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The Reliability Prediction of Structures with Random Parameters Subjected to Stationary Stochastic Input

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The structural elements reliability prediction problem is investigated. At random loads, structure parameter randomness and fatigue failures are taken into account in this paper. External loads are assumed to be a vector space-time random field, and the parameters of the structures investigated are assumed to be random variables with known probabilistic characteristics. The reliability prediction problem includes three stages. At the first stage, the stochastic dynamics problem is solved using correlation theory relations. It concerns the definition of the stress-strain state (SSS) characteristics conditional correlation function and power spectral density (PSD) taking account of external random loads with the fixed parameters of the investigated structure. At the second stage, the SSS correlation functions and PSDs are defined taking account of structural element randomness. At the third stage, the reliability characteristics definition problem due to fatigue failures is solved.

1 Introduction

Many civil engineering structures are subjected to cyclic random loads (Bolotin, 1979), which may lead to fracture due to fatigue damage accumulation. Besides, the structure parameters are random as a consequence of manufacture imperfection, structure material physical properties non-homogenity and other random factors. The mentioned factors lead to the necessity in the structure's design to take into account the randomness of external loads and structural properties. At present the Finite Element Method (FEM) is widely used to investigate various complicated structural elements. This method may be used effectively to solve the reliability problems due to fatigue failures. These failures occur in the structural elements under external loads given as a random field and structure random parameters given as a random variables vector.

The phenomenological approach using various kinetic equations for harmonic loads is to describe a cumulative fatigue damage measure (Pavlov, 1988; Bolotin, 1984). Kinetic equations may be utilized in the case of broadband random loads if an initial process leads to a process with an equal-in-damage effect. The Markov process mathematical means are widely used to solve the reliability problem using kinetic equations describing a damage measure.

Accordingly, in this paper the structural elements reliability prediction approach is being worked out taking into account external loads and structure property randomness using FEM and Markov process theory.

2 The Stochastical Dynamics Problem Solution of a Structure with Deterministic Parameters

The stochastical dynamics problem dealt with here is for deterministic structures subjected to an external input as a vector space-time random field. The external load field intensity vector $\mathbf{F}(r,t)$ is assumed to be stationary on a time coordinate and homogeneous on a space coordinate. It is defined by the mean value $\mathbf{m}_F = <\mathbf{F}(r,t)>$ and the correlation tensor

$$\mathbf{K}_F(r_1,t_1;r_2,t_2) = <\tilde{\mathbf{F}}(r_1,t_1)\,\tilde{\mathbf{F}}(r_2,t_2)> = \mathbf{K}_F(\rho,\tau) = \int_{-\infty}^{+\infty}\mathbf{S}_F(\rho,\omega)e^{i\omega\tau}d\omega \tag{2.1}$$

with $\tau = t_2 - t_1$; $\rho = r_2 - r_1$; and where $<\cdot>$ denotes the expected value operation; $\overline{\mathbf{F}}(r,t) = \mathbf{F}(r,t) - \mathbf{m}_F$; $\mathbf{S}_F(\rho,\omega)$ denotes the time PSD, possessing the correlation tensor property with respect to the variable ω.

One of the simplest models of a random external load vector field $\mathbf{F}(r,t)$ is

$$\mathbf{F}(r,t) = \mathbf{F}_1(t)\ \mathbf{F}_2(r) \tag{2.2}$$

where $\mathbf{F}(r,t)$ is a separable function of t and r. In the particular case the external loads depend on the time coordinate only $\mathbf{F}(r,t) = \mathbf{F}(t)$ which is a vector random function.

After the finite element discretization a matrix differential equation describing a structural element random vibration is given by

$$\mathbf{M}\ \ddot{\mathbf{Y}}(t) + \mathbf{C}\ \dot{\mathbf{Y}}(t) + \mathbf{K}\ \mathbf{Y}(t) = \mathbf{X}(t) \tag{2.3}$$

where **M, C, K** denote mass, damping, and stiffness matrices respectively having random parameters $S_i(i = 1,\ldots,m)$, while $\mathbf{X}(t)$, $\mathbf{Y}(t)$ are the external nodal input and generalized nodal displacements n-dimensional vectors, respectively, components of which are random functions with respect to a time coordinate.

A damping matrix **C** is assumed to be present as a linear combination of the matrices **K** and **M.**

$$\mathbf{C} = \beta_1\mathbf{M} + \beta_2\mathbf{K} \tag{2.4}$$

where β_1 and β_2 denote the internal and external friction factors.

In equation (2.3) mass, damping, and stiffness matrix coefficients have the random parameters S_i as independent random variables with given probabilistic characteristics: mean value m_{s_i}, variance $\sigma^2_{s_i}$, and probability density function (PDF) $f_i(s_i)$. The parameters $S_i(i = 1,\ldots,m)$ mentioned form the vector **S** and relate the inertia, stiffness and geometric properties or material physical properties of each structural element considered (Gallagher, 1984).

If a structure is subjected to a vector distributed load characterized by the intensity vector $\mathbf{F}(r,t)$, a component of which is to be the random stationary homogeneous field with given probabilistic characteristics, one can obtain the equivalent nodal load vector $\mathbf{X}(t)$ according to FEM general theory.

Let us consider an i-th finite element which is assumed to be subjected to a uniformly distributed surface load $\mathbf{F}(r_i,t)$ (r_i is the center of gravity coordinate). In this case the equivalent nodal load vector for the i-th element may be presented as

$$\mathbf{X}_i = \int_{\delta_i} \mathbf{B}_i^T(r)\ \mathbf{F}(r,t)\,dr \tag{2.5}$$

where δ_i is the element cross-section, $\mathbf{B}_i(r)$ denotes the matrix connected the vector $\mathbf{u}_i$ of the generalized displacements in any i-th element point with the generalized nodal displacements vector $\mathbf{Y}_i$. Since the load intensity $\mathbf{F}(r,t)$ in the i-th element limits is assumed to be constant, the expression (2.5) can be written as

$$\mathbf{X}_i(t) = \left\{\int_{\delta_i} \mathbf{B}_i^T(r)\right\} \mathbf{F}(r_i,t) = \overline{\mathbf{B}}_i^T\ \mathbf{F}(r_i,t) \tag{2.6}$$

The full vector $\mathbf{X}(t)$ of the structure nodal loads is formed by the vectors $\mathbf{X}_i(t)$ defined by equation (2.5), and furthermore it may be presented in the form

$$\mathbf{X}(t) = \mathbf{A}\ \mathbf{F}(t) \tag{2.7}$$

where **A** is the matrix formed by integrating equation (2.6) and passing from the local coordinate system to the global one, while $\mathbf{F}(t)$ denotes the random process vector components of which to be the values of the vector field $\mathbf{F}(r,t)$ in the i-th element center of gravity components $r = r_i$. So, the random process vector $\mathbf{F}(r,t)$ correlation matrix is given by the correlation tensor $\mathbf{K}_F(\rho, t)$ according to equation (2.1).

Let us assume that the mean value of the external load field is equal to zero, $\mathbf{m}_F = \mathbf{0}$. Then in accordance with equation (2.7) $\mathbf{m}_X = \mathbf{0}$ and its correlation matrix may be presented by

$$\mathbf{K}_X(t_1, t_2) = \mathbf{K}_X(\tau) = <\mathbf{X}(t_1)\mathbf{X}(t_2)> = \mathbf{A}\ \mathbf{K}_F(\rho, \tau)\mathbf{A}^T \tag{2.8}$$

The adopted expression allows a possibility to form the generalized nodal load $\mathbf{X}(t)$ correlation matrix with respect to given distributed load field $\mathbf{F}(r,t)$ characteristics (these loads $\mathbf{X}(t)$ are in the right hand side of general equation (2.3)). The vector $\mathbf{X}(t)$ PSD is defined as Fourier transform of the correlation matrix $\mathbf{K}_X(\tau)$

$$S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{K}_X(\tau)\,e^{-i\omega\tau}d\tau = \mathbf{A}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{K}_F(\tau)\,e^{-i\omega\tau}d\tau\right]\mathbf{A}^T = \mathbf{A}\ S_F(\omega)\mathbf{A}^T \tag{2.9}$$

Expression (2.9) allows to determine the usual and mutual PSDs of the nodal load vector $\mathbf{X}(t)$ components using the field $\mathbf{F}(r, t)$ time PSD.

The stochastical dynamics problem solution for the construction described by equation (2.3) is derived on the basis of a nondamping structure mode shape series expansion

$$\mathbf{Y}(t) = \mathbf{\Phi}\ \mathbf{q}(t) \tag{2.10}$$

where $\mathbf{q}(t)$ is the generalized coordinate vector, the components of which are random variables, while $\mathbf{\Phi} = [\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_n]$ is the mode shape matrix, the column $\mathbf{\Phi}_k$ of which is the k-th standardized mode shape obtained from the eigenvalue problem solution

$$\left(\mathbf{K} - \omega_k^2\mathbf{M}\right)\mathbf{\Phi}_k = \mathbf{0} \tag{2.11}$$

where ω_k is the k-th fundamental frequency.

Substituting equation (2.10) into equation (2.3) and multiplying it by $\mathbf{\Phi}^T$ and taking into account the aforementioned assumptions, one can obtain the separated ordinary differential simultaneous equations relative to the generalized coordinates

$$\ddot{q}_k(t) + \left(\beta_1\omega_k^2 + \beta_2\right)\dot{q}_k(t) + \omega_k^2 q_k(t) = \left(\mathbf{\Phi}_k^T . \mathbf{X}(t)\right) = \chi_k(t) \qquad k = 1,\ldots,n \tag{2.12}$$

where n denotes the number of the retained mode shapes.

To solve the statistical dynamics problem it is necessary for equation (2.12) to contain the external load $X_k(t)$ correlation functions (or PSDs) using the process $\mathbf{X}(t)$ similar characteristics. Using the random functions $X_k(t)$ definition one can write the expressions for the vector $\mathbf{X}(t)$ correlation matrix and the PSD matrix

$$K_{x_k x_m}(\tau) = \sum_{i,j=1}^{n}\Phi_{ki}\Phi_{mj}K_{X_i X_j}(\tau)$$

$$S_{x_k x_m}(\omega) = \sum_{i,j=1}^{n}\Phi_{ki}\Phi_{mj}S_{X_i X_j}(\omega) \tag{2.13}$$

where the Φ_{ki} are elements of the vector $\mathbf{\Phi}_k$.

Using the spectral decomposition representation method (Bolotin, 1979), one can write the expression for the mutual PSD matrix elements $q_k(t)$ for a stationary case.

$$S_{q_k q_m}(\omega) = H_k(-i\omega)\, H_m(i\omega) S_{x_k x_m}(\omega) \qquad (k,m = 1,\dots,n) \qquad (2.14)$$

where $H_k(i\omega)$ denotes the frequency response matrix element. They are defined with respect to formulae

$$H_k(i\omega) = \frac{1}{\omega_k^2 + 2i\varepsilon_k \omega - \omega} \qquad 2\varepsilon_k = \beta_1 \omega_k^2 + \beta_2 \qquad (2.15)$$

With equation (2.15) the expression for $S_{q_k q_m}(\omega)$ is given by

$$S_{q_k q_m}(\omega) = \frac{S_{x_k x_m}(\omega)}{\left(\omega_k^2 - 2i\varepsilon_k \omega - \omega^2\right)\,\left(\omega_m^2 - 2i\varepsilon_m \omega - \omega^2\right)} \qquad (2.16)$$

The correlation matrix may be defined with respect to the PSD obtained.

$$K_{q_k q_m}(\tau) = \frac{S_{x_k x_m}(\omega) e^{i\omega\tau} d\omega}{\left(\omega_k^2 - 2i\varepsilon_k \omega - \omega^2\right)\,\left(\omega_m^2 - 2i\varepsilon_m \omega - \omega^2\right)} \qquad (2.17)$$

For civil engineering structures the situation is often such that the external loads field is broadband, the damping is sufficiently small and the fundamental frequencies are dispersed. In this case the mutual correlation between the generalized coordinates $q_k(t)$ may be neglected. The external loads $\mathbf{X}_k(t)$ PSDs are assumed to be constant in the limits of the admission band of the system described by equation (2.12). With these assumptions

$$S_{q_k}(\omega) = \frac{S_{x_k}(\omega_k)}{\left(\omega_k^2 - \omega^2\right)^2 + 4\varepsilon_k^2 \omega^2} \qquad (2.18)$$

$$K_{q_k}(\tau) = \sigma_{q_k}^2 e^{-\varepsilon_k |\tau|} \left(\cos\beta_k \tau + \frac{\varepsilon_k}{\beta_k} \sin\beta_k |\tau| \right)$$

$$\sigma_{q_k}^2 = \frac{\pi S_{x_k}(\omega_k)}{2\varepsilon_k \omega_k^2} \qquad \beta_k = \sqrt{\omega_k^2 - \varepsilon_k^2} \qquad (2.19)$$

If aforementioned conditions are not fulfilled then the integral in equation (2.17) is computed numerically. Equations (2.18) and (2.19) describe the probabilistic characteristics of the narrowband random process.

The FEM general relations are used to obtain the correlation functions and PSDs of the generalized nodal displacements. So the nodal displacements $\mathbf{Y}(t)$ correlation matrix on the basis of equation (2.10) may be presented by

$$\mathbf{K}_y(\tau) = \langle \mathbf{Y}(t)\mathbf{Y}^T(t+\tau) \rangle = \langle \mathbf{\Phi q}(t)\mathbf{q}^T(t+\tau)\mathbf{\Phi}^T \rangle = \mathbf{\Phi} \langle \mathbf{q}(t)\mathbf{q}^T(t+\tau) \rangle \mathbf{\Phi}^T = \mathbf{\Phi K}_q(\tau)\mathbf{\Phi}^T \qquad (2.20)$$

The stress correlation matrix is obtained similarly. The PSDs are obtained as the correlation matrices from a Fourier transform.

3 The Stochastical Dynamics Problem Solution for Structures with Random Parameters

The stochastical dynamics problem solution for the structures with random parameters is executed on the basis of the aforementioned approach using the sensitivity theory (Haug et al., 1988). The structure parameters in the expressions for mass, damping and stiffness matrices are random variables and so are the fundamental frequencies and mode shapes as well. For civil engineering structures the S_i parameter dispersion is small, therefore the structure fundamental frequencies depending on the $\mathbf{S}$ vector may be expanded into Taylor series in the neighborhood of the mean value $\mathbf{m}_s$ having linear terms only.

$$\omega_k(\mathbf{S}) = \omega_k(\mathbf{m}_s) + \sum_{i=1}^{m} \left. \frac{\partial \omega_k}{\partial S_i} \right|_{S_i = m_{s_i}} \left(S_i - m_{s_i}\right) \qquad (k = 1,\ldots,n) \qquad (3.1)$$

The assumption that vector $\mathbf{S}$ is distributed in accordance with Gaussian Law or that m is great enough yields the normalized PDF for $\omega_k(\mathbf{S})$. The mean values m_{ω_k} and variances $\sigma^2_{\omega_k}$ are defined as

$$m_{\omega_k} = \omega_k(\mathbf{m}_s) \qquad \sigma^2_{\omega_k} = \sum_{i=1}^{m} \left. \frac{\partial \omega_k}{\partial S_i} \right|_{S_i = m_{s_i}} \sigma^2_{s_i} \qquad (k = 1,\ldots,n) \qquad (3.2)$$

The mean value is defined by the eigenvalue problem solution (2.11) for the structure with deterministic parameters $\mathbf{S} = \mathbf{m}_s$. To compute the variances $\sigma^2_{\omega_k}$ the sensitivity theory is applied (Haug et al., 1988). Therefore, to determine the derivatives $\partial \omega_k / \partial S_i$ in the expression (3.2) equation (2.11) must differentiated with respect to parameters S_i , and the adopted expression multiplied by the mode shape vector $\mathbf{\Phi}_k$.

$$\left(\frac{\partial}{\partial_{S_i}} \left[\left(\mathbf{K} - \omega_k^2 \mathbf{M}\right) \mathbf{\Phi}_k \right], \mathbf{\Phi}_k \right) = \left(\frac{\partial \mathbf{K}}{\partial_{S_i}} \mathbf{\Phi}_k , \mathbf{\Phi}_k \right) - \omega_k^2 \left(\frac{\partial \mathbf{M}}{\partial_{S_i}} \mathbf{\Phi}_k , \mathbf{\Phi}_k \right) = 0$$

Taking account of the normalizing condition and reducing this equation respectively one can obtain the partial derivative

$$\frac{\partial \omega_k}{\partial_{S_i}} = \frac{1}{2\omega_k} \left(\frac{\partial \mathbf{K}}{\partial_{S_i}} \mathbf{\Phi}_k , \mathbf{\Phi}_k \right) - \frac{\omega_k}{2} \left(\frac{\partial \mathbf{M}}{\partial_{S_i}} \mathbf{\Phi}_k , \mathbf{\Phi}_k \right) \qquad (k = 1,\ldots,n) \qquad (i = 1,\ldots,m) \qquad (3.3)$$

The right hand side expression (3.3) is computed for $S_i = m_{S_i}$. The obtained matrices of mass and stiffness derivatives are created making use of their additivity property.

The solution obtained in the previous chapter is used to solve the stochastical dynamics problem for the structure with random parameters as the random variables. But the fundamental frequencies $\omega_k (k = 1,\ldots,n)$ are assumed to be deterministic hence the PSD and correlation function defined from equations (2.18) and (2.19) to be conditional ones and they are denoted by $S_{q_k}(\omega / \omega_k)$ and $K_{q_k}(\tau / \omega_k)$.

The generalized coordinates unconditional PSDs and correlation functions taking account of structure parameters randomness are obtained as

$$K_{q_k}(\tau) = \int_0^{\infty} K_{q_k y}(\tau / \omega_k) \; f(\omega_k) \; d\omega_k \qquad (k = 1,\ldots,n) \qquad (3.4)$$

and

$$S_{q_k}(\omega) = \int_0^{\infty} S_{q_k x}(\omega / \omega_k)\ f(\omega_k)\ d\omega_k \tag{3.5}$$

In the case the PDF $f(\omega_n)$ is Gaussian, equation (3.4) for the normalized correlation function and variance of q_k may be written

$$\begin{aligned} R_k y(\tau) &= \frac{1}{\sqrt{2\pi}\ \sigma_{\omega_k}} \int_0^{\infty} e^{-\varepsilon_k |\tau|} \left(\cos\omega_k \tau + \frac{\varepsilon_k}{\omega_k} \sin\omega_k \tau \right) \exp\left[-\frac{\left(\omega_k - m_{\omega_k}\right)^2}{2\sigma_\omega^2} \right] d\omega_k \\ &= e^{-\varepsilon_k |\tau|} \exp\left(-\frac{\tau^2 \sigma_{\omega_k}^2}{2} \right) \left(\cos m_{\omega_k} \tau + \frac{\varepsilon_k}{\omega_k} \sin m_{\omega k} \tau \right) \end{aligned} \tag{3.6}$$

$$\sigma_{q_k}^2 = \frac{S_{x_k}\left(m_{\omega_k}\right)}{2\varepsilon_k \sqrt{2\pi}\ \sigma_{\omega_k}} \int_0^{\infty} \frac{1}{\omega_k^2} \exp\left[-\frac{\left(\omega_k - m_{\omega_k}\right)^2}{2\sigma_{\omega_k}^2} \right] d\omega_k \tag{3.7}$$

The integral in equation (3.7) is computed numerically. The generalized coordinate PSD $S_{q_k}(\omega)$ is defined as the correlation function inverse Fourier transform.

The generalized displacements probabilistic characteristics may be obtained on the adopted probabilistic characteristics of the generalized coordinates $\mathbf{q}\ (t)$ vector using the method presented above.

4 Reliability Characteristics Definition Due to Fatigue Failures

As shown in chapter 2 for the chosen finite elements discretization of a structural element the generalized nodal displacements (stress or strain) vector components $y_i(t), (i = 1,\dots,m)$ in accordance with equation (2.10) may be introduced as a linear combination of the structure generalized coordinates $q_k(t), (k = 1,\dots,n)$.

$$y_1(t) = \sum_{k=1}^{n} b_{ik}\ q_k(t) \tag{4.1}$$

For broadband random loads and small damping the generalized coordinates $q_k(t)$ vector components are considered to be the narrowband quasiharmonic random processes having the PSDs (2.18) and correlation functions (2.19) and may be presented (Bolotin, 1979) by

$$q_k(t) = A_k(t) \sin\left[\omega_k t + \varphi_k(t)\right] \tag{4.2}$$

where $A_k(t)$ and $\varphi_k(t)$ are the slowly changing amplitude and phase respectively in comparison with $\sin(\omega_k t)$, while ω_k is the k-th fundamental frequency. Substituting equation (4.2) into equation (4.1) gives

$$y_i(t) = \sum_{k=1}^{n} b_{ik}\ A_k(t)\ \sin\left[\omega_k t + \varphi_k(t)\right] = \sum_{k=1}^{n} y_{ik}(t) \tag{4.3}$$

In equation (4.3) the narrowband processes $y_{ik}(t)$ introduced are the i-th components of the stress (strain) vector $y\ (\ t\ )$ corresponding to the k-th mode shape

$$y_{ik}(t) = b_{ik} q_k(t) = b_{ik}\ A_k(t)\ \sin[\omega_k t + \varphi_k(t)] \tag{4.4}$$

Various approximate approaches for the complex SSS and deterministic regular loads allow to reduce the complex SSS to a simple one on the basis of the classic strength hypotheses and tests results under a state of plane stress generalization. As a rule in this case the stress (strain) tensor components are assumed to change synchronously and synphasely. The conditions mentioned are fulfilled for the narrowband processes $y_{ik}(t)$ introduced in accordance with equation (4.4). These processes have the same fundamental frequencies $\omega_k(t)$ and phases $\varphi_k(t)$. In this case an equivalent narrowband process $y_{ek}(t)$ with amplitude $\lambda_k(t)$ and frequency ω_k may be introduced corresponding to the linear SSS in a given structure point and this process is considered to be a square root from some components of the quadratic form $y_{ik}(t)$.

$$y_{ek}(t) = \left[\sum_{i,r} C_{ir}\ y_{ik}(t)\ y_{rk}(t)\right]^{1/2} \tag{4.5}$$

where C_{ir} are the factors assigned from the corresponding strength hypothesis. For example, the stress intensities are $y_{ek}(t)$. Substituting equation (4.5) in expression (4.4) gives

$$\begin{aligned} y_{ek}(t) &= \left[\sum_{i,r} C_{ir}\ b_{ik}\ b_{rk}\ q_k^2(t)\right]^{1/2} = C_k\ q_k(t) \\ C_k &= \left[\sum_{i,r} C_{ir}\ b_{ik}\ b_{rk}\right]^{1/2} \qquad (k = 1,\dots,n) \end{aligned} \tag{4.6}$$

On adopted components $y_{ek}(t)$ one can define some equivalent stress for a given structure point considering all excited vibration mode shapes

$$y_e(t) = \sum_{k=1}^{n} y_{ek}(t) = \sum_{k=1}^{n} C_k\ q_k(t) = \sum_{k=1}^{n} C_k\ A_k(t)\ \sin[\omega_k t + \varphi_k(t)] \tag{4.7}$$

The broadband random process $y_e(t)$ is the narrowband random processes $y_{ek}(t)$ superposition. Further on basis of the random processes schematization method the process $y_e(t)$ reduces to the narrowband process $y_M(t)$ as the equivalent one by a damage effect measure.

$$y_M(t) = \lambda(t)\ \sin[\omega t + \varphi(t)] = A(t)\ \cos(\omega t) + D(t)\ \sin(\omega t) \tag{4.8}$$

The key factors of the processes $y_e(t)$ and $y_M(t)$ equivalence are:

- coincidence of the envelope $\lambda(t)$ PDF in equation (4.8) and amplitude $y_e(t)$ PDF defined in accordance with one of the random processes schematization methods (Kogaev, 1977);
- coincidence of the narrowband process $y_M(t)$ frequency ω and the process $y_e(t)$ zero crossing or peaks mean values;
- coincidence of the processes $y_e(t)$ and $y_M(t)$ correlation time.

The filter equation for the envelope $\lambda(t)$ is formed on the basis of the key factors mentioned. The envelope λ is obtained from

$$d\lambda / dt = \Phi_1(\lambda) + \Phi_2(\lambda)\, n(t) \tag{4.9}$$

where $n(t)$ is normal white noise with the correlation function $K(\tau) = 0.5\,N_0\delta(\tau)$ (δ -correlated process with intensity N_0); $\Phi_1(\lambda)$, $\Phi_2(\lambda)$ are known deterministic functions satisfied to the Lipschitz condition.

For various engineering structures the failures due to low and high cycle fatigue is a typical feature. These phenomena are described in the phenomenological model limits by means of kinetic equations for the fatigue damage measure which varies from zero to one. An equation may be presented for the quasiharmonic loads process

$$dz(t)/dt = C(\lambda)\,\lambda(t)^{m(\lambda)} \tag{4.10}$$

where $\lambda(t)$ is the stress (strain) amplitude described by equation (4.8), while $C(\lambda)$ and $m(\lambda)$ are the factors and slopes of stress-life curve which are step functions of $\lambda(t)$.

Considering jointly equations (4.9) and (4.10) one can state on the basis of the Doob theorem that $[z(t), \lambda(t)]$ is a two-dimensional Markov process, the one-dimensional PDF $f(z, \lambda, t)$ of which satisfies to the Fokker-Planck-Kolmogorov (FPK) equation (Tichonov, 1977)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \lambda}\left[A_1(\lambda)f\right] - \frac{\partial}{\partial z}\left[A_2(\lambda)f\right] + \frac{1}{2}\frac{\partial^2}{\partial \lambda^2}\left[B_2(\lambda)f\right] \tag{4.11}$$

with boundary conditions

$$\lim f(\lambda, z, t) = 0 \qquad (\lambda, z) \to 0, \infty \tag{4.12}$$

and initial condition

$$\lim f(\lambda, z, t) = f(\lambda)f(z) \qquad t \to 0 \tag{4.13}$$

The FPK equation coefficients are formed by the damage equation and filter equation coefficients

$$A_1(\lambda) = \Phi_1(\lambda) + \frac{N_0}{4}\Phi_2(\lambda)\ d\Phi_2(\lambda)/d\lambda$$

$$A_2(\lambda) = C(\lambda)\lambda^{m(\lambda)} \tag{4.14}$$

$$B(\lambda) = \frac{N_0}{2}\Phi^2(\lambda)$$

where N_0 is the white noise intensity in filter equation (4.9).

To solve equation (3.11) the function $\theta(\lambda, \omega, t)$ is introduced as a characteristic function with respect to z and the PDF with respect to λ.

$$\theta(\lambda, \omega, t) = \int_0^\infty f(\lambda, z, t)e^{i\omega z}dz \tag{4.15}$$

Accordingly equation (4.15) from the FPK equation taking account of boundary conditions (4.12) has partial derivatives for the $\theta(\lambda, \omega, t)$ function with only two independent variables t and λ

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial \lambda}\left[A_1(\lambda)\theta\right] + i\omega A_2(\lambda)\theta + \frac{1}{2}\frac{\partial^2}{\partial \lambda^2}\left[B(\lambda)\theta\right] \tag{4.16}$$

In equation (4.16) the variable ω is a parameter. Considering $\theta(\lambda, \omega, t)$ as the PDF with respect to the variable $\lambda \geq 0$ one can expand the series by the orthogonal polynomials $Q_n(\lambda)$ with the weight function $f(\lambda)$ as the known PDF of $\lambda(t)$ (Lewin, 1989)

$$\theta(\lambda, \omega, t) = f(\lambda) \sum_{n=0}^{\infty} \alpha_n(\omega, t) Q_n(\lambda) \tag{4.17}$$

Unknown complex factors $\alpha_n(\omega, t)$ entered in equation (4.17) are defined by substituting (4.17) in equation (4.16) taking account of the polynomials $Q_n(\lambda)$ orthogonality condition. Further one can obtain the ordinary simultaneous differential equations relative to the unknown factors $\alpha_n(\omega, t)$ which are presented in the complex form

$$\frac{\partial \alpha_n(\omega, t)}{\partial t} = \sum_{k=0}^{n} \alpha_k(\omega, t) u_{nk} \qquad (n = 1, \ldots, N) \tag{4.18}$$

One can show that $\alpha_0(\omega, t)$ is the characteristic function of $z(t)$. The one-dimensional PDF of $f(z, t)$ is defined to be the $\alpha_0(\omega, t)$ inverse Fourier-transform. Hence the structure main reliability characteristics are defined, for example, by the probability of survival

$$P(t) = \int_0^1 f(z, t) dz \tag{4.19}$$

5 Numerical Investigations

On the basis of the approaches developed to solve the stochastical dynamics problem of beam structures with random parameters presented in chapters 2 and 3 the influence of most typical components of the structural parameters vector $\mathbf{S}(t)$ random dispersion on the probabilistic characteristics of the beam structure state vector $\mathbf{Y}(t)$ is investigated. At first one considers a solution of the model test for a ring cross-section beam rigidly closed on the left end with point mass on the right end and an elastic support in the middle. The external load is a point force to be a centered normal stationary process as „truncated white noise" (the PSD is constant in the frequency range [0, 160] Hz). The random parameters - beam external diameter D, point mass value M, support stiffness **C**-influence is investigated. Given parameters are assumed to be the normal random variables having mean values and variances.

The material physical characteristics and beam parameters are:

- length 1 m;
- external diameter/internal diameter ratio 1.67
- modulus of elasticity (Young's modulus) $E = 1.96 \cdot 10^5$ MPa
- Poisson's ratio $\nu = 0.3$
- material density $\rho = 7.8 \cdot 10^5 \text{ kg/m}^3$
- white noise intensity $S_0 = 481.18 \text{ N}^2\text{m}$

This structure presented in Figure 1 consists of 10 finite elements. In given frequency range three vibration mode shapes are excited in the external loads action plane (the ordinal numbers are 2-nd, 4-th, 6-th).

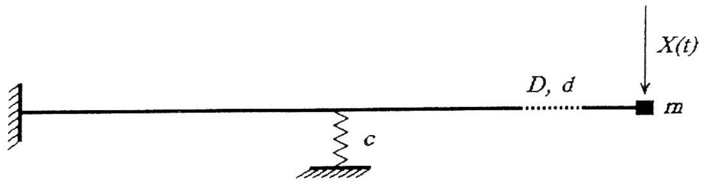

Figure 1. Model Structure

Let us study the corresponding random parameter influence on the structure fundamental frequencies. The mean values and root mean squares of the random parameters are respectively:

1) elastic support stiffness

$m_c = 198$ kN / m $\qquad \sigma_c = 19.8$ kN / m

2) point mass on the right end

$m_c = 9.8 \cdot 10^{-2}$ kg $\qquad \sigma_N = 9.8 \cdot 10^{-3}$ kg

3) 9-th finite element external diameter

$m_D = 10^{-2}$ m $\qquad \sigma_D = 10^{-3}$ m

4) 9-th finite element internal diameter

$m_d = 6 \cdot 10^{-3}$ m $\qquad \sigma_d = 6 \cdot 10^{-4}$ m

The given parameters distribution law is considered to be normal and they are assumed to be independent. In accordance with equation (3.3) the fundamental frequency derivatives with respect to the mentioned varying parameters have been defined in the point relating to the parameter mean values. The derivatives are given in Table 1.

Fundamental frequencies derivatives with respect to the varying parameters	$\frac{\partial\omega_1}{\partial S_k}$	$\frac{\partial\omega_2}{\partial S_k}$	$\frac{\partial\omega_3}{\partial S_k}$	$\frac{\partial\omega_4}{\partial S_k}$	$\frac{\partial\omega_5}{\partial S_k}$	$\frac{\partial\omega_6}{\partial S_k}$
Support stiffness c	3.28788E-15	9.42326E-03	1.24149E-15	2.45411E-01	9.26052E-12	8.75077E-01
Point mass m	-9.41352E+04	-2.76960E+05	-2.32904E+05	-5.26345E+05	-3.45316E+05	-2.38994E+03
External diameter D	-7.03035E+00	-1.01577E+01	2.29855E+01	1.51401E+02	1.31715E+02	-3.06113E+00
Internal diameter d	4.41961E+00	9.43927E+00	8.22859E+00	1.73419E+02	1.24345E+02	2.95116E+00

Table 1. Fundamental Frequency Derivatives

As one can see, the point mass on the right end affects the fundamental frequencies most significantly, the support in the middle affects on the fundamental frequencies most weakly. This is connected with the fact that the vibration mode node is in the middle of the structure (especially for the 1-st, 3-rd, 5-th vibration modes) and the greatest mode distance is at the structure's right end. That is why the parameter M influence on the 2-nd, 4-th, 6-th vibration modes was studied in detail.

The fundamental frequency variances have been computed (since only one parameter is being changed only one term in equation (3.2) remains). Then the generalized coordinates correlation functions with respect to equation (3.6) are defined. The expressions obtained are compared with the deterministic parameter correlation functions. The normalized correlation functions for the 2-nd frequency are given in Figure 2 and the corresponding PSDs are given in Figure 3. Here and further on the deterministic parameter curve is denoted by number 1, the random parameter curve is denoted by number 2.

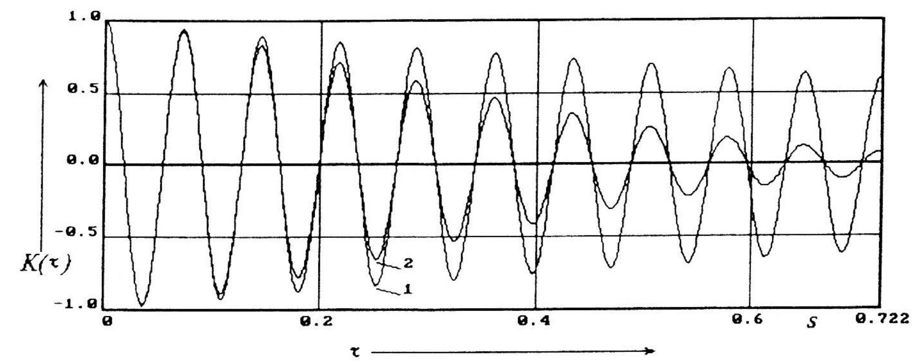

Figure 2. The 2-nd Generalized Coordinate Normalized Correlation Function

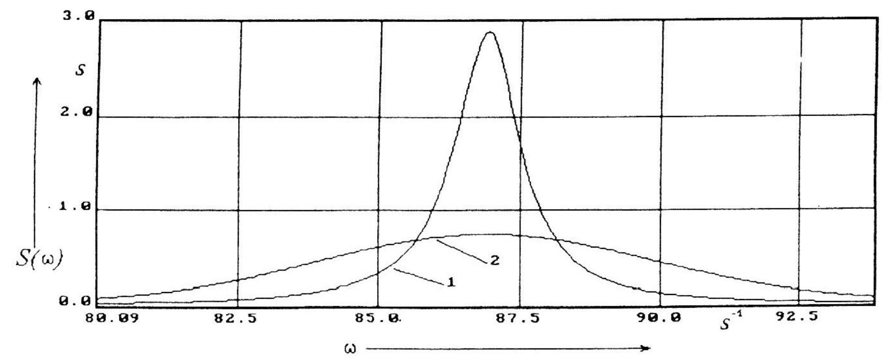

Figure 3. The 2-nd Generalized Coordinate Normalized Power Spectral Density

For the 2-nd vibration mode when bending stress is dominant the stress variance values have been obtained

- $\sigma_d^2 = 9.47 \cdot 10^3$ (MPa)^2 (deterministic mass case)
- $\sigma_r^2 = 9.71 \cdot 10^3$ (MPa)^2 (random mass case)

It is necessary to notice the generalized coordinates variances non-sensitivity to any parameter dispersion. It may be explained by the circumstance that the variance is a random process integral characteristic. The PSDs and correlation functions differ most significantly.

The random vibration numerical investigations have been for a complex branching space tube structure of an airplane control system element. The structure consists of 156 finite elements, the number of nodes is 153. The nodes are enumerated to minimize the mass and stiffness matrices band width. The structure parameters are

- modulus of elasticity $E = 7 \cdot 10^4$ MPa
- Poisson's ratio $\nu = 0.3$
- material density $\rho = 2.7 \cdot 10^5$ kg / m^3
- external diameter/internal diameter ratio 1.13

The tube structure is filled with a liquid having a density of $\rho = 10^3\ kg/m^3$, and fastened with six elastic supports to a base structure. The experimental investigations for the supports show that only vertical axial stiffness and angle torsional stiffness components are needed to be taken into account, and the remaining stiffness components are assumed to be infinite. The axial and angle stiffnesses are random variables, the mean values of which are $m_{ax} = 933.45\ kN/m$ and $m_{an} = 0.273\ kN/m$.

The structure random vibrations are excited through the elastic supports. Six supports are assumed to be random vibrated in accordance with a given law being stationary normal. As calculations show the external load spectrum includes the structure's six fundamental frequencies. This structure plot and 1-st and 2-nd fundamental modes are shown in Figure 4.

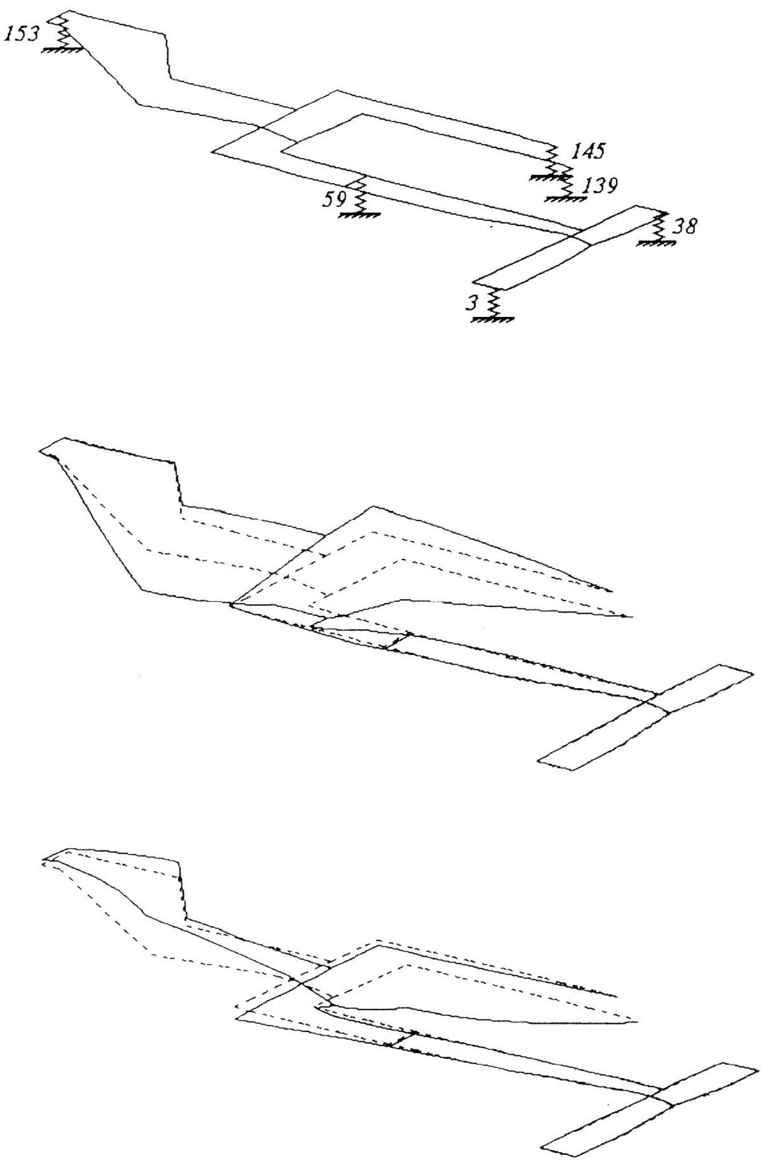

Figure 4. Structure Plot and 1-st and 2-nd Fundamental Modes

Fundamental frequency sensitivity to the structure random parameters changing analysis shows that the parameter $N\ 7\ C_7$-axial vertical stiffness in the 59-th node affects the fundamental frequencies most significally (Table 2). Therefore, the most detailed calculations have been carried out for these random parameters. The first generalized coordinates' normalized correlation functions and PSDs are shown in Figure 5 and Figure 6 respectively. Root-Mean-Square $\sigma_7 = 196$ kN/m.

Relations of the various frequency sensitivities, PSD values on these frequencies, displacement absolute values lead to the largest variances of the stress intensities to appear at the 1-st fundamental frequency. The highest frequencies have practically no effect on the stress variances.

Fundamental frequency derivatives with respect to the parameters s	N	$\frac{\partial\omega_1}{\partial s_k}$	$\frac{\partial\omega_2}{\partial s_k}$	$\frac{\partial\omega_3}{\partial s_k}$
Axial stiffness into point 153	s_1	4.81530E-03	1.45530E-03	2.56983
Angle torsional stiffness into point 153	s_2	7.59090E-03	6.39397E-04	4.66385
Axial stiffness into point 145	s_3	1.34491E-03	1.38171E-03	5.68937
Angle torsional stiffness into point 145	s_4	7.73184E-03	3.06824E-03	2.37573
Axial stiffness into point 139	s_5	2.35170E-05	6.99471E-04	1.36709
Angle torsional stiffness into point 139	s_6	1.11013E-03	5.30892E-03	9.89055
Axial stiffness into point 59	s_7	1.89074E-02	1.93690E-02	1.16597
Angle torsional stiffness into point 59	s_8	2.13541E-03	1.04967E-03	1.95248
Axial stiffness into point 38	s_9	7.13677E-04	4.60507E-04	1.01971
Angle torsional stiffness into point 38	s_{10}	3.40855E-05	2.09092E-05	5.02191
Axial stiffness into point 3	s_{11}	2.83265E-04	1.81709E-04	6.69719
Angle torsional stiffness into point 3	s_{12}	4.36764E-05	2.62375E-05	6.75474

Table 2. Fundamental Frequency Derivatives

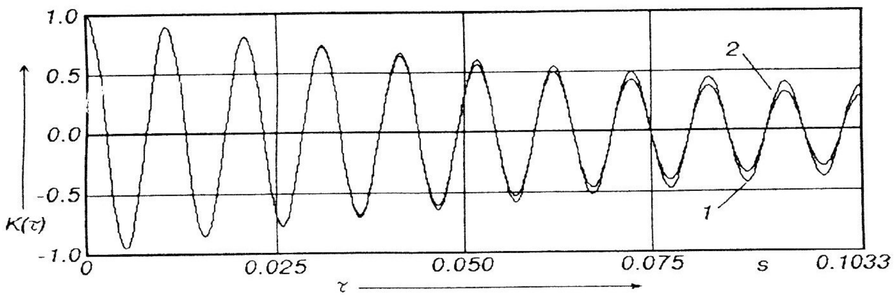

Figure 5. The 1-st Generalized Coordinate Normalized Correlation Function

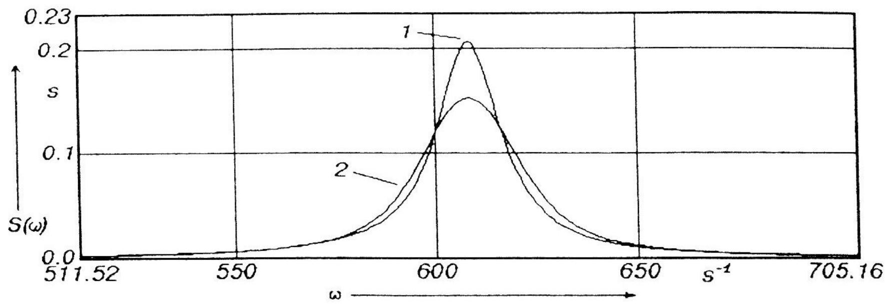

Figure 6. The 1-st Generalized Coordinate Normalized Power Spectral Density

The reliability characteristics computing problem for the model test is solved with the following material fatigue parameter data:

- endurance limit $\sigma_{-1} = 33$ MPa;
- curve stress-life slopes are $m_1 = 8$ ($10^5 < N < 10^7$cycles) and $m_2 = 2$ ($N \le 10^5$cycles).

As the 2-nd mode shape stress exceeds stresses on remaining mode shapes significantly, only this mode shape is taken into account to solve the reliability problem. The mean resource values (in cycles) have been obtained for deterministic and stochastic problems respectively

- $T_d = 18524$
- $T_r = 18058$

In Figure 7 the damage measure PDFs are presented for three various times. In Figure 8 probability of survival and failures PDF plots are shown. The results obtained allow to make the following conclusions:

- if only generalized coordinates variances are needed for computing it makes no sense to take into account structure parameter randomness; in the PSD and correlation function needed for computing structure parameters not accounting for randomness gives distorted results
- mean resource decreasing (3 percent) for random mass case test is defined by difference between stress variance values
- survival time and failure PDF depend on the PSD and correlation functions of the structure SSS parameter forms.

The reliability characteristics computing problem for the aforementioned structure is solved with the same material fatigue parameter data. In Figure 9 the damage measure PDFs are presented for three different times. In Figure 10 probability of survival and failure PDF plots are shown.

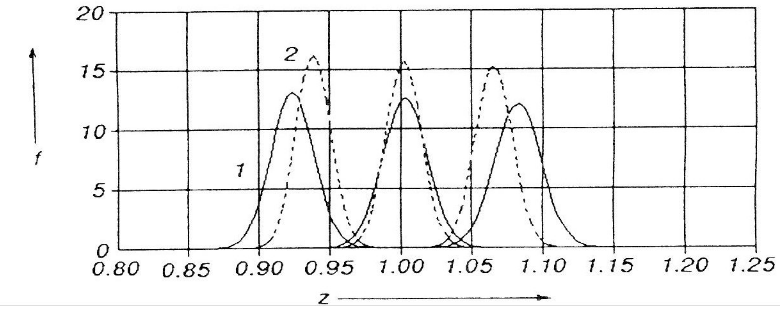

Figure 7. Damage Measure Probability Density Functions

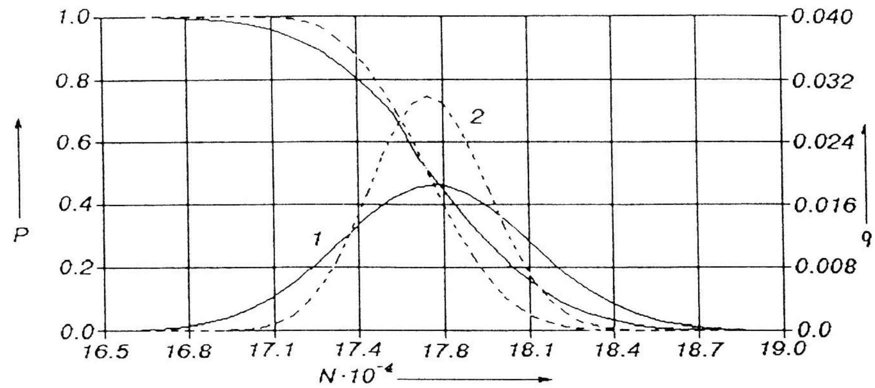

Figure 8. Probability of Survival (P) and Failure Probability Density Function (q)

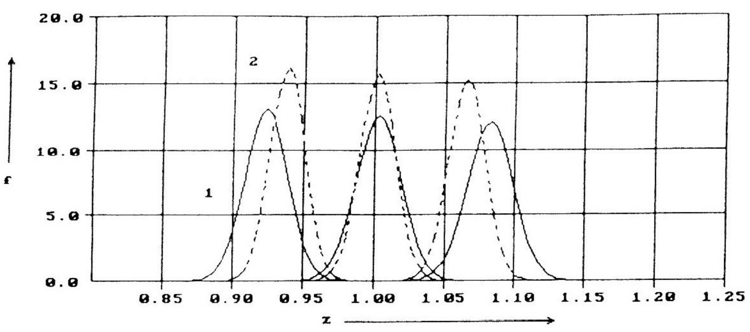

Figure 9. Damage Measure Probability Density Functions

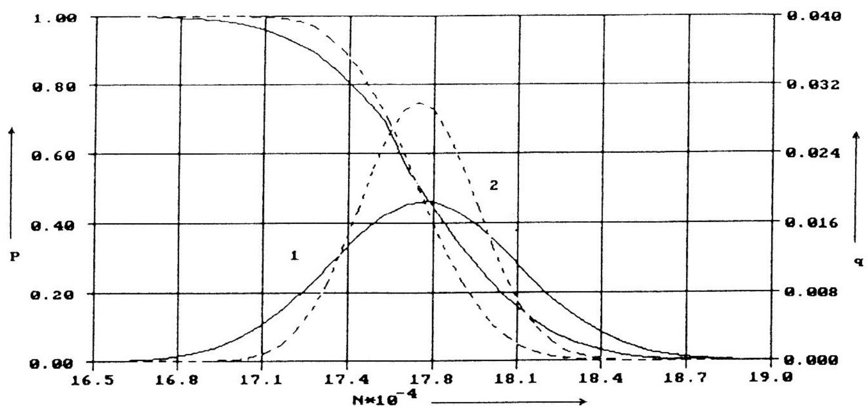

Figure 10. Probability of Survival (P) and Failures Probability Density Function (q)

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