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Mathematical Modelling of Spatial Contact Interaction of a System of Finite Cylindrical Bodies

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This paper deals with the mathematical modelling of spatial contact interaction of a system of finite cylindrical bodies. The approach significantly reduces the complexity of investigating the contact stress in practice. It allows to define the influence of the form, the loading character and the material properties of the interacting bodies on the distribution of contact stresses.

1 Introduction

A model of elastic contact of a system of finite cylindrical bodies is widely used in applications of engineering mechanics. It forms a base for calculating a stress state of some parts of machines, such as plain bearings, hinges (Jonson, 1989; Kovalenko, 1995; Levina, 1971).

For the first time the contact problem was solved by Hertz. He supposed that for the calculation of local stresses each body may be considered as an elastic half-space loaded along a small elliptical area on its surface. Within this assumption generally accepted by the theory of contact problems, the stresses which are concentrated near the area of contact are investigated independently of their shape and the method of attaching. The results received on the basis of Hertz's theory describe a stress state in a narrow set of the contact problem (Jonson, 1989; Muskhelishvily, 1966; Shtaerman, 1949). But its distinctive features are widely and easily used in engineering calculation.

Further development of the contact interaction mechanics was connected, mainly, with lifting these restrictions and with the broad implementation of analytical methods (method of complex potential and etc.). But existenting investigations show that these methods can't be used in many cases of contact interaction of finite cylindrical bodies due to the bulky character of mathematical transformations (Kovalenko, 1995; Teply, 1983; Chernets,1996).

Significant progress in solving the contact problem is connected with the application of numerical methods. Several software programs have been developed now to analyze a stress state of any bodies. However, certain difficulties arise when using these programs to solve three-dimensional contact problems. In particular one must take into account local geometric peculiarities of the investigated bodies, high cost of this software and the required equipment. Therefore these investigations are too expensive to be widely used in engineering calculations.

The present paper considers a method which allows to reduce the contact problems for cylindrical bodies to a system of a two-dimension boundary problem and generalizes Hertz's solution for two cylinders to the case of variable load intensitiy. It considerably reduces complexity and cost for calculating the contact stress in practice. The approach uses the method of complex potentials for an explicit approximate solution of the two-dimension contact problem of an isotropic elastic disk and a plate with a cylindrical hole.

2 Statement of the Problem

Consider a system of elastic isotropic cylinders of finite length. Its axes are parallel and belong to the plane YOZ. The axis OZ is the axis one of the cylinders. Force is acting on this system in the plane YOZ too. The system is in elastic equilibrium. It is assumed that any cross-section of any cylinder remains plane after load. Friction in areas of contact is not considered (Figure 1).

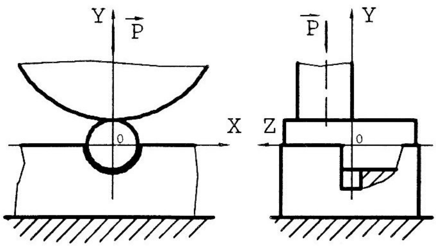

Figure 1. System of Three Elastic Finite Length Cylinders

3 Solving Auxiliary Problems

Any cross-section of two neighbouring cylinders realize the following interactions schemes:

- contact as specified by Hertz's theory (Figure 2);
- interior contact of cylinders with similar radii (Figure 3).

In the first case the contact stress is described by the following equation (Jonson, 1989; Muskhelishvily, 1966):

$$\sigma(x) = \frac{R+r}{2Rr}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)^{-1} \sqrt{\left(l^2 - x^2\right)}$$

$$l = 2\sqrt{\frac{P\,rR}{\pi(r+R)}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)}$$

Here P is the intensity of load, R and r are the radii of curvature of the interacting bodies, $E_i(i = 1, 2)$ are the Young's moduli, $\nu_i(i = 1, 2)$ are the Poisson's ratios, l specifies the area of contact, $\sigma(x)$ is the normal contact stress.

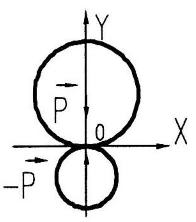

Figure 2. Scheme of Cylinders Location in Hertz's Theory

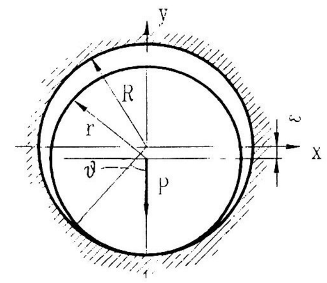

Figure 3. Scheme of Bodies Location in Non-Hertz's Theory of Interacting Cylinders

Consider an elastic isotropic plate with a cylindrical hole of radius R. An elastic isotropic disk of radius r is put into the hole. It will be assumed that ε^2, ε / R $(\varepsilon = R - r > 0)$ are small values which can be neglected. Force P acts along the $y-$ axis (Figure 3). Due to the fact that displacements are negligible in comparison with the dimensions of the bodies in the area of contact specified by L one obtains:

$$\left(x_1 + u_1\right)^2 + \left(y_1 + \upsilon_1\right)^2 = \left(x_2 + u_2\right)^2 + \left(y_2 + \upsilon_2 - \delta\right)^2 \qquad (1)$$

where

$$x_1 = R\cos(\zeta) \qquad y_1 = R\sin(\zeta)$$

$$x_2 = r\cos(\zeta) \qquad y_2 = r\sin(\zeta) - \varepsilon$$

u_m, $\upsilon_m \left(m = \overline{1,2}\right)$ are components of displacements of plate with the hole $(m = 1)$ and for the elastic disk $(m = 2)$; δ is the displacement of the disk center. Its easy to see that equation (1) reduces to

$$\varepsilon + u_1 \cos(\zeta) + \upsilon_1 \sin(\zeta) = u_2 \cos(\zeta) + (\upsilon_2 - \delta)\sin(\zeta)$$

and after transforming

$$\begin{aligned} &\varepsilon - 2\frac{\partial u_1}{\partial \zeta}\sin(\zeta) + 2\frac{\partial \upsilon_2}{\partial \zeta}\cos(\zeta) + \left(\frac{\partial^2 u_1}{\partial \zeta^2}\cos(\zeta) + \frac{\partial^2 \upsilon_1}{\partial \zeta^2}\sin(\zeta)\right) \\ &= -2\frac{\partial u_2}{\partial \zeta}\sin(\zeta) + 2\frac{\partial \upsilon_2}{\partial \zeta}\cos(\zeta) + \left(\frac{\partial^2 u_2}{\partial \zeta^2}\cos(\zeta) + \frac{\partial^2 \upsilon_2}{\partial \zeta^2}\sin(\zeta)\right) \end{aligned} \tag{2}$$

But on the contour of the hole we arrive at (Prusov, 1978)

$$\frac{1}{R_m}\left(\frac{\partial \upsilon_{\theta m}}{\partial \theta} + \upsilon_{rm}\right) = \frac{1}{E_m}\left(\left(1 - \nu_m^2\right)\sigma_{\theta m} - \upsilon_m\left(1 + \nu_m\right)\sigma_r\right) \tag{3}$$

where $R_m = R\,(m = 1)$ and $R_m = r\,(m = 2)$, $\nu_i, \left(i = \overline{1,2}\right)$ are Poissons's ratios, $E_i, \left(i = \overline{1,2}\right)$ are Young's moduli, $\sigma_{\theta n}$, σ_r are normal components of stress. Then, using equations (2) and (3), we obtain

$$\begin{aligned} &\varepsilon + \frac{R}{E_1}\left(\left(1 - \nu_1^2\right)\sigma_{\theta 1} - \nu_1\left(1 + \nu_1\right)\sigma_r\right) + \frac{\partial}{\partial \zeta}\left(\frac{\partial u_1}{\partial \zeta}\cos(\zeta) + \frac{\partial \upsilon_1}{\partial \zeta}\sin(\zeta)\right) \\ &= \frac{r}{E_2}\left(\left(1 - \nu_2^2\right)\sigma_{\theta 2} - \nu_2\left(1 + \nu_2\right)\sigma_r\right) + \frac{\partial}{\partial \zeta}\left(\frac{\partial u_2}{\partial \zeta}\cos(\zeta) + \frac{\partial \upsilon_2}{\partial \zeta}\sin(\zeta)\right) \end{aligned} \tag{4}$$

It's known that (Muskhelishvily, 1966)

$$\begin{aligned} &\sigma_{\theta n} + \sigma_r = 2\left[\Phi_m(w) + \overline{\Phi_m(w)}\right] \\ &\sigma_{\theta n} - \sigma_r + 2i\tau_{r\zeta m} = 2e^{2i\zeta}\left[\overline{w}\,\Phi'(w) + \Psi(w)\right] \\ &2\mu_m\left(u_m + i\upsilon_m\right) = \kappa_m\varphi_m(w) - w\overline{\Phi_m(w)} - \overline{\psi_m(w)} \end{aligned} \tag{5}$$

here $w = z\,(m = 1)$, $w = s\,(m = 2)$, $\mu_m\left(m = \overline{1,2}\right)$ are Lame's coefficients; $i = \sqrt{-1}$; $\kappa_m = 3 - 4\nu_m$, $\varphi_m(w)$, $\psi_m(w)$ are Kolosov- Muskhelishvily complex potentials $\varphi'(w) = \Phi_m(w)$, $\psi'(w) = \psi_m(w)$.

Then using equality (Teply, 1983)

$$\begin{aligned} \Phi(z) &= \frac{\kappa_1}{2\pi(1+\kappa_1)}\frac{iP}{z} - \frac{1}{2\pi i}\int_L \frac{\sigma_r(\tau)d\tau}{\tau - z} \\ \Phi(s) &= \frac{-iP}{2\pi(1+\kappa_2)}\frac{1}{s} - \frac{iP}{2\pi(1+\kappa_2)}\frac{s}{r^2} + \frac{1}{2\pi i}\int_L \frac{\sigma_r(\xi)d\xi}{\xi - s} - \frac{1}{4\pi i}\int_L \frac{\sigma_r(\xi)d\xi}{\xi} \\ \xi &= \tau\, r / R \end{aligned} \tag{6}$$

and with equations (4) and (5), we arrive at the integral equation

$$\frac{t}{\pi i}\int_L \frac{\sigma_r'(\tau)d\tau}{\tau - t} = \gamma_1\sigma_r(\tau) - \frac{iP}{\pi}\gamma_2\left(\frac{1}{t} - \frac{t}{R^2}\right) - \gamma_3\frac{P}{\pi} - \gamma_4 b - \gamma_5\varepsilon \tag{7}$$

where

$$\gamma_1 = \frac{\left(1-\nu_2-2\nu_2^2\right)E_1 Rr - \left(1-\nu_1-2\nu_1^2\right)E_2 R^2}{2\left(R^2 E_2\left(1-\nu_1^2\right) + r^2 E_1\left(1-\nu_2^2\right)\right)}$$

$$\gamma_2 = \frac{\left(1+\nu_2\right)E_1 Rr + \left(3-4\nu_1\right)\left(1+\nu_1\right)E_2 R^2}{4\left(R^2 E_2\left(1-\nu_1^2\right) + r^2 E_1\left(1-\nu_2^2\right)\right)}$$

$$\gamma_3 = \frac{\left(1+\nu_2\right)\varepsilon R E_1}{8r\left(R^2 E_2\left(1-\nu_1^2\right)+r^2 E_1\left(1-\nu_2^2\right)\right)}$$

$$\gamma_4 = \frac{\left(1-\nu_1^2\right)E_2}{\left(R^2 E_2\left(1-\nu_1^2\right)+r^2 E_1\left(1-\nu_2^2\right)\right)}$$

$$\gamma_5 = \frac{E_1 E_2}{2\left(R^2 E_2\left(1-\nu_1^2\right)+r^2 E_1\left(1-\nu_2^2\right)\right)}$$

$$\frac{b}{R^2} = -\frac{1}{2\pi i}\int_L \frac{\sigma_r}{\tau}d\tau$$

$$t = \mathrm{R}e^{i\zeta} \qquad c = \mathrm{R}e^{i\alpha_0}$$

where α_0 is a contact half-angle.

The results of the investigations (Levina, 1971; Narodetzki, 1943) show that the approximate solution of equation (7) can be expressed in the following form

$$\sigma_r(\theta) = -P\frac{\sqrt{2}}{R}\left[\gamma_2\frac{2}{\pi}+\frac{\gamma_1}{\alpha_0-\cos(\alpha_0)\sin(\alpha_0)}\right]\sqrt{\cos(\theta)-\cos(\alpha_0)}\cos\left(\frac{\theta}{2}\right)$$
$$+2\left[P\left(\frac{\gamma_3}{\pi}+\frac{\gamma_1\cos(\alpha_0)}{R(\alpha_0-\cos(\alpha_0)\sin(\alpha_0))}\right)+\gamma_4 b+\gamma_5\varepsilon\right]\ln\left[\frac{\sqrt{1+\cos(\theta)}-\sqrt{\cos(\theta)-\cos(\alpha_0)}}{\sqrt{1+\cos(\alpha_0)}}\right] \quad (8)$$

$$P = -2R\int_0^{a_0}\sigma_r(\theta)\cos(\theta)d\theta$$

$$b = -\frac{R^2}{\pi}\int_0^{a_0}\sigma_r(\theta)d\theta$$

It's necessary to emphasize that for elastic constants of isotropic materials, which are widely used in machines, the error of approximation (8) of the solution of the equation (7) with respect to $\sigma_r^{\max}(\theta)$ is less than 4 %.

It has been established that the received dependency of a half-angle of contact on the non-dimensional parameter, introduced by I. Y. Staerman, is analogous to the dependencies, estabilished by M. I. Teply for the state of plane deformation (Teply, 1983). This confirms a high efficiency of the approximate solution (8). In addition it has been proved, that equation (8) describes a contact stress in the case of interaction of the cylinder and a half-plane with a cylindrical cavity, when α_0 is less than $\frac{\pi}{6}$.

4 Solution of the Initial Problem

Let's return to the consideration of the initial problem. The principal approach is based on the superposition of the boundary conditions. That means it is sufficient to represent a cylindrical body in terms of cross sections, connected by an elastic brace, the flexural of which is defined by the parameters of the investigated body. Then this allows to represent spatial distributions of the contact stress by the following equations:

- when the cross section of the area of contact is small in comparison with the dimensions of the interacting bodies:

$$\sigma(x,z) = \frac{R+r}{2Rr}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)$$

$$l(z) = 2\sqrt{\frac{q(z)\,rR}{\pi(r+R)}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)}$$

- when the cylinders have similar radii:

$$\sigma_r(\theta,z) = -q(z)\frac{\sqrt{2}}{R}\left[\gamma_2\frac{2}{\pi} + \frac{\gamma_1}{\alpha_0 - \cos(\alpha_0)\sin(\alpha_0)}\right]\sqrt{\cos(\theta)-\cos(\alpha_0)}\cos\left(\frac{\theta}{2}\right)$$

$$+2\left[q(z)\left(\frac{\gamma_3}{\pi} + \frac{\gamma_1\cos(\alpha_0)}{R(\alpha_0-\cos(\alpha_0)\sin(\alpha_0))}\right) + \gamma_4 b(z) + \gamma_5\varepsilon\right]\ln\left[\frac{\sqrt{1+\cos(\theta)} - \sqrt{\cos(\theta)-\cos(\alpha_0)}}{\sqrt{1+\cos(\alpha_0)}}\right]$$

$$q(z) = -2R\int_0^{\alpha_0(z)}\sigma_r(\theta,z)\cos(\theta)d\theta$$

$$b(z) = -\frac{R^2}{\pi}\int_0^{\alpha_0(z)}\sigma_r(\theta,z)d\theta$$

where $q(z)$ is the load intensity, defined from the solution of the problem of the theory of elasticity for the section of the system by the plane YOZ.

The proposed approach simplifies the analysis of the investigated values, enables to take into account the material properties, the loading features and to use a well-developed apparatus of the two-dimensional elasticity theory. Three-dimensional picture of the contact stress distribution, normal to the bodies surface, can be drawn in accordance with the developed calculation scheme (Figure 4, Figure 5).

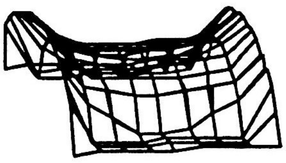

Figure 4. Surface of Stress Distribution for an Exterior Contact of Cylinders (see Figure 1 and Figure 2)

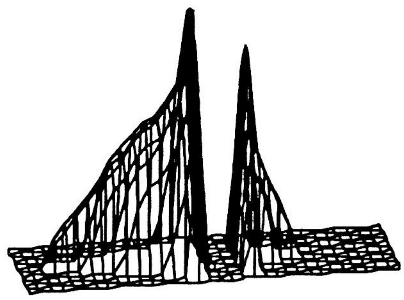

Figure 5. Stress Distribution for an Interior Contact of Cylinders (see Figure 1 and Figure 3)

5 Conclusions

A method which allows to reduce the contact problem for cylindrical bodies to a system of a two-dimension boundary value problem has been considered. It generalizes Hertz's solution for two cylinders to case of variable load intensity. It considerably reduces the complexity and cost of investigation contact stress in practice. Furthermore, with the help of the complex potentials method it delivers an explicit approximate solution of the two-dimension contact problem for an isotropic elastic disk and a plate with a cylindrical hole.

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