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Axisymmetric Vibrations of Thin Shells of Revolution Joint at a Small Angle

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Free axisymmetric vibrations of thin shells of revolution joined at a small angle are analyzed. The asymptotic integration method for equations with a small parameter is used. Vibration frequencies of a cylindrical shell joined with a conical shell are found. A comparison of numerical and asymptotic results is performed.

1 Introduction

Assemblies containing shells of revolution joined along parallel circles find wide use in the modern engineering, especially in aircraft and spacecraft industry. In many papers, the vibration of connected shells are analyzed by means of numerical methods (see, for example, Anderson at al., 1971; Bushnell, 1974; Hu and Raney, 1967). The asymptotic solutions (Filippov, 1975, 1977, 1981) amplify the numerical results and clarify qualitatively the allocation of the vibration frequencies and behavior of the vibration modes.

The approximate formulae for the lowest frequencies of thin connected shells, obtained in papers of Filippov (1977, 1981), do not allow to consider axisymmetric vibrations of the shells joint at a small angle. In this paper more general asymptotic formulae are found. By means of these formulae the lowest frequencies of axisymmetric vibrations for all values of the connection angle can be calculated. As an example the axisymmetric vibrations of a cylindrical shell joined with a conical shell are considered. A comparison of the asymptotic and numerical results is performed.

2 Basic Equations

Let us consider two shells of revolution of equal material and thickness, connected at an angle φ (see Figure 1).

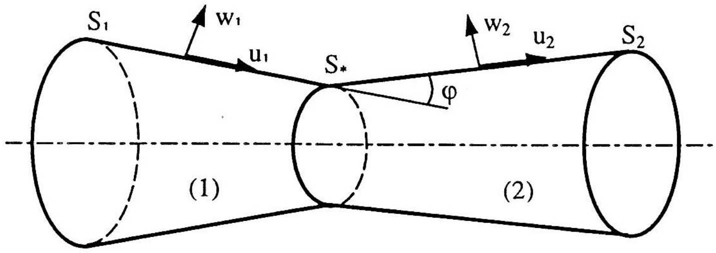

Figure 1. Two Thin Shells of Revolution Joint at a Small Angle

The free axisymmetric vibration of either of the shells are described by the following dimensionless equations:

$$\left(\frac{(Bu)'}{B}\right)' + \left(\lambda + \frac{1-\nu}{R_1 R_2}\right)u + \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)w' + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)' w = 0$$

$$\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)u' + \left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)\frac{B'}{B}u - \left(\lambda - \frac{1}{R_1^2} - \frac{2\nu}{R_1 R_2} - \frac{1}{R_2^2}\right)w + \frac{\mu^4}{B}\left(B\left(\frac{(Bw')'}{B}\right)'\right)' = 0 \qquad (1)$$

where

$$\frac{1}{R_1}=\frac{B''}{\sqrt{1-(B')^2}} \qquad \frac{1}{R_2}=\frac{\sqrt{1-(B')^2}}{B} \qquad \mu^4=\frac{h^2}{12}$$

The superscript $(\,)'$ denotes the derivative respect to the meridian co-ordinate s, u and w are the components of displacement, B is the distance between a point of a middle surface and the axis of revolution, R_1 and R_2 are the radii of curvature, $\lambda = (1-\nu^2)\rho\omega^2 E^{-1}$ is the frequency parameter, ν is Poisson's ratio, E is Young's modulus, ρ is the mass density, ω is the vibration frequency. The dimensionless thickness h is a small parameter.

For the first shell s changes from s_1 to s_* and for the second shell it changes from s_* to s_2. We suppose that the shell edges $s = s_1$ and $s = s_2$ are clamped and on these,

$$u^{(k)}=w^{(k)}=\theta^{(k)}=0 \qquad (k=1,2) \tag{2}$$

At the connection lines $s = s_*$ the following continuity conditions are to be satisfied:

$$\begin{aligned}
&T_1^{(1)}=T_1^{(2)}\cos\varphi+N_1^{(2)}\sin\varphi && N_1^{(1)}=N_1^{(2)}\cos\varphi-T_1^{(2)}\sin\varphi\\
&u^{(1)}=u^{(2)}\cos\varphi+w^{(2)}\sin\varphi && w^{(1)}=w^{(2)}\cos\varphi-u^{(2)}\sin\varphi\\
&\theta^{(1)}=\theta^{(2)} && M_1^{(1)}=M_1^{(2)}
\end{aligned} \tag{3}$$

Here T_1, N_1 and M_1 are the dimensionless stress-resultants and stress-couple, θ is the angle of rotation. The superscript $k(k = 1, 2)$ denotes variables corresponding to the first and the second shells, respectively. We shall omit superscripts in formulae like equations (1), which are valid for both shells. The connections between T_1, N_1, M_1, θ and u, w have the form

$$T_1=u'+\frac{w}{R_1}+\nu\left(\frac{B'}{B}u+\frac{w}{R_2}\right) \qquad M_1=\mu^4\left(\theta'+\nu\frac{B'}{B}\theta\right)$$

$$N_1=M_1'+\frac{B'}{B}(M_1-M_2) \qquad M_2=\mu^4\left(\frac{B'}{B}\theta+\nu\theta'\right)$$

$$\theta=-w'+\frac{u}{R_1}$$

3 Asymptotic Analysis

System (1) can be reduced to the following equation:

$$\mu^4\left(\frac{d^6w}{ds^6}+d_5\frac{d^5w}{ds^5}+\ldots\right)-b\frac{d^2w}{ds^2}-b_1\frac{dw}{ds}-b_0w=0$$

where $b(s) = \lambda - (1-\nu^2)R_2^{-2}(s)$. When $\mu \to 0$ the integrals of this equation have the following asymptotic expansions:

$$w_n = W_n(s)\exp\left(\frac{1}{\mu}\int q_n\,ds\right)+\ldots \quad n = 1, 2, 3, 4 \qquad w_m = w_{m0}+\ldots \quad m = 5, 6$$

where q_n is a root of the equation $q^4 = b$, w_{50} and w_{60} are the solutions of the momentless equation

$$b\frac{d^2w}{ds^2} + b_1\frac{dw}{ds} + b_0 w = 0$$

If $b(s) < 0$ for all s then $\Re(q_n) \neq 0$, functions $w_n(s)$ ($n = 1, 2, 3, 4$) increase or decrease rapidly and are called boundary effect functions (see Gol'denveizer at al., 1979).

Therefore the approximate solution of the boundary value problem equations (1) to (3) for

$$\lambda < \lambda_* = \min_{s \in [s_1, s_2]} \left(1 - \nu^2\right) R_2^{-2}(s)$$

can be expressed as

$$x = \mu^{\gamma_0(x)} x_0 + \mu^{\gamma_b(x)} x_b \qquad \lambda = \lambda_0 + \mu\lambda_1 + \ldots \tag{4}$$

where x denotes one of the variables u, w, T_1, N_1, θ or M_1. The first term x_0 is the solution of the momentless system, which can be derived from system (1) by choosing $\mu = 0$. The second term x_b is the linear combination of the boundary effect functions. The intensity coefficients $\gamma_0(x)$ and $\gamma_b(x)$ are presented in Table 1.

function	u	w	T_1	N_1	θ	M_1
γ_0	0	0	0	4	0	4
γ_b	1	0	1	1	-1	2

Table 1. The Intensity Coefficients

The order of the momentless system is 2 while the order of system (1) is 6. Therefore the solution x_0 can not satisfy all conditions (2) and (3). To find function x_0 and an eigenvalue λ_0 of the momentless boundary value problem for the momentless system it is necessary to choose one main condition from three boundary conditions (2) and two main conditions from six continuity conditions (3). The others, additional, conditions can be satisfied by the choice of the boundary effect functions x_b . The separation into main and additional conditions is called the splitting of the boundary conditions.

Splitting of the boundary conditions (2) is carried out in the book of Gol'denveizer at al. (1979). The main conditions on the shell edges are:

$$u^{(k)}(s_k) = u_0^{(k)}(s_k) + \mu u_b^{(k)}(s_k) = 0 \qquad k = 1, 2$$

Discarding minor terms $\mu u_b^{(k)}$ (s_k) we obtain the following conditions for the momentless system:

$$u_0^{(k)}(s_k) = 0 \qquad k = 1, 2 \tag{5}$$

In the papers of Filippov (1975, 1977) it is shown that for $\varphi \sim 1$ the conditions for the momentless system at the connection line have the form

$$T_{10}^{(1)} = T_{10}^{(2)} = 0 \qquad s = s_* \tag{6}$$

Conditions (6) are not valid for small values of the angle φ .
Let us consider

$$\varphi = \varphi_0 \mu^{\alpha} \qquad \varphi_0 \sim 1 \qquad 0 < \alpha < 1 \tag{7}$$

Taking into account equations (6) and (7) and the data from Table 1, the continuity condition (3) may be represented as follows:

$$T_{10}^{(1)} + \mu T_{1b}^{(1)} = T_{10}^{(2)} + \mu T_{1b}^{(2)} + \varphi_0 \mu^{\alpha}\left(\mu^4 N_{10}^{(2)} + \mu N_{1b}^{(2)}\right) \tag{8.1}$$

$$\mu^4 N_{10}^{(1)} + \mu N_{1b}^{(1)} = \mu^4 N_{10}^{(2)} + \mu N_{1b}^{(2)} - \varphi_0 \mu^{\alpha}\left(T_{10}^{(2)} + \mu T_{1b}^{(2)}\right) \tag{8.2}$$

$$u_0^{(1)} + \mu u_b^{(1)} = u_0^{(2)} + \mu u_b^{(2)} + \varphi_0 \mu^{\alpha}\left(w_0^{(2)} + w_b^{(2)}\right) \tag{8.3}$$

$$w_0^{(1)} + w_b^{(1)} = w_0^{(2)} + w_b^{(2)} - \varphi_0 \mu^{\alpha}\left(u_0^{(2)} + \mu u_b^{(2)}\right) \tag{8.4}$$

$$\mu \theta_0^{(1)} + \theta_b^{(1)} = \mu \theta_0^{(2)} + \theta_b^{(2)} \tag{8.5}$$

$$\mu^2 M_{10}^{(1)} + M_{1b}^{(1)} = \mu^2 M_{10}^{(2)} + M_{1b}^{(2)} \qquad s = s_* \tag{8.6}$$

From these six continuity conditions it is necessary to choose two main and four additional ones. We consider the differences $\Delta_i = \gamma_{oi} - \gamma_{bi}$, where γ_{oi} and γ_{bi} are the orders in μ of the main terms among the momentless and boundary effect functions respectively in the equation (8.i). For example, in equation (8.2) the main momentless term is $\mu^{\alpha}\varphi_0 T_{10}^{(2)}$ and its order $\gamma_{02} = \alpha$. The main terms among the boundary effect functions in equation (8.2) are $\mu N_{1b}^{(k)}$. Therefore $\gamma_{b2} = 1$, $\Delta_2 = \alpha - 1$. It follows from equations (8) that

$$\Delta_1 = -1 \qquad \Delta_2 = \alpha - 1 \qquad \Delta_3 = -\alpha \qquad \Delta_4 = 0 \qquad \Delta_5 = 1 \qquad \Delta_6 = 2$$

The values Δ_i for the main boundary conditions must be strictly less than for the additional ones. The condition (8.1) ist the main condition for any values of α. Discarding minor terms in equation (8.1) we obtain the condition

$$T_{10}^{(1)} = T_{10}^{(2)} \qquad s = s_* \tag{9}$$

If $\alpha < 1/2$ then equation (8.2) is the second main condition. The main conditions (8.1) and (8.2) give the conditions (6) for the momentless system. If $\alpha > 1/2$ then equation (8.3) becomes the second main condition and the conditions for the momentless system have the form

$$T_{10}^{(1)} = T_{10}^{(2)} \qquad u_0^{(1)} = u_0^{(2)} \qquad s = s_* \tag{10}$$

In case $\alpha = 1/2$ we have the equality $\Delta_2 = \Delta_3$. To obtain the second main condition for $\alpha = 1/2$ we shall compose the linear combination of the boundary conditions (8).

The boundary effect functions for the first and the second shell close to line $s = s_*$ are:

$$w_b^{(1)} = e^{-\xi_1}\left(c_1 \cos\xi_1 + c_2 \sin\xi_1\right) \qquad w_b^{(2)} = e^{-\xi_2}\left(c_3 \cos\xi_2 + c_4 \sin\xi_2\right) \tag{11}$$

where

$$\xi_1 = \frac{q}{\mu\sqrt{2}}\left(s - s_*\right) \qquad \xi_2 = -\frac{q}{\mu\sqrt{2}}\left(s - s_*\right) \qquad q = \left|b\left(s_*\right)\right|^{\frac{1}{4}}$$

The analogous formulae are valid for the functions u_b, T_{1b}, N_{1b}, θ_b and M_{1b} (see Gol'denveizer at al., 1979).

By substituting expressions (11) into the conditions (8.2) to (8.6) and discarding of insignificant terms we obtain

$$
\begin{aligned}
&2^{-1/2}\ \mu^{1/2}\ b^{3/4}\left(-c_1 + c_2 + c_3 - c_4\right) + \varphi_0\ T_{10}^{(2)} = 0 \\
&u_0^{(1)} - u_0^{(2)} - \mu^{1/2}\ \varphi_0\ c_3 = 0 \\
&w_0^{(1)} - w_0^{(2)} + c_1 - c_3 = 0 \\
&c_1 + c_2 + c_3 + c_4 = 0 \\
&c_2 - c_4 = 0
\end{aligned}
\tag{12}
$$

The elimination of c_1, c_2, c_3 and c_4 from the system (12) gives the second main condition:

$$
A\left(u_0^{(1)} - u_0^{(2)}\right) + \varphi_0^2\ T_{10}^{(2)} = 0 \qquad\qquad s = s_* \tag{13}
$$

where $A = 2^{3/2}|b|^{3/4}$.

If $\varphi_0 << 1$ then condition (13) takes the from $u_0^{(1)}(s_*) = u_0^{(2)}(s_*)$ and we have conditions (10) for the momentless system. If $\varphi_0 >> 1$ then it follows from equation (13) that $T_{10}^{(2)}(s_*) = 0$ and the momentless solution satisfies conditions (6). Consequently conditions (9) and (13) are more general than conditions (6). Conditions (9) and (13) can be used for any values of the angle φ .

4 Numerical Results

As an example let us consider the axisymmetric vibrations of a cylindrical shell joined with a conical shell at a small angle. The solution of the momentless system for the cylindrical shell is:

$$
u_0^{(1)} = a_1 \cos(ks) + a_2 \sin(ks) \qquad w_0^{(1)} = \frac{\nu}{\lambda - 1}\left(u_0^{(1)}\right)' \qquad k^2 = \frac{\lambda(1-\lambda)}{1-\nu^2-\lambda} \tag{14}
$$

In the general case the momentless system for the conical shell has no analytical solution. For the problem at issue the middle surface of the conical shell is close to the middle surface of the cylindrical shell. Therefore we can use the solution of the momentless system for the cylindrical shell

$$
u_0^{(2)} = a_3 \cos(ks) + a_4 \sin(ks) \qquad w_0^{(2)} = \frac{\nu}{\lambda - 1}\left(u_0^{(2)}\right)' \tag{15}
$$

as the approximate solution for the conical shell. By substituting equations (14) and (15) into equations (5), (9) and (13) we obtain the system of linear algebraic equations containing the four unknown constants a_1, a_2, a_3, a_4:

$$
\mathbf{M}(\lambda) \cdot \mathbf{a} = 0
$$

Here $\mathbf{a} = (a_1, a_2, a_3, a_4)$. The nonzero elements of the matrix $\mathbf{M}$ are:

$$
\begin{aligned}
&m_{11} = \cos(ks_1) \quad m_{12} = \sin(ks_1) \quad m_{23} = \cos(ks_2) \quad m_{24} = \sin(ks_2) \\
&m_{31} = -\sin(ks_*) \quad m_{32} = \cos(ks_*) \quad m_{33} = -m_{31} \quad m_{34} = -m_{32} \\
&m_{41} = \cos(ks_*) \quad m_{42} = \sin(ks_*) \quad m_{43} = -\cos(ks_*) - r m_{33} \quad m_{44} = -\sin(ks_*) + r m_{32} \\
&r = \varphi_0^2\left(1-\nu^2\right)k / \left[A(\lambda - 1)\right]
\end{aligned}
$$

The parameter λ_0 is the root of the equation

$$\det \mathbf{M}(\lambda) = 0 \tag{16}$$

For shells with the paramters $h = 0.001$, $l_1 = s_* - s_1 = 2$, $l_2 = s_2 - s_* = 2$, $\nu = 0.3$ the radius of the cylindrical shell is taken as unity. Table 2 contains the least values of the frequency parameter λ for different values of connection angle φ are given.

φ in degrees	λ_0 (approximate value of λ)	λ (numerical calculations)
0	0.5048	0.5069
5	0.4617	0.4640
10	0.3997	0.4070
15	0.3540	0.3602
20	0.3231	0.3259

Table 2. The Frequency Parameter λ vs. the Angle φ

In the second column the least root λ_0 of equation (16) is shown. Numerical results (third column) were obtained by using Finite Element Method. Asymptotic and numerical results are in good agreement with each other.

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