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# Chaotic Motion of a Magnetic Rigid Satellite in an Orbit Near the Equatorial Plane of the Earth

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*The planar motion of a magnetic rigid satellite moving in a circular orbit near the equatorial plane of the earth, under the action of the gravitational and the magnetic force, is investigated in the present paper. It is found that the motion is chaotic in the sense of Smale's horseshoe by using Melnikov's method. Numerical simulations in conjunction with the Poincare map show that the periodic disturbance of the magnetic field of the earth can cause chaotic motion of the satellite and the increase of the disturbance can intensify the chaotic motion.*

## 1 Introduction

The attitude motion of spacecraft in the gravitational field has been extensively studied over the past several decades. In recent years, the chaotic attitude motion of satellites has attracted the attention of many researchers. Seisl and Steindl (1989) studied the chaotic planar motion of a satellite moving in an elliptic orbit and subjected to air drag force. Tong and Rimrott (1991) discussed the chaotic planar motion of a non-spinning satellite with structural damping. Tong et al. (1993, 1995) investigated the chaotic attitude motion of gyrostat satellites. Peng and Liu (1996) discussed the chaotic motion of a tethered satellite system. Chaotic attitude motion of satellites under the action of other forces besides the gravitational force is also studied. Beletsky (1995) investigated the planar chaotic motion of satellites subjected to both gravitational and magnetic forces with numerical methods. In this paper we study the chaotic planar motion of a magnetic non-spinning satellite moving in a circular orbit near the equatorial plane of the earth, under the action of both gravitational and magnetic forces, with the analytical method due to Melnikov (1963) and numerical computations.

## 2 Equation of Motion

Consider the attitude motion of a magnetic rigid satellite whose center of mass moves in a circular orbit near the equatorial plane of the earth with angular velocity $\omega_c$. Introduce an inertial coordinate frame $(0_e - X_0 Y_0 Z_0)$, the center of the earth $0_e$ as the origin, $\mathbf{X}^0, \mathbf{Y}^0, \mathbf{Z}^0$ as bases $Z_0$ − axis along the polar axis of the earth and $X_0$ − axis pointing to the ascending node of the satellite (see Figure 1).

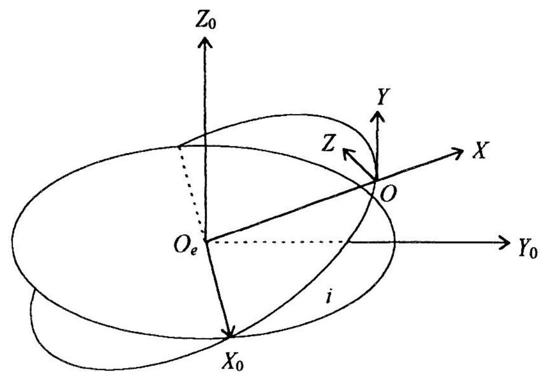



Figure 1. Inertial Coordinate System and Orbit Coordinate System

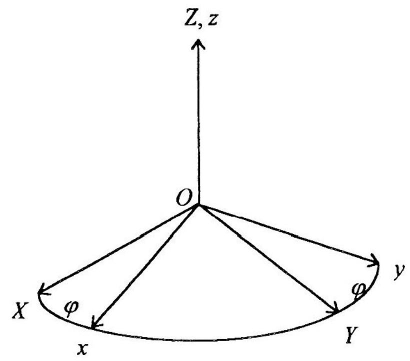



Figure 2. Generalized Coordinate φ

Set up a principal coordinate frame $(0-xyz)$ of the body, with the center of mass $0$ as the origin and $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ as bases. The magnetic moment of the rigid body $\mathbf{I}$ lies along axis $-x$. Introduce an orbital coordinate frame $(0-XYZ)$ with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as bases, $X$- and $Z$-axes along the position vector of $0$ relative to $0_e$ and the normal of the orbital plane respectively. Assume that the angle of inclination of the orbital plane $i << 1$ and that the principal axes $x$, $y$ of the satellite are restricted in the orbital plane, inclined by angle $\varphi$ with respect to $X$- and $Y$-axes (see Figure 2). Decompose the magnetic density $\mathbf{H}_m$ of the magnetic field of the earth in the coordinate system $(0_e - X_0\, Y_0\, Z_0)$

$$\mathbf{H}_m = H_{mX}\mathbf{X}^0 + H_{mY}\mathbf{Y}^0 + H_{mZ}\mathbf{Z}^0 \tag{1}$$

where

$$\begin{aligned} H_{mX} &= -\frac{3}{2} iH_{m0}\sin 2\omega_c t \\ H_{mY} &= -\frac{3}{2} iH_{m0}(1-\cos 2\omega_c t) \\ H_{mZ} &= H_{m0} \end{aligned} \tag{2}$$

in which $H_{m0} = \mu_m / r^3$, $\mu_m$ is the magnetic moment constant of the earth and $r$ is the radius of the circular orbit. The magnetic moment of the satellite is

$$\mathbf{I} = -I\,\mathbf{i}' = -I(\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}) \tag{3}$$

The satellite is subjected to the magnetic torque

$$\mathbf{M}_m = \mathbf{I} \times \mathbf{H}_m \tag{4}$$

and the gravitational torque

$$\mathbf{M}_g = -3\omega_c^2(B-A)\sin\varphi\cos\varphi\,\mathbf{k} \tag{5}$$

Write the angular momentum of the satellite about its center of mass as

$$\mathbf{H} = C(\omega_c + \dot{\varphi})\mathbf{k} \tag{6}$$

In equations (5) and (6), $A$, $B$, $C$ indicate the principal inertia moments of the rigid body. Without loss of generality, we assume $C > B > A$. Decompose the base vectors $(\mathbf{X}^0, \mathbf{Y}^0, \mathbf{Z}^0)$ along $(\mathbf{i}, \mathbf{j}, \mathbf{k})$:

$$\begin{aligned} \mathbf{X}^0 &= \cos\omega_c t\,\mathbf{i} - \sin\omega_c t\,\mathbf{j} \\ \mathbf{Y}^0 &= \sin\omega_c t\,\mathbf{i} + \cos\omega_c t\,\mathbf{j} - i\mathbf{k} \\ \mathbf{Z}^0 &= i\sin\omega_c t\,\mathbf{i} + i\cos\omega_c t\,\mathbf{j} + \mathbf{k} \end{aligned} \tag{7}$$

Employing the theorem of angular momentum about the center of mass

$$\frac{d}{dt}\mathbf{H} = \mathbf{M}_m + \mathbf{M}_g \tag{8}$$

and using equations (4) – (7), we obtain the following equation of motion

$$C\ddot{\varphi} = -3\omega_c^2(B-A)\sin\varphi\cos\varphi - iIH_{m0}(2\sin\varphi\sin\omega_c t + \cos\varphi\cos\omega_c t) \tag{9}$$

By introducing the dimensionless time variable $\tau = \omega_c t$ and dimensionless parameters $\sigma = (B-A)/C$, $\varepsilon = iIH_{m0}/C\omega_c^2$, equation (9) can be simplified as

$$\ddot{\varphi} + 3\sigma \sin\varphi\cos\varphi + \varepsilon(2\sin\varphi\sin\tau + \cos\varphi\cos\tau) = 0 \tag{10}$$

## 3 Melnikov Function

Let $x = \varphi$, then we have the equivalent system of equation (10):

$$\dot{x} = y \tag{11a}$$

$$\dot{y} = -3\sigma\sin x\cos x - \varepsilon(2\sin x\sin\tau + \cos x\cos\tau) \tag{11b}$$

where $\varepsilon(<<1)$ is a small parameter. When $\varepsilon = 0$, equations (11a, 11b) represent an integrable Hamiltonian system, whose Hamiltonian is

$$H(x,y) = -\frac{3}{4}\sigma\cos 2x + \frac{1}{2}y^2 = h \tag{12}$$

In case of $\varepsilon = 0$, dividing equations (11b) by (11a) yields the differential equation in the phase plane as follows

$$\frac{dy}{dx} = -\frac{3\sigma\sin 2x}{2y} \tag{13}$$

For $x$ lying in $[-\pi/2, \pi/2]$, there exist three equilibria for equation (13), namely

$$\begin{aligned} &P_1: x = -\pi/2, && y = 0 \text{ (saddle point)} \\ &P_2: x = \pi/2, && y = 0 \text{ (saddle point)} \\ &P_3: x = 0, && y = 0 \text{ (center)} \end{aligned} \tag{14}$$

And the saddle points $P_1$, $P_2$ are connected by two heteroclinic orbits, whose parametric equations can be derived as follows

$$\begin{aligned} x_\pm(\tau) &= \pm \mathrm{tg}^{-1}\left(\mathrm{sh}\sqrt{3\sigma}\tau\right) \\ y_\pm(\tau) &= \pm\sqrt{3\sigma}\,\mathrm{sech}\left(\sqrt{3\sigma}\tau\right) \end{aligned} \tag{15}$$

Figure 3 shows the phase trajectories of the system in case of $\varepsilon = 0$.

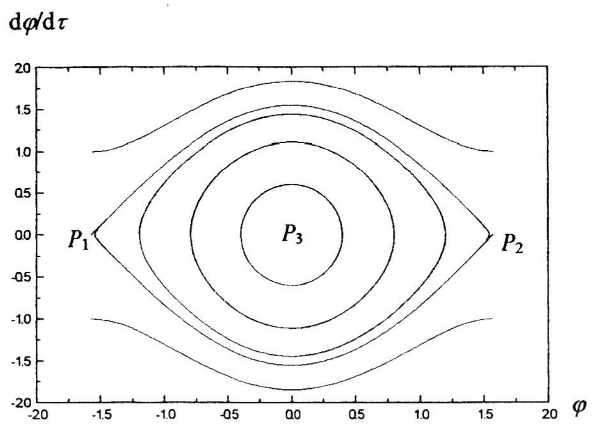



Figure 3. Phase Trajectories of Unperturbed System

When $\varepsilon \neq 0$, the satellite is subjected to the periodic disturbance of the magnetic field of the earth. Calculating the Melnikov function corresponding to the heteroclinic orbits

$$M_{\pm}(\tau_0) = \int_{-\infty}^{\infty} -Y_{\pm}(\tau)[2\sin x_{\pm}(\tau)\sin(\tau+\tau_0) + \cos x_{\pm}(\tau)\cos(\tau+\tau_0)]d\tau \tag{16}$$

we have

$$M_{\pm}(\tau_0) = -2\sqrt{3\sigma}a\cos\tau_0 \tag{17}$$

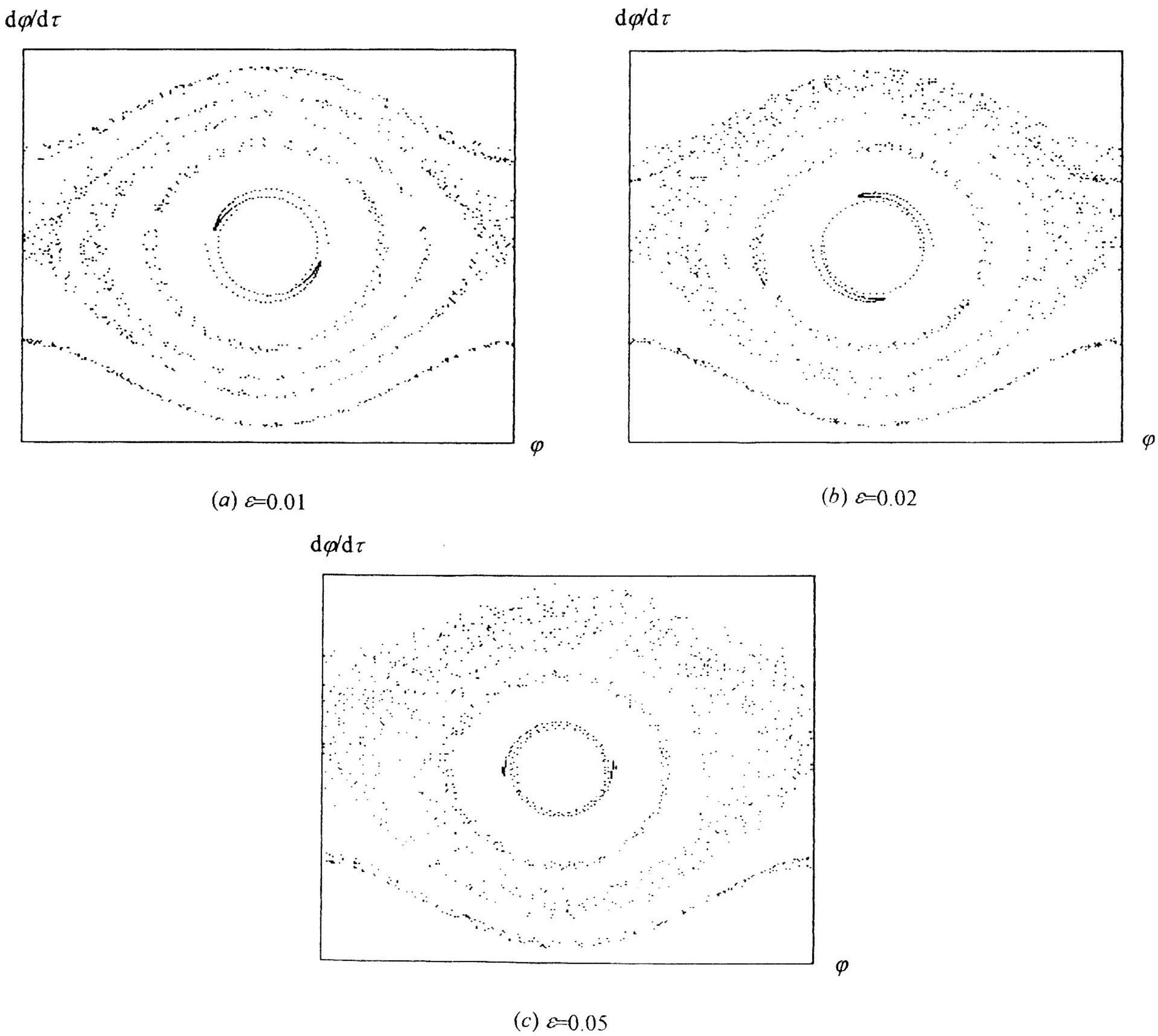
where

$$a = \pm\frac{\pi}{12\sigma}\operatorname{csch}\left(\frac{\pi}{4\sqrt{3\sigma}}\right)\operatorname{sech}\left(\frac{\pi}{4\sqrt{3\sigma}}\right) + \frac{\pi}{6\sigma}\operatorname{Re}\left[-\cot\frac{\left(-i+\sqrt{3\sigma}\right)\pi}{4\sqrt{3\sigma}} + \cot\frac{\left(-i+3\sqrt{3\sigma}\right)\pi}{2\sqrt{3\sigma}}\right] \tag{18}$$

It can be seen from equation (17) and (18) that the Melnikov function has simple zeros. Thus the motion of the perturbed system is chaotic in the sense of Smale's horseshoe.

## 4 Numerical Computations and Conclusions

For the purpose of verifying the above analysis, we numerically integrate the equation of motion (11) and plot the results on Poincaré maps. The Poincaré map is defined as $\sum\{(\varphi, d\varphi/d\tau)|\tau = 2k\pi,\ k = 1, 2,\ldots\}$. Computations were performed with parameter $\sigma = 0.8$. A fourth-order Runge-Kutta integration algorithm was used to give points on the Poincaré maps for each of the total 7 different initial conditions. The results are shown in Figure 4. It can be seen that under the action of the periodic disturbance, the motion of the satellite rapidly becomes irregular and turns into chaotic motion, and as the disturbance increases, the chaotic area enlarges.




(a) $\varepsilon$=0.01 (b) $\varepsilon$=0.02 (c) $\varepsilon$=0.05

Figure 4. The Poincaré Map $\varphi$ vs. $d\varphi/d\tau$

Thus we can conclude that the periodic disturbance of the magnetic field of the earth can cause chaotic motion of the satellite and an increase of the disturbance can intensify the chaotic motion.

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