TECHNISCHE MECHANIK, Band 18, Heft 2, (1998), 153-160
Manuskripteingang: 16. September 1997

# A New Method of Vibration Analysis of Elastic Systems, Based on the Lagrange Equations of the First Kind

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*The new method is offered for the vibration analysis of elastic systems, consisting of elements whose natural frequencies and shapes are known. The equations, describing the conditions of constraint of the system's elements with each other, are considered as holonomic constraints. It is shown, that the application of the Lagrange equations of the first kind allows to find the natural frequencies and shapes of the system as a whole from the natural frequencies and shapes of the system's elements. The quasistatic account of the maximum shapes of system elements natural oscillations enables to simplify essentially calculations by the method offered. This method is most effective to use for elastic systems, consisting of concentrated masses, rods, rings and plates. These elements can be connected to each other rigidly or by means of linear springs. The method is demonstrated on a specific example, allowing to show all its basic characteristics.*

## 1 The New Method

In Figure 1 an elastic system, consisting of three homogeneous elastic straight rods and one linear compliance $\delta = 1/c$ is represented. It is supposed, that the rods lie in one plane, the system has small oscillations, the rod 1 executes longitudinal oscillations, while rods 2 and 3 execute bending oscillations.

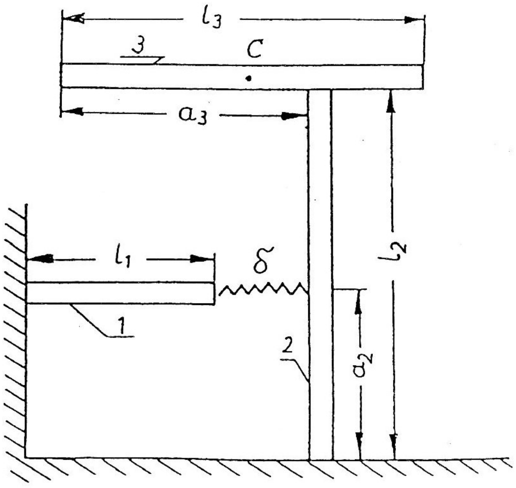


Figure 1. A Three-rod One-spring System

From the principle of system releasability from constraints it follows, that the rod oscillations can be described by

$$u(x_1,t) = \sum_{\sigma=1}^{\infty} q_{1\sigma}(t) X_{1\sigma}(x_1) \qquad X_{1\sigma}(x_1) = \sin\frac{(2\sigma-1)\pi x_1}{2l_1}$$

$$y_2(x_2,t) = \sum_{\sigma=1}^{\infty} q_{2\sigma}(t) X_{2\sigma}(x_2) \tag{1}$$

$$y_3(x_3,t) = \eta(t) + \left(x_3 - \frac{l_3}{2}\right)\varphi(t) + \sum_{\sigma=1}^{\infty} q_{3\sigma}(t) X_{3\sigma}(x_3)$$

$$0 \le x_i \le l_i \qquad i = 1, 2, 3$$

Here $X_{2\sigma}(x_2)$ and $X_{3\sigma}(x_3)$ are accordingly beam functions of consoles and free rod (Timoshenko, 1955; Babakov, 1965). The first two components in the expression for $y_3(x_3,t)$ correspond to a movement of rod 3 as absolutely rigid body. The value $\eta$ is equal to displacement of the mass centre $C$ of rod 3 lengthways along axis $y_3$, and $\varphi$ is the angle of its turn. Let us also enter into consideration the displacement $\xi$ of the mass centre $C$ of rod 3 along an axis $x_3$ and we shall consider the quantities $\xi$, $\eta$, $\varphi$, $q_{\nu\sigma}$ $(\nu = 1, 2, 3;\ \sigma = 1, 2, \ldots)$ as generalized Lagrange coordinates. Let $\delta = 0$, then all constraints between the coordinates are holonomic and are given by the following equations:

$$\begin{aligned}
f_1 &= u(l_1,t) - y_2(a_2,t) = \sum_{\sigma=1}^{\infty} q_{1\sigma} X_{1\sigma}(l_1) - \sum_{\sigma=1}^{\infty} q_{2\sigma} X_{2\sigma}(a_2) = 0 \\
f_2 &= y_2(l_2,t) - \xi = \sum_{\sigma=1}^{\infty} q_{2\sigma} X_{2\sigma}(l_2) - \xi = 0 \\
f_3 &= y_3(a_3,t) = \eta + \left(a_3 - \frac{l_3}{2}\right)\varphi + \sum_{\sigma=1}^{\infty} q_{3\sigma} X_{3\sigma}(a_3) = 0 \\
f_4 &= \left.\frac{\partial y_2}{\partial x_2}\right|_{x_2=l_2} + \left.\frac{\partial y_3}{\partial x_3}\right|_{x_3=a_3} = \sum_{\sigma=1}^{\infty} q_{2\sigma} X'_{2\sigma}(l_2) + \varphi + \sum_{\sigma=1}^{\infty} q_{3\sigma} X'_{3\sigma}(a_3) = 0
\end{aligned} \tag{2}$$

The kinetic energy of the rods and the potential energy of their deformation can be expressed as (Timoshenko, 1955; Babakov, 1965)

$$\begin{aligned}
T &= \frac{m_3\left(\dot{\xi}^2 + \dot{\eta}^2\right)}{2} + \frac{m_3 l_3^2 \dot{\varphi}^2}{24} + \sum_{\nu=1}^{n} \sum_{\sigma=1}^{\infty} \frac{M_{\nu\sigma} \dot{q}_{\nu\sigma}^2}{2} \\
\Pi &= \sum_{\nu=1}^{n} \sum_{\sigma=1}^{\infty} \frac{M_{\nu\sigma} \omega_{\nu\sigma}^2 q_{\nu\sigma}^2}{2} \qquad M_{1\sigma} = \frac{m_1}{2} \\
M_{\mu\sigma} &= \frac{m_\mu}{l_\mu} \int_0^{l_\mu} X_{\mu\sigma}^2(x)\,dx = \frac{m_\mu X_{\mu\sigma}^2(l_\mu)}{4}
\end{aligned} \tag{3}$$

$$n = 3 \qquad \sigma = 1, 2, \ldots \qquad \nu = 1, 2, 3 \qquad \mu = 2, 3$$

In these formulas the $\omega_{\nu\sigma}$ are the natural frequencies of the rods in the absence of constraints, and $m_\nu$ their masses. Let us take advantage of the Lagrange equations of the first kind, written down in generalized coordinates (Butenin and Fufaev, 1991).

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} - \frac{\partial L}{\partial q_p} = \sum_{i=1}^{k} \Lambda_i \frac{\partial f_i}{\partial q_p} \qquad L = T - \Pi \tag{4}$$

Here $k$ is the number of constraints, and $q_1, q_2, \ldots, q_p, \ldots$ are the generalized coordinates. Applying equations (4) we obtain

$$m_3\ddot{\xi} = -\Lambda_2 \qquad m_3\ddot{\eta} = \Lambda_3 \qquad \frac{m_3 l_3^2}{12}\ddot{\varphi} = \Lambda_3\left(a_3 - \frac{l_3}{2}\right) + \Lambda_4$$

$$\begin{aligned} M_{1\sigma}\left(\ddot{q}_{1\sigma} + \omega_{1\sigma}^2 q_{1\sigma}\right) &= \Lambda_1 X_{1\sigma}(l_1) \\ M_{2\sigma}\left(\ddot{q}_{2\sigma} + \omega_{2\sigma}^2 q_{2\sigma}\right) &= -\Lambda_1 X_{2\sigma}(a_2) + \Lambda_2 X_{2\sigma}(l_2) + \Lambda_4 X_{2\sigma}'(l_2) \\ M_{3\sigma}\left(\ddot{q}_{3\sigma} + \omega_{3\sigma}^2 q_{3\sigma}\right) &= \Lambda_3 X_{3\sigma}(a_3) + \Lambda_4 X_{3\sigma}'(a_3) \end{aligned} \tag{5}$$

The generalized reactions $\Lambda_1, \Lambda_2, \Lambda_3$ are equal to forces of rod interaction in points of their junction with each other, and $\Lambda_4$ is equal to the interactive moment between rods 2 and 3. From the principle of releasability from constraints it follows, that equations (5) can be considered also as the equations of dynamics of the system's elements under the action of forces $\Lambda_1, \Lambda_2, \Lambda_3$ and moment $\Lambda_4$ applied to them from the constraints and irrespective of whether they are rigid or elastic. This makes possible an application of the Lagrange equations of the first kind (4) from a case where there are rigid constraints to a case, when all or some constraints are elastic. First it is necessary to consider all constraints as rigid and, to use them to write equations (4). Then it is necessary to enter into the equations for elastic constraints, the reactions, appropriate to these constraints. In the example considered an elastic constraint is the first one. The reaction $\Lambda_1$ is equal to the force of a stretching (compression) of the springs of compliance $\delta$, and if the spring is stretched then $\Lambda_1 > 0$. Therefore from the first of the equations of system (2) there will come the equation

$$f_1 = \sum_{\sigma=1}^{\infty} q_{1\sigma} X_{1\sigma}(l_1) - \sum_{\sigma=1}^{\infty} q_{2\sigma} X_{2\sigma}(a_2) + \Lambda_1\delta = 0 \tag{6}$$

Let us note, that if we multiply it with negative unity, i.e. to write it down as $f_1^* = -f_1 = 0$, then the new Lagrange multiplier $\Lambda_1^*$ will be, obviously, such, that $\Lambda_1^* = -\Lambda_1$. Hence, a quantity $\Lambda_1^*\delta$ will enter an equation $f_1^* = 0$, also with a plus sign.

Thus, if the $i$-th constraint is elastic and its compliance is $\delta$, that, by writing it down at first as holonomic, we pass to an elastic constraint by an addition of the quantity $\Lambda_i\delta_i$.

Let the elastic system considered oscillate with a given natural frequency $p$. Then reactions $\Lambda_i$ and coordinates $q_\rho$ can be written

$$\Lambda_i = \tilde{\Lambda}_i \cos(pt + \alpha) \qquad \text{and} \qquad q_\rho = \tilde{q}_\rho \cos(pt + \alpha) \tag{7}$$

From here and from equations (5) there follows, that

$$\begin{aligned} \tilde{\xi} &= \frac{\tilde{\Lambda}_2}{m_3 p^2} \qquad \tilde{\eta} = -\frac{\tilde{\Lambda}_3}{m_3 p^2} \\ \tilde{\varphi} &= -\frac{12}{l_3^2}\,\frac{(a_3 - l_3/2)\tilde{\Lambda}_3 + \tilde{\Lambda}_4}{m_3 p^2} \\ \tilde{q}_{1\sigma} &= \frac{X_{1\sigma}(l_1)\tilde{\Lambda}_1}{M_{1\sigma}\left(\omega_{1\sigma}^2 - p^2\right)} \\ \tilde{q}_{2\sigma} &= \frac{-X_{2\sigma}(a_2)\tilde{\Lambda}_1 + X_{2\sigma}(l_2)\tilde{\Lambda}_2 + X_{2\sigma}'(l_2)\tilde{\Lambda}_4}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} \\ \tilde{q}_{3\sigma} &= \frac{X_{3\sigma}(a_3)\tilde{\Lambda}_3 + X_{3\sigma}'(a_3)\tilde{\Lambda}_4}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)} \end{aligned} \tag{8}$$

By substituting expressions (7) into the equations of constraints (2) and (6), and then the formulas (8), we obtain

$$\sum_{j=1}^{4}\alpha_{ij}(p^2)\tilde{\Lambda}_j = 0 \qquad \alpha_{ij} = \alpha_{ji} \qquad i = 1,\ldots, 4 \tag{9}$$

Here the index $i$ corresponds to the number of constraint. From system (9) follows that the equation of frequencies is

$$\det\left[\alpha_{ij}\left(p^2\right)\right] = 0 \tag{10}$$

It is expedient to represent the coefficients $\alpha_{ij}$ as

$$\alpha_{ii} = \delta_i + \beta_{ii} + \gamma_{ii}$$

$$\alpha_{ij} = \beta_{ij} + \gamma_{ij} \qquad i \neq j$$

Here $\delta_i$ is the compliance of $i$-th constraint in case it is elastic. The quantities $\beta_{ij}$ are inversely proportional to $p^2$ and can be called the factors of compliance of the inertia forces. In the case considered we have

$$\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0 \qquad \beta_{ij} = \beta_{ji}$$

$$\beta_{22} = -\frac{1}{m_3 p^2} \qquad \beta_{33} = -\frac{1+12(a_3 - l_3/2)^2/l_3^2}{m_3 p^2} \tag{11}$$

$$\beta_{34} = -\frac{12(a_3 - l_3/2)}{m_3 l_3^2 p^2} \qquad \beta_{44} = -\frac{12}{m_3 l_3^2 p^2}$$

Coefficients $\gamma_{ij} = \gamma_{ji}$ are the infinite sums of simple fractions.

$$\gamma_{11}\left(p^2\right) = \sum_{\sigma=1}^{\infty}\frac{X_{1\sigma}^2(l_1)}{M_{1\sigma}\left(\omega_{1\sigma}^2 - p^2\right)} + \sum_{\sigma=1}^{\infty}\frac{X_{2\sigma}^2(a_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)}$$

$$\gamma_{12}\left(p^2\right) = -\sum_{\sigma=1}^{\infty}\frac{X_{2\sigma}(a_2)X_{2\sigma}(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} \qquad \gamma_{13}\left(p^2\right) = 0$$

$$\gamma_{14}\left(p^2\right) = -\sum_{\sigma=1}^{\infty}\frac{X_{2\sigma}(a_2)X'_{2\sigma}(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)}$$

$$\gamma_{22}\left(p^2\right) = \sum_{\sigma=1}^{\infty}\frac{X_{2\sigma}^2(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} \qquad \gamma_{23}\left(p^2\right) = 0 \tag{12}$$

$$\gamma_{24}\left(p^2\right) = \sum_{\sigma=1}^{\infty}\frac{X_{2\sigma}^2(l_2)X'_{2\sigma}(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} \qquad \gamma_{33}\left(p^2\right) = \sum_{\sigma=1}^{\infty}\frac{X_{3\sigma}^2(a_3)}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)}$$

$$\gamma_{34}\left(p^2\right) = \sum_{\sigma=1}^{\infty}\frac{X_{3\sigma}(a_3)X'_{3\sigma}(a_3)}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)}$$

$$\gamma_{44}\left(p^2\right) = \sum_{\sigma=1}^{\infty}\frac{\left[X'_{2\sigma}(l_2)\right]^2}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} + \sum_{\sigma=1}^{\infty}\frac{\left[X'_{3\sigma}(a_3)\right]^2}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)}$$

When $p^2 = 0$ the values $\gamma_{ij}$ can be presented as

$$\gamma_{ij}(0) = \frac{\partial^2 \Pi}{\partial \Lambda_i \partial \Lambda_j} \tag{13}$$

Here $\Pi$ is the total potential energy of the system element deformations under action of the generalized reactions $\Lambda_i$. To be convinced of the validity of formulas (13), we will return to expression (3) for the potential energy of rod deformations and to the Lagrange equations (5). When the account of all shapes of systems element natural oscillations is quasistatic, i.e. when $\ddot{q}_{\nu\sigma} = 0$ ($\nu = 1, 2, 3$; $\sigma = 1, 2, \ldots$), we then have

$$\Pi = \frac{1}{2}\sum_{\sigma=1}^{\infty}\frac{\left[\Lambda_1 X_{1\sigma}(l_1)\right]^2}{M_{1\sigma}\omega_{1\sigma}^2} + \frac{1}{2}\sum_{\sigma=1}^{\infty}\frac{\left[-\Lambda_1 X_{2\sigma}(a_2) + \Lambda_2 X_{2\sigma}(l_2) + \Lambda_4 X'_{2\sigma}(l_2)\right]^2}{M_{2\sigma}\omega_{2\sigma}^2} + \frac{1}{2}\sum_{\sigma=1}^{\infty}\frac{\left[\Lambda_3 X_{3\sigma}(a_3) + \Lambda_4 X'_{3\sigma}(a_3)\right]^2}{M_{3\sigma}\omega_{3\sigma}^2} \tag{14}$$

Using this expression and formulas (12), the validity of formulas (13) has been established. It is essential that the potential energy of rod deformation can be written not only as infinite series (14), but also by finite expressions.

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3 \qquad \Pi_1 = \frac{\Lambda_1^2 l_1}{2E_1 S_1} \qquad \Pi_\mu = \frac{1}{2}\int_0^{l_\mu}\frac{M_\mu^2(x)dx}{E_\mu J_\mu} \qquad \mu = 2, 3 \tag{15}$$

Here $E$ is Young's modulus, $J$ is the moment of inertia of the rod cross section, and $S$ is the cross section area. The bending moments $M_2(x)$ and $M_3(x)$ are linear functions of the generalized reactions $\Lambda_i$. When calculating $M_3(x)$ it is necessary to take into account that the force $\Lambda_3$ and the moment $\Lambda_4$, applied to the third rod, are counterbalanced with its quasistatic deformation by forces of inertia of translatory and rotary motion. Using formulas (13) and (15), we obtain

$$\gamma_{11}(0) = \frac{l_1}{E_1 S_1} + \frac{a_2^3}{3E_2 J_2} \qquad \gamma_{12}(0) = -\frac{a_2^2(3l_2 - a_2)}{6E_2 J_2}$$

$$\gamma_{12}(0) = 0 \qquad \gamma_{14}(0) = -\frac{a_2^2}{2E_2 J_2} \qquad \gamma_{22}(0) = \frac{l_2^3}{3E_2 J_2}$$

$$\gamma_{23}(0) = 0 \qquad \gamma_{24}(0) = \frac{l_2^2}{2E_2 J_2} \qquad \gamma_{33}(0) = \frac{f_{33}(z) l_3^3}{E_3 J_3}$$

$$\gamma_{34}(0) = \frac{f_{34}(z) l_3^2}{E_3 J_3} \qquad \gamma_{44}(0) = \frac{f_{44}(z) l_3}{E_3 J_3} + \frac{l_2}{E_2 J_2} \tag{16}$$

$$f_{33} = \frac{1}{105} - \frac{11z}{105} + \frac{13z^2}{35} - \frac{z^3 + z^4}{3} + \frac{3z^5 - z^6}{5}$$

$$f_{34} = -\frac{11}{210} + \frac{13z}{35} - \frac{z^2}{2} - \frac{2z^3}{3} + \frac{3z^4}{2} - \frac{3z^5}{5}$$

$$f_{44} = \frac{13}{35} - z + 2z^3 - z^4 \qquad z = \frac{a_3}{l_3}$$

For an approximate calculation of frequencies by equation (10) it is possible to propose an approach, based on the dynamic calculation of the $N$ first natural forms of system oscillations with the quasistatic calculation of all other natural oscillation maximum shapes. Efficiency of the quasistatic account of natural oscillation maximum shapes in dynamic problems of the theory of elasticity was shown by Zegzhda (1979), Zegzhda (1986), and Vernigor (1990). According to this approach formulas $\gamma_{ij}(p^2)$ can be approximately calculated with a high degree of accuracy as

$$\gamma_{11}(p^2) = \sum_{\sigma=1}^{N} \frac{X_{1\sigma}^2(l_1)}{M_{1\sigma}\left(\omega_{1\sigma}^2 - p^2\right)} + \sum_{\sigma=1}^{N} \frac{X_{2\sigma}^2(a_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} +$$

$$+\gamma_{11}(0) - \sum_{\sigma=1}^{N} \frac{X_{1\sigma}^2(l_1)}{M_{1\sigma}\omega_{1\sigma}^2} - \sum_{\sigma=1}^{N} \frac{X_{2\sigma}^2(a_2)}{M_{2\sigma}\omega_{2}^2}$$

$$\gamma_{12}(p^2) = -\sum_{\sigma=1}^{N} \frac{X_{2\sigma}(a_2)X_{2\sigma}(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} + \gamma_{12}(0) + \sum_{\sigma=1}^{N} \frac{X_{2\sigma}(a_2)X_{2\sigma}(l_2)}{M_{2\sigma}\omega_{2\sigma}^2}$$

$$\gamma_{13}(p^2) = 0$$

$$\gamma_{14}(p^2) = -\sum_{\sigma=1}^{N} \frac{X_{2\sigma}(a_2)X'_{2\sigma}(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} + \gamma_{14}(0) + \sum_{\sigma=1}^{N} \frac{X_{2\sigma}(a_2)X'_{2\sigma}(l_2)}{M_{2\sigma}\omega_{2\sigma}^2}$$

$$\gamma_{22}(p^2) = \sum_{\sigma=1}^{N} \frac{X_{2\sigma}^2(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} + \gamma_{22}(0) - \sum_{\sigma=1}^{N} \frac{X_{2\sigma}^2(l_2)}{M_{2\sigma}\omega_{2\sigma}^2} \tag{17}$$

$$\gamma_{23}(p^2) = 0$$

$$\gamma_{24}(p^2) = \sum_{\sigma=1}^{N} \frac{X_{2\sigma}(l_2)X'_{2\sigma}(l_2)}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} + \gamma_{24}(0) - \sum_{\sigma=1}^{N} \frac{X_{2\sigma}(l_2)X'_{2\sigma}(l_2)}{M_{2\sigma}\omega_{2\sigma}^2}$$

$$\gamma_{33}(p^2) = \sum_{\sigma=1}^{N} \frac{X_{3\sigma}^2(a_3)}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)} + \gamma_{33}(0) - \sum_{\sigma=1}^{N} \frac{X_{3\sigma}^2(a_3)}{M_{3\sigma}\omega_{3\sigma}^2}$$

$$\gamma_{34}(p^2) = \sum_{\sigma=1}^{N} \frac{X_{3\sigma}(a_3)X'_{3\sigma}(a_3)}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)} + \gamma_{34}(0) - \sum_{\sigma=1}^{N} \frac{X_{3\sigma}(a_3)X'_{3\sigma}(a_3)}{M_{3\sigma}\omega_{3\sigma}^2}$$

$$\gamma_{44}(p^2) = \sum_{\sigma=1}^{N} \frac{\left[X'_{2\sigma}(l_2)\right]^2}{M_{2\sigma}\left(\omega_{2\sigma}^2 - p^2\right)} + \sum_{\sigma=1}^{N} \frac{\left[X'_{3\sigma}(a_3)\right]^2}{M_{3\sigma}\left(\omega_{3\sigma}^2 - p^2\right)}$$

$$+\gamma_{44}(0) - \sum_{\sigma=1}^{N} \frac{\left[X'_{2\sigma}(l_2)\right]^2}{M_{2\sigma}\omega_{2\sigma}^2} - \sum_{\sigma=1}^{N} \frac{\left[X'_{3\sigma}(a_3)\right]^2}{M_{3\sigma}\omega_{3\sigma}^2}$$

The static coefficients $\gamma_{ij}(0)$ are found in formulas (16). Let us now calculate the eigenfunctions. We shall designate the multipliers $\tilde{\Lambda}_j$, satisfying system (9) with natural frequencies $p_\rho$, as $\tilde{\Lambda}_{\rho j}$. From expressions (1) and (8) follows that the natural shapes of oscillations of the system considered are described by functions

$$u_\rho(x_1) = \sum_{\sigma=1}^{\infty} \frac{X_{1\sigma}(l_1)\tilde{\Lambda}_{\rho 1}}{M_{1\sigma}(\omega_{1\sigma}^2 - p_\rho^2)} X_{1\sigma}(x_1) \qquad 0 \le x_1 \le l_1$$

$$y_{\rho 2}(x_2) = \sum_{\sigma=1}^{\infty} \frac{-X_{2\sigma}(a_2)\tilde{\Lambda}_{\rho 1} + X_{2\sigma}(l_2)\tilde{\Lambda}_{\rho 2} + X'_{2\sigma}(l_2)\tilde{\Lambda}_{\rho 4}}{M_{2\sigma}(\omega_{2\sigma}^2 - p_\rho^2)} X_{2\sigma}(x_2) \qquad 0 \le x_2 \le l_2 \qquad (18)$$

$$y_{\rho 3}(x_3) = \sum_{\sigma=1}^{\infty} \frac{X_{3\sigma}(a_3)\tilde{\Lambda}_{\rho 3} + X'_{3\sigma}(a_3)\tilde{\Lambda}_{\rho 4}}{M_{3\sigma}(\omega_{3\sigma}^2 - p_\rho^2)} X_{3\sigma}(x_3) \qquad 0 \le x_3 \le l_3$$

So we find the appearance of the natural shapes of oscillations of initially complex elastic system through natural shapes of its separate elements.

It is expedient to enter into consideration the functions $u_\rho^{st}(x_1)$ and $y_{\rho\mu}^{st}(x_\mu)$ $(\mu = 2,3)$, describing the deformation of the rods in a quasistatic mode under action of the generalized reactions $\tilde{\Lambda}_{pj}$. These functions can be found in finite form by methods of resistance from the formulas (1) and (8) and can be supplied as infinite series.

$$u_\rho^{st}(x_1) = \sum_{\sigma=1}^{\infty} \frac{X_{1\sigma}(l_1)\tilde{\Lambda}_{\rho 1}}{M_{1\sigma}\omega_{1\sigma}^2} X_{1\sigma}(x_1) \qquad 0 \le x_1 \le l_1$$

$$y_{\rho 2}^{st}(x_2) = \sum_{\sigma=1}^{\infty} \frac{-X_{2\sigma}(a_2)\tilde{\Lambda}_{\rho 1} + X_{2\sigma}(l_2)\tilde{\Lambda}_{\rho 2} + X'_{2\sigma}(l_2)\tilde{\Lambda}_{\rho 4}}{M_{2\sigma}\omega_{2\sigma}^2} X_{2\sigma}(x_2) \qquad 0 \le x_2 \le l_2$$

$$y_{\rho 3}^{st}(x_3) = \sum_{\sigma=1}^{\infty} \frac{X_{3\sigma}(a_3)\tilde{\Lambda}_{\rho 3} + X'_{3\sigma}(a_3)\tilde{\Lambda}_{\rho 4}}{M_{3\sigma}\omega_{3\sigma}^2} X_{3\sigma}(x_3) \qquad 0 \le x_3 \le l_3$$

From here and from expressions (18) there follows, that the required forms of oscillations can be presented as

$$u_\rho(x_1) = u_\rho^{st}(x_1) + \sum_{\sigma=1}^{\infty} \frac{X_{1\sigma}(l_1)\tilde{\Lambda}_{\rho 1} p_\rho^2}{M_{1\sigma}(\omega_{1\sigma}^2 - p_\rho^2)\omega_{1\sigma}^2} X_{1\sigma}(x_1)$$

$$y_{\rho 2}(x_2) = y_{\rho 2}^{st}(x_2) + \sum_{\sigma=1}^{\infty} \frac{\left[-X_{2\sigma}(a_2)\tilde{\Lambda}_{\rho 1} + X_{2\sigma}(l_2)\tilde{\Lambda}_{\rho 2} + X'_{2\sigma}(l_2)\tilde{\Lambda}_{\rho 4}\right] p_\rho^2}{M_{2\sigma}(\omega_{2\sigma}^2 - p_\rho^2)\omega_{2\sigma}^2} X_{2\sigma}(x_2) \qquad (19)$$

$$y_{\rho 3}(x_3) = y_{\rho 3}^{st}(x_3) + \sum_{\sigma=1}^{\infty} \frac{\left[X_{3\sigma}(a_3)\tilde{\Lambda}_{\rho 3} + X'_{3\sigma}(a_3)\tilde{\Lambda}_{\rho 4}\right] p_\rho^2}{M_{3\sigma}(\omega_{3\sigma}^2 - p_\rho^2)\omega_{3\sigma}^2} X_{3\sigma}(x_3)$$

$$0 \le x_i \le l_i \qquad i = 1,\ 2,\ 3$$

## 2 Conclusion

The frequencies of longitudinal oscillations $\omega_{1\sigma}$ grow as $\sigma$, and frequencies of cross oscillations $\omega_{2\sigma}$ and $\omega_{3\sigma}$ as $\sigma^2$. The values $X'_{2\sigma}(l_2)$ and $X'_{3\sigma}(a_3)$ grow proportionally to $\sigma$. Therefore the series which is included in the first of formulas (19) converges with $1/\sigma^4$, and other two sums with $1/\sigma^7$. Such fast convergence of the series is explained by an allocation of quasistatic shapes of deformation of the system's elements in solution (19).

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