TECHNISCHE MECHANIK, Band 18, Heft 2, (1998), 135-139
Manuskripteingang: 16. September 1997

Buckling of Spherical Shells under Concentrated Load and Internal Pressure

S.M.Bauer, P.E.Tovstik

Stability of a complete spherical shell under concentrated load was first considered by D.Bushnell (1967) by means of a finite difference method. In the present paper the effect of the internal pressure on the critical value of a concentrated load is studied by means of a combination of asymptotic and numerical methods.

1 The Equation of the Axisymmetric Deformation

We start by examining the pre-buckling axisymmetric stress-strain state of a spherical shell. Let the shell with radius R be subjected to the concentrained load P applied at two poles $\theta_0 = 0$, $\theta_0 = \pi$, where θ_0 is the angle between the shell normal and the axis of symmetry (see Figure 1), and to the internal pressure p_n.

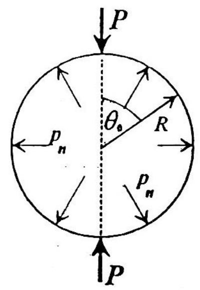

Figure 1. Sperical Shell

We use the dimensionless system of equations (Tovstik, 1996).

$$
\begin{aligned}
&(bV)' = bp(1+\mu\varepsilon_1)(1+\mu\varepsilon_2)\cos\theta \\
&\mu(bU)' = \varepsilon_2 + \mu\nu(U\cos\theta + V\sin\theta) - bp(1+\mu\varepsilon_1)(1+\mu\varepsilon_2)\sin\theta \\
&\mu(b\varepsilon_2)' = (1+\mu\varepsilon_1)\cos\theta - \cos\theta_0 \\
&\mu(bM_1)' = b(1+\mu\varepsilon_1)(U\sin\theta - V\cos\theta) + \mu M_2\cos\theta \\
&\mu\theta' = (1+\mu\varepsilon_1)\big(M_1 + \mu(1-\nu\kappa_2)\big)
\end{aligned}
\qquad (1)
$$

where

$$
(\ )' \equiv \frac{d(\ )}{d\theta_0} \qquad \mu^4 = \frac{h^2}{12(1-\nu^2)R^2} \qquad b = \frac{B}{R} = \sin\theta_0 \qquad \kappa_2 = \frac{\sin\theta - b}{b}
$$

$$
\varepsilon_1 = -\nu\varepsilon_2 + \mu(1-\nu^2)(U\cos\theta + V\sin\theta)
$$

$$
M_2 = \mu(\kappa_2 + \nu\kappa_1) \qquad \kappa_1 = \frac{\theta'}{1+\mu\varepsilon_1} - 1
$$

Here V and U are the projections of the stress resultants onto the axial and onto the normal to its directions, correspondingly,

$$
T_1 = U\cos\theta + V\sin\theta \qquad Q_1 = U\sin\theta - V\cos\theta
$$

where θ is the angle between the shell normal and the axis of symmetry after deformation, $\mu\varepsilon_1$ and $\mu\varepsilon_2$ are the tensile deformations of the meridian and of the parallel, M_1 and M_2 are the stress couples, h is the shell thickness, ν is the Poisson ratio, $\mu > 0$ is a small parameter, B is the distance between the point in the neutral surface and the axis of symmetry.

Dimensionless variables in equations (1) are related to the corresponding dimensional variables (marked by bars) as

$$\bar{U} = UEh\mu^2 \qquad \bar{V} = VEh\mu^2 \qquad \bar{M}_i = M_i EhR\mu^3 \qquad p_n = p\frac{Eh\mu^2}{R}$$

Firstly, we study the dependence of axisymmetric deformation of the shell on the applied load P. It is clear that the large deformations occur at the neigbourhood of the points where the load is applied. We study only the neigbourhood of the point $\theta_0 = 0$. From the first of equations (1) we get

$$V \simeq \frac{C}{\theta_0} + \frac{p\theta_0}{2} + O(\theta_0^3) \qquad \text{where} \qquad P = 2\pi REh\mu^2 C \tag{2}$$

in that neigbourhood.

System (1) contains the small parameter μ at the derivatives. Therefore we can use asymptotic method. We rescale the variables

$$\theta_0 = \mu\xi \qquad \theta = \mu\psi \qquad U = \frac{u}{\xi} \qquad C = \mu^2 c \qquad \text{or} \qquad P = \frac{\pi Eh^3 c}{6R(1-\nu^2)} \tag{3}$$

By taking into consideration formulas (2) and (3) and by expanding the trigonomical functions, system (1) is transformed with error of the order μ^2 into two nonlinear equations of the second order.

$$\begin{aligned} (\xi\psi^\cdot)^\cdot - \frac{\psi}{\xi} &= u\psi - c - \frac{p\xi^2}{2} \\ (\xi u^\cdot)^\cdot - \frac{u}{\xi} &= \frac{1}{2}(\xi^2 - \psi^2) \end{aligned} \qquad (\ )^\cdot = \frac{d(\ )}{d\xi} \tag{4}$$

The study of a system similar to equations (4) is reported in Shilkrut (1974).

2 Integration of System (4)

For an arbitrary load parameter $c < 0$ we seek a solution of system (4) which satisfies the following conditions

i) limited at the point where the concentrated load is applied ,

$$u = \psi = 0 \qquad \text{when} \qquad \xi = 0 \tag{5}$$

ii) which turns into the membrane solution when we go away from the domain of large deflections

$$\psi \to \xi \qquad u \to \frac{c}{\xi} + \frac{p\xi}{2} \qquad \text{as} \qquad \xi \to \infty \tag{6}$$

In order to get such a solution, we construct its asymptotic expansion in the neighbourhood of the point where the load is applied $\xi = 0$ and in a domain which is far away.

Taking into account conditions (5) and (6) we get as $\xi \to 0$

$$\psi^{(-)} = -\frac{c}{2}\xi\log\xi + C_1\xi + O(\xi^3\log\xi) \qquad u^{(-)} = C_2\xi + O(\xi^3\log^2\xi) \tag{7}$$

and as $\xi \to \infty$,

if $|p| < 4$ then

$$\psi^{(+)} = \xi + C_3 \Re\Big(\frac{1}{\sqrt{\xi}} e^{\lambda\xi + iC_4}\Big)\,(1 + O(\xi^{-1}))$$
$$u^{(+)} = \frac{c}{\xi} + \frac{p\xi}{2} - C_3 \Re\Big(\frac{1}{\sqrt{\xi}\lambda^2} e^{\lambda\xi + iC_4}\Big)\,(1 + O(\xi^{-1})) \tag{8}$$

and if $|p| > 4$ then

$$\psi^{(+)} = \xi + \Big(C_3 \frac{1}{\sqrt{\xi}} e^{\lambda_1\xi} + C_4 \frac{1}{\sqrt{\xi}} e^{\lambda_2\xi}\Big)\,(1 + O(\xi^{-1}))$$
$$u^{(+)} = \frac{c}{\xi} + \frac{p\xi}{2} - \Big(C_3 \frac{1}{\sqrt{\xi}\lambda_1^2} e^{\lambda_1\xi} + C_4 \frac{1}{\sqrt{\xi}\lambda_2^2} e^{\lambda_2\xi}\Big)\,(1 + O(\xi^{-1})) \tag{8'}$$

where

$$\lambda_{1,2}^2 = \frac{p}{4} \pm \sqrt{\frac{p^2}{16} - 1} \qquad \Re\{\lambda\} < 0$$

Both expansions depend on two unknown constants C_i, $i = 1, 2, 3, 4$. Using the condition that these two solutions $\psi^{(-)}(\xi)$ and $\psi^{(+)}(\xi)$, $u^{(-)}(\xi)$ and $u^{(+)}(\xi)$ (and their derivatives) should be equal at some intermediate point $\xi = \xi_*$, the constants may be evaluated numerically.

After determination of the constants C_i, using expansions (7) and (8) or (8$'$) as boundary conditions, we can get the general solution of system (4).

The plot of the deflection w_0 at the pole vs the load parameter c for some values of internal pressure p is shown in Figure 2.

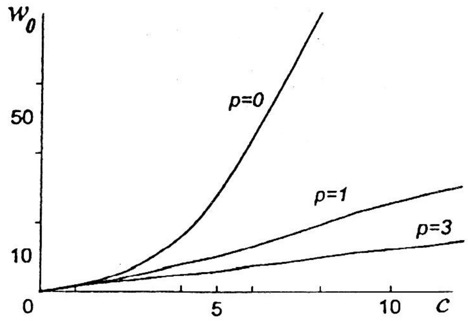

Figure 2. The Deflection at the Pole vs the Load Parameter

3 Equations of Axisymmetric State Bifurcation

We search the adjacent nonaxisymmetric equilibrium mode with m waves in the circurmferential direction in the form

$$w(\xi, \varphi) = w(\xi)\cos(m\varphi)$$

In order to construct the adjacent equillibrium mode we use the Donnell system of equations for shallow shells. After separation of variables and introduction of the dimensionless variables this system may be written as

$$\Delta\Delta w - \Delta_t w - \Delta_k \Phi = 0 \qquad \Delta\Delta\Phi + \Delta_k w = 0 \tag{9}$$

In equations (9) the quantities $w(\xi)$ and $\Phi(\xi)$ are the additional displacement and the stress function, and the differential operators Δ, Δ_k and Δ_t are given by formulas

$$\Delta w = \xi^{-2}\left[\xi(\xi w^{\cdot})^{\cdot} - m^2 w\right]$$
$$\Delta_k w = \xi^{-2}\left[\xi(\xi k_2 w^{\cdot})^{\cdot} - m^2 k_1 w\right] \tag{10}$$
$$\Delta_t w = \xi^{-2}\left[\xi(\xi t_1 w^{\cdot})^{\cdot} - m^2 t_2 w\right]$$

$$k_1 = \psi^{\cdot} \qquad k_2 = \psi/\xi \qquad t_1 = u/\xi \qquad t_2 = u^{\cdot} \tag{11}$$

where k_i (i=1,2) are the curvatures of the neutral surface after deformation, t_i are the main prebuckling stresses.

4 Integration of System (9)

Taking into account relations (11) and expansions (7) we can construct asymptotic expansions of four linearly independent solutions of system (9) in the neighbourhood of the point where the load is applied $\xi = 0$ and in the domain which is far away ($\xi = \infty$).

These solutions should be limited at the point where the concentrated load is applied , $\{w^{(-)}, \Phi^{(-)}\} \to 0$ as $\xi \to 0$.

$$\begin{aligned}
w_1^{(-)} &= \xi^{m+2} + O\left(\xi^{m+4}\right) & \Phi_1^{(-)} &= O\left(\xi^{m+4}\right)\\
w_2^{(-)} &= O\left(\xi^{m+4}\right) & \Phi_2^{(-)} &= \xi^{m+2} + O\left(\xi^{m+4}\right)\\
w_3^{(-)} &= \xi^{m} + O\left(\xi^{m+4}\log\xi\right) & \Phi_3^{(-)} &= -a\xi^{m+2}\log\xi + O\left(\xi^{m+4}\log\xi\right)\\
w_4^{(-)} &= a\xi^{m+2}\log\xi + O\left(\xi^{m+4}\log\xi\right) & \Phi_4^{(-)} &= \xi^{m}\log\xi + O\left(\xi^{m+4}\log\xi\right)
\end{aligned}$$

where $a = c(m-1)/[16(m+1)]$. To construct the expansions as $\xi \to \infty$ we can take $\psi \simeq \xi$, $u \simeq c/\xi + p\xi/2$. Then in equations (9)

$$\Delta w = \Delta_k w = w^{\cdot\cdot} + \frac{w^{\cdot}}{\xi} - \frac{m^2 w}{\xi^2} \qquad \Delta_t w = \frac{p}{2}\Delta w + \frac{c}{\xi^2}\left(w^{\cdot\cdot} - \frac{w^{\cdot}}{\xi} + \frac{m^2 w}{\xi^2}\right)$$

and the solutions of system (9) satisfying the conditions of damping $\{w^{(+)}, \Phi^{(+)}\} \to 0$ as $\xi \to \infty$, have the form

$$\begin{aligned}
w_1^{(+)} &= \Re\{Ze^{\lambda\xi}\} & \Phi_1^{(+)} &= \Re\{\lambda^2 Ze^{\lambda\xi}\}\\
w_2^{(+)} &= \Im\{Ze^{\lambda\xi}\} & \Phi_2^{(+)} &= \Im\{\lambda^2 Ze^{\lambda\xi}\}\\
w_3^{(+)} &= \xi^{-m} + O\left(\xi^{-m-2}\right) & \Phi_3^{(+)} &= O\left(\xi^{-m-2}\right)\\
w_4^{(+)} &= O\left(\xi^{-m-2}\right) & \Phi_4^{(+)} &= \xi^{-m} + O\left(\xi^{-m-2}\right)
\end{aligned} \qquad Z = \frac{1}{\sqrt{\xi}} + O\left(\frac{1}{\xi\sqrt{\xi}}\right)$$

here λ is the same as in equations (8).

The axisymmetric state bifurcation takes place for load parameter c, if there exist eight nonzero constants $C_i^{(-)}$, $C_i^{(+)}$, $i = 1,2,3,4$, such that at some intermediate point $\xi = \xi_*$ the functions

$$w^{(-)} = \sum_{i=1}^{4} C_i^{(-)} w_i^{(-)} \qquad \text{and} \qquad w^{(+)} = \sum_{i=1}^{4} C_i^{(+)} w_i^{(+)}$$

$$\Phi^{(-)} = \sum_{i=1}^{4} C_i^{(-)} \Phi_i^{(-)} \qquad \text{and} \qquad \Phi^{(+)}\} = \sum_{i=1}^{4} C_i^{(+)} \Phi_i^{(+)}$$

(and their three derivatives) are equal. The smallest (by the wave number m) load parameter $|c|$ coresponds to the critical value of concentrated load.

5 Results

In Table the values of the load parameter c are given for the different values of the internal pressure p and of the wave numbers m.

m	p=0	p=1	p=3	p=5
2	-10.876	-31.374	-82.437	-107.121
3	- 10.834	-27.624	-64.127	-94.632
4	-11.871	-29.992	-64.941	-93.606
5	-13.094	-33.582	-69.356	-96.206
6	-14.339	-37.774	-75.007	-100.111

Table 1. Load Parameter c as Function of Internal Pressure p and Wave Number m

For the case when the internal pressure is equal to zero the results agree well with those obtained by Bushnell (1967). The smallest parameter of load $|c|$ coresponds to the three waves mode. It is natural that the value of critical concentrated load increases with the internal pressure. When $p < 3.56$ the smallest load parameter $|c|$ coresponds to the three waves mode, when $p = 3.56$ the load parameter is the same for the three waves mode and for the four waves mode, if $p > 3.56$ the smallest load parameter $|c|$ coresponds to the four waves mode.

It is interesting to note that for different values of internal pressure the sizes of the large deformation zones are almost equal. The dent halfangle is approximately equal to

$$\theta_{0*} = 6\mu = \frac{6}{12^{1/4}(1-\nu^2)^{1/4}}\sqrt{\frac{h}{R}} \tag{12}$$

Let be $\nu = 0.3$. For $h/R = 0.02$ formula (12) gives $\theta_{0*} = 27^o$, and for $h/R = 0.05$ it gives $\theta_{0*} = 42^o$.

Acknowledgement
The research described in this publication was possible in part by Grants N 98 – 01 – 01006 and N 98 – 01 – 01010 from Russian Foundation for Fundamental Research.

Literature

1. Bushnell D. Bifurcation Phenomena in Spherical Shells under Concentrated and Ring Loads. AIAA Journ., Vol.5., (Nov. 1967), No 11, pp. 2034 -2040.
2. Koroteeva P.V., Tovstik P.E., Shuvalkin S.P.: On the Axial Force Value at the Post-buckling Axisymmetric Deflections of a Shell of Revolution. Vestnik S.-Peterburg. Univ. Mat.Mekh. Astron, (1995), No. 3, p. 58-61.
3. Shilkrut D.I. The Problems of Qualitative Theory of Nonlinear Shells. Kishinev, (1974), 144 p.
4. Tovstik P.E. The Post-buckling Axisymmetrical Deflections of Thin Shells of Revolution under Axial Loading. Technische Mechanik, 16, 2, (1996), 117–132.

Address: Professor Svetlana M. Bauer, Professor Peter E. Tovstik, Department of Theoretical and Applied Mechanics, Faculty of Mathematics and Mechanics, St. Petersburg University, Bibliotechnaya sq. 2, Stary Peterhof, RUS-198904 St. Petersburg.
E-mail: Svetlana.Bauer@pobox.spbu.ru, Peter.Tovstik@pobox.spbu.ru