot{\Sigma}_1 + 2\dot{\Sigma}_2 + 3\gamma_1\dot{\Sigma}_3\right) > 0 \tag{38}$$

represents the necessary and sufficient condition for the damage loading in the hardening case $(h > 0)$. The inequality

$$\dot{\sigma}_e\left(\alpha_1\Sigma_1 + 2\Sigma_2 + 3\gamma_1\Sigma_3\right) + \sigma_e\left(\alpha_1\dot{\Sigma}_1 + 2\dot{\Sigma}_2 + 3\gamma_1\dot{\Sigma}_3\right) < 0 \tag{39}$$

represents only a necessary but not sufficient condition for the damage loading in the softening case $(h < 0)$, because the inequality (39) can be satisfied during unloading as well.

4 Particular Cases

For initially orthotropic material the constitutive equation (11) for elastic deformation can be written in a coordinate system whose axes coincide with the principal directions of anisotropy as follows:

$$\begin{aligned}
\varepsilon_{11} = \sigma_e\Bigg(&\frac{\tilde{a}_{1111}\sigma_{11} + \tilde{a}_{1122}\sigma_{22} + \tilde{a}_{1133}\sigma_{33}}{\sigma_2} + \alpha\tilde{b}_{11} + \\
&+ \gamma\frac{\tilde{c}_{111111}\sigma_{11}^2 + \tilde{c}_{112222}\sigma_{22}^2 + \tilde{c}_{113333}\sigma_{33}^2}{\sigma_3^2} + \\
&+ 2\gamma\frac{\tilde{c}_{111122}\sigma_{11}\sigma_{22} + \tilde{c}_{111133}\sigma_{11}\sigma_{33} + \tilde{c}_{112233}\sigma_{22}\sigma_{33}}{\sigma_3^2} + \\
&+ 4\gamma\frac{\tilde{c}_{111212}\sigma_{12}^2 + \tilde{c}_{112323}\sigma_{23}^2 + \tilde{c}_{111313}\sigma_{13}^2}{\sigma_3^2}\Bigg) \quad (1,2,3)
\end{aligned} \tag{40a}$$

$$\begin{aligned}
\varepsilon_{12} = 2\sigma_e\Bigg(&\frac{\tilde{a}_{1212}\sigma_{12}}{\sigma_2} + 2\gamma\frac{\tilde{c}_{121211}\sigma_{12}\sigma_{11} + \tilde{c}_{121222}\sigma_{12}\sigma_{22} + \tilde{c}_{121233}\sigma_{12}\sigma_{33}}{\sigma_3^2} + \\
&+ 4\gamma\frac{\tilde{c}_{122313}\sigma_{23}\sigma_{13}}{\sigma_3^2}\Bigg) \quad (1,2,3)
\end{aligned} \tag{40b}$$

Here

$$\sigma_1 = \tilde{b}_{11}\sigma_{11} + \tilde{b}_{22}\sigma_{22} + \tilde{b}_{33}\sigma_{33} \tag{41a}$$

$$\begin{aligned}
\sigma_2^2 = &\tilde{a}_{1111}\sigma_{11}^2 + \tilde{a}_{2222}\sigma_{22}^2 + \tilde{a}_{3333}\sigma_{33}^2 + 2\tilde{a}_{1122}\sigma_{12}^2 + 2\tilde{a}_{2323}\sigma_{23}^2 + 4\tilde{a}_{1212}\sigma_{12}^2 + \\
&+ 4\tilde{a}_{2323}\sigma_{23}^2 + 4\tilde{a}_{1313}\sigma_{13}^2
\end{aligned} \tag{41b}$$

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$$\sigma_3^3 = \widetilde{c}_{111111}\sigma_{11}^3 + \widetilde{c}_{222222}\sigma_{22}^3 + \widetilde{c}_{333333}\sigma_{33}^3 + 3\widetilde{c}_{111122}\sigma_{11}^2\sigma_{22} + 3\widetilde{c}_{111133}\sigma_{11}^2\sigma_{33} +$$
$$+ 3\widetilde{c}_{222211}\sigma_{22}^2\sigma_{11} + 3\widetilde{c}_{222233}\sigma_{22}^2\sigma_{33} + 3\widetilde{c}_{333311}\sigma_{33}^2\sigma_{11} + 3\widetilde{c}_{333322}\sigma_{33}^2\sigma_{22} +$$
$$+ 6\widetilde{c}_{112233}\sigma_{11}\sigma_{22}\sigma_{33} + 12\widetilde{c}_{121211}\sigma_{12}^2\sigma_{11} + 12\widetilde{c}_{121222}\sigma_{12}^2\sigma_{22} + 12\widetilde{c}_{121233}\sigma_{12}^2\sigma_{33} + \qquad (41c)$$
$$+ 12\widetilde{c}_{232311}\sigma_{23}^2\sigma_{11} + 12\widetilde{c}_{232322}\sigma_{23}^2\sigma_{22} + 12\widetilde{c}_{232333}\sigma_{23}^2\sigma_{33} + 12\widetilde{c}_{131311}\sigma_{13}^2\sigma_{11} +$$
$$+ 12\widetilde{c}_{131322}\sigma_{13}^2\sigma_{22} + 12\widetilde{c}_{131333}\sigma_{13}^2\sigma_{33} + 48\widetilde{c}_{122313}\sigma_{12}\sigma_{23}\sigma_{13}$$

The constitutive equations (40a,b) for elastic deformation must be written together with equations (28a,b,c), (23a,b,c), (30), (35) for the damage evolution, and the number of the different parameters $\widetilde{b}_{ij}, \beta_{ij} - 3, \widetilde{a}_{ijkl}, \xi_{ijkl} - 9, \widetilde{c}_{ijklmn}, \zeta_{ijklmn} - 20$.

The following relations are valid for an isotropic medium:

$$\widetilde{b}_{ij} = \widetilde{B}\delta_{ij} \qquad (42a)$$

$$\beta_{ij} = \Delta\,\delta_{ij} \qquad (42b)$$

$$\widetilde{a}_{ijkl} = \widetilde{A}\delta_{ij}\delta_{kl} + \frac{\widetilde{C}}{2}(\delta_{ik}\delta_{jl} + \delta_{li}\delta_{jk}) \qquad (42c)$$

$$\xi_{ijkl} = \Xi\,\delta_{ij}\delta_{kl} + \frac{\Lambda}{2}(\delta_{ik}\delta_{jl} + \delta_{li}\delta_{jk}) \qquad (42d)$$

$$\widetilde{c}_{ijklmn} = \widetilde{D}\delta_{ij}\delta_{kl}\delta_{mn} + \frac{\widetilde{K}}{6}(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{kl}\delta_{in}\delta_{jm} +$$
$$+ \delta_{mn}\delta_{ik}\delta_{jl} + \delta_{mn}\delta_{il}\delta_{jk}) + \frac{\widetilde{L}}{8}(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{km}\delta_{jn} + \qquad (42e)$$
$$+ \delta_{il}\delta_{kn}\delta_{jm} + \delta_{im}\delta_{kj}\delta_{ln} + \delta_{im}\delta_{kn}\delta_{lj} + \delta_{in}\delta_{kj}\delta_{lm} + \delta_{in}\delta_{km}\delta_{lj})$$

$$\zeta_{ijklmn} = \Theta\delta_{ij}\delta_{kl}\delta_{mn} + \frac{\Omega}{6}(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{kl}\delta_{in}\delta_{jm} +$$
$$+ \delta_{mn}\delta_{ik}\delta_{jl} + \delta_{mn}\delta_{il}\delta_{jk}) + \frac{\Psi}{8}(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{km}\delta_{jn} + \qquad (42f)$$
$$+ \delta_{il}\delta_{kn}\delta_{jm} + \delta_{im}\delta_{kj}\delta_{ln} + \delta_{im}\delta_{kn}\delta_{lj} + \delta_{in}\delta_{kj}\delta_{lm} + \delta_{in}\delta_{km}\delta_{lj})$$

These relations make it possible to obtain from equations (40a,b) the equation:

$$\varepsilon_{ij} = \sigma_e \left[\frac{\widetilde{A}J_1\delta_{ij} + \widetilde{C}\sigma_{ij}}{\sigma_2} + \alpha\widetilde{B}\delta_{ij} + \gamma\frac{\widetilde{D}J_1^2\delta_{ij} + \widetilde{L}\sigma_{ik}\sigma_{kj} + \frac{\widetilde{K}}{3}(J_2\delta_{ij} + 2J_1\sigma_{ij})}{\sigma_3^2} \right] \qquad (43)$$

where the equivalent stress

$$\sigma_e = \sigma_e(\sigma, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{K}, \widetilde{L}) \qquad (44)$$

contains the six damage dependent functions. In the case under consideration the evolution equations (28a,b,c), (23a,b,c), (30), (35) use the six materials parameters $\Delta, \Xi, \Lambda, \Theta, \Omega, \Psi$.

5 Basic Experiments

Now we assume that $\alpha = \alpha_1 = 1$ and $\gamma = \gamma_1 = 0$ in equations (40a,b), (25), (28a,c), and we consider the particular case of plane stress with $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. Let us discuss the determination of the parameters in the constitutive equations. For this purpose we use the results of the basic experiments on standard specimens whose orientations coincide with the principal directions of anisotropy. Standard specimens, in which a homogeneous stress state is obtained, are made from initially orthotropic material. We will also assume that the principal directions of damage coincide with the principal directions of the orthotropy.

In the case of uniaxial tension in the principal direction 1, we have strain in the direction of loading

$$\varepsilon_{11} = \frac{\sigma_{11}}{\widetilde{E}_1^+} \tag{45a}$$

and in transverse direction

$$\varepsilon_{22} = -\widetilde{\nu}_2^+ \varepsilon_{11} \tag{45b}$$

Here $\widetilde{E}_1^+$ is the elastic secant modulus in tension in direction 1, and $\widetilde{\nu}_2^+$ is the ratio of transverse strain in tension. These data are the functions of the damage in this uniaxial case. On the other hand, using equations (40a), (41a,b), it has been found that

$$\varepsilon_{11} = \left(\sqrt{\widetilde{a}_{1111}} + \widetilde{b}_{11}\right)^2 \sigma_{11} \tag{46a}$$

$$\varepsilon_{22} = \left(\sqrt{\widetilde{a}_{1111}} + \widetilde{b}_{11}\right)\left(\frac{\widetilde{a}_{1122}}{\sqrt{\widetilde{a}_{1111}}} + \widetilde{b}_{22}\right) \sigma_{11} \tag{46b}$$

Now, equating relations (45a,b) and (46a,b), we have

$$\left(\sqrt{\widetilde{a}_{1111}} + \widetilde{b}_{11}\right)^2 = \frac{1}{\widetilde{E}_1^+} \tag{47a}$$

$$\frac{\widetilde{a}_{1122}}{\sqrt{\widetilde{a}_{1111}}} + \widetilde{b}_{22} = -\frac{\widetilde{\nu}_2^+}{\left(\widetilde{E}_1^+\right)^{1/2}} \tag{47b}$$

Using equations (28a,b), (23a,b), (30), (47a), it is not difficult to obtain the following equations

$$\widetilde{b}_{11} = \omega\beta_{11} \tag{48a}$$

$$\sqrt{\widetilde{a}_{1111}} = \frac{1}{\sqrt{E_1}} + \omega\sqrt{\xi_{1111}} \tag{48b}$$

leading to the expression for the cumulative damage parameter

$$\omega = \frac{\left(\sqrt{\widetilde{E}_1^+}\right)^{-1} - \left(\sqrt{E_1}\right)^{-1}}{\sqrt{\xi_{1111}} + \beta_{11}} \tag{49}$$

where E_1 is the initial elastic modulus in the direction 1. If the initial damage threshold stress in uniaxial tension in the direction 1 is σ_1^+, then the initial damage criterion can be written as

$$\frac{\left(\sigma_1^+\right)^2}{\sqrt{E_1}}\left(2\sqrt{\xi_{1111}} + \beta_{11}\right) = r_0 \tag{50}$$

Without loss of generality we can assume that

$$\frac{\left(\sigma_1^+\right)^2}{\sqrt{E_1}} = r_0 \tag{51}$$

and we get from equation (50):

$$2\sqrt{\xi_{1111}} + \beta_{11} = 1 \tag{52}$$

Under uniaxial compression in the principal direction 1 the following relation can be written:

$$\varepsilon_{11} = -\frac{|\sigma_{11}|}{\widetilde{E}_1^-} \tag{53}$$

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where $\widetilde{E}_1^-$ is the elastic secant modulus in compression in the direction 1. Then we obtain an analogous relation from equations (40a), (41a):

$$\varepsilon_{11} = -\left(\sqrt{\widetilde{a}_{1111}} - \widetilde{b}_{11}\right)^2 |\sigma_{11}| \tag{54}$$

Equating relations (53) and (54) we have

$$\left(\sqrt{\widetilde{a}_{1111}} - \widetilde{b}_{11}\right)^2 = \frac{1}{\widetilde{E}_1^-} \tag{55}$$

Using equations (48a,b), (55) we obtain

$$\omega = \frac{\left(\sqrt{\widetilde{E}_1^-}\right)^{-1} - \left(\sqrt{E_1}\right)^{-1}}{\sqrt{\xi_{1111}} - \beta_{11}} \tag{56}$$

If the initial damage threshold stress in uniaxial compression in the direction 1 is σ_1^-, then the initial damage criterion can be written in the following form

$$\frac{\left(\sigma_1^-\right)^2}{\sqrt{E_1}} (2\sqrt{\xi_{1111}} - \beta_{11}) = r_0 \tag{57}$$

Taking into account equation (51) we then find

$$(2\sqrt{\xi_{1111}} - \beta_{11}) = \frac{\left(\sigma_1^+\right)^2}{\left(\sigma_1^-\right)^2} \tag{58}$$

In the case of uniaxial tension in the principal direction 2, the following relation

$$\varepsilon_{22} = \frac{\sigma_{22}}{\widetilde{E}_2^+} \tag{59}$$

holds. Here $\widetilde{E}_2^+$ is the elastic secant modulus in tension in the direction 2. On the other hand, using equations (40a), (41a,b) it has been found that

$$\varepsilon_{22} = \left(\sqrt{\widetilde{a}_{2222}} + \widetilde{b}_{22}\right)^2 \sigma_{22} \tag{60}$$

Now, equating relations (59) and (60), we have

$$\left(\sqrt{\widetilde{a}_{2222}} + \widetilde{b}_{22}\right)^2 = \frac{1}{\widetilde{E}_2^+} \tag{61}$$

Using equations (28a), (28b), (30), (47a) we obtain the following equations:

$$\widetilde{b}_{22} = \omega\beta_{22} \tag{62}$$

and

$$\sqrt{\widetilde{a}_{2222}} = \frac{1}{\sqrt{E_2}} + \omega\sqrt{\xi_{2222}} \tag{63}$$

leading to the expression for the cumulative damage parameter under uniaxial tension in the direction 2 as

$$\omega = \frac{\left(\sqrt{\widetilde{E}_2^+}\right)^{-1} - \left(\sqrt{E_2}\right)^{-1}}{\sqrt{\xi_{2222}} + \beta_{22}} \tag{64}$$

where E_2 is the initial elastic modulus in the direction 2. If the initial damage threshold stress in uniaxial tension in the direction 2 is σ_2^+, then the initial damage criterion can be written as

$$\frac{\left(\sigma_2^+\right)^2}{\sqrt{E_2}}\left(2\sqrt{\xi_{2222}} + \beta_{22}\right) = r_0 \tag{65}$$

Substituting equation (51) into equation (65) we have

$$2\sqrt{\xi_{2222}} + \beta_{22} = \frac{\left(\sigma_1^+\right)^2 \sqrt{E_2}}{\sqrt{E_1}\left(\sigma_2^+\right)^2} \tag{66}$$

By analogy with equations above formulas for uniaxial compression in the direction 2 will be written in the form:

$$\left(\sqrt{\widetilde{a}_{2222}} - \widetilde{b}_{22}\right)^2 = \frac{1}{\widetilde{E}_2^-} \tag{67}$$

$$\omega = \frac{\left(\sqrt{\widetilde{E}_2^-}\right)^{-1} - \left(\sqrt{E_2}\right)^{-1}}{\sqrt{\xi_{2222}} - \beta_{22}} \tag{68}$$

$$2\sqrt{\xi_{2222}} - \beta_{22} = \frac{\left(\sigma_1^+\right)^2 \sqrt{E_2}}{\sqrt{E_1}\left(\sigma_2^-\right)^2} \tag{69}$$

Here $\widetilde{E}_2^-$ is the elastic secant modulus in tension in the direction 2, σ_2^- is the initial damage threshold stress in uniaxial tension in the direction 2.

Then, it follows from equations (52),(58),(66), (69) that

$$4\sqrt{\xi_{1111}} = 1 + \frac{\left(\sigma_1^+\right)^2}{\left(\sigma_1^-\right)^2} \tag{70a}$$

$$2\beta_{11} = 1 - \frac{\left(\sigma_1^+\right)^2}{\left(\sigma_1^-\right)^2} \tag{70b}$$

$$4\sqrt{\xi_{2222}} = \left(\sigma_1^+\right)^2 \sqrt{\frac{E_2}{E_1}}\left[\frac{1}{\left(\sigma_2^+\right)^2} + \frac{1}{\left(\sigma_2^-\right)^2}\right] \tag{70c}$$

$$2\beta_{22} = \left(\sigma_1^+\right)^2 \sqrt{\frac{E_2}{E_1}}\left[\frac{1}{\left(\sigma_2^+\right)^2} - \frac{1}{\left(\sigma_2^-\right)^2}\right] \tag{70d}$$

For pure torsion we take the following relation

$$\varepsilon_{12} = \frac{\sigma_{12}}{2\widetilde{G}_{12}} \tag{71}$$

where $\widetilde{G}_{12}$ is the secant shear modulus. An analogous formula to equations (40b), (41b) is

$$\varepsilon_{12} = 2\widetilde{a}_{1212}\sigma_{12} \tag{72}$$

Therefore we obtain:

$$\widetilde{a}_{1212} = \frac{1}{4\widetilde{G}_{12}} \tag{73}$$

Using dependencies (28b), (30), (73), we can write the following relation describing the damage in this basic experiment

$$\omega = \frac{\left(\sqrt{\widetilde{G}_{12}}\right)^{-1} - \left(\sqrt{G_{12}}\right)^{-1}}{2\sqrt{\xi_{1212}}} \tag{74}$$

where G_{12} is the initial shear modulus. If the initial damage threshold stress in pure torsion is τ_{12}, then it follows from the initial damage criterion that

$$8\sqrt{\xi_{1212}} = \frac{\left(\sigma_1^+\right)^2 \sqrt{G_{12}}}{\sqrt{E_1}\left(\tau_{12}\right)^2} \tag{75}$$

Thus, the parameters in the constitutive equations (40a,b) and the evolution equations (28a,b), (30) may be determined on the basis of experimental data from stress-strain diagrams obtained from basic experiments. Also note that dependencies (40a,b) describe the difference between the elastic processes in tension and compression and the change of the elastic moduli when the stresses change their signs, the initial anisotropy and the damage induced anisotropy.

7 Conclusion

The proposed model with monotonic strain-stress relations is able to reproduce initial anisotropy, damage induced anisotropy, the difference between the elastic processes in tension and compression as well as the change of the elastic stiffness when the stresses are changing their signs. In a particular case, for example, of plane stress with $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$, and with $\alpha = \alpha_1 = 1$ and $\gamma = \gamma_1 = 0$ in equations (40a,b), (25), (28a,c), we can rewrite those dependencies in the following form

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1112} \\ D_{1122} & D_{2222} & D_{2212} \\ D_{1112} & D_{2212} & D_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} D_{11} \\ D_{22} \\ D_{12} \end{Bmatrix} \tag{76}$$

It can be seen that the secant stiffness matrix D is symmetric. Therefore the Onsager Principle holds in this case. The damage loading surface in the plane $\sigma_{11} - \sigma_{22}$ has an elliptic form. The loading surface is both continuous and continuously differentiable, and furthermore convex. Thus, the proposed model satisfies all requirements which are formulated by Chaboche (1992). A comparison between the theoretical results and experimental data under multiaxial loading will be a subject of a forthcoming paper.

Acknowledgement

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