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An Accelerated Iterative Procedure for Solving Optimizational Problems through the Finite Element Method

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The calculation procedure offered works with the initial stiffness matrix and the changes caused by the changes in the discretized body are obtained by iteration. Each consecutive iteration is carried out by approximation of the sought precise change, constructed on the basis of the simple increments of nodal displacements known at that moment. When the approximation does not satisfy the accuracy requirements, the increment is found on the basis of the results obtained and it is also taken into considerations when the next approximation is found.

1 Introduction. Some Basic Dependences

The adaptation of the finite element method for solving dimensional or optimizational problems is done through purposeful alterations in the geometry of the discretized body or of the mechanical characteristics of its material (Baltov, 1981). The stiffness matrix $\mathbf{K}_E$ of the body changes into $\mathbf{K}_{EP} = \mathbf{K}_E - \mathbf{K}_P$ and the nodal displacements from $\mathbf{u}_0 = \mathbf{K}_E^{-1}\varphi$ to $\mathbf{u} = \mathbf{K}_{EP}^{-1}\varphi$

$$\mathbf{x}_n = \mathbf{K}_E^{-1}\psi_{n-1} \qquad \psi_n = \mathbf{K}_P\mathbf{x}_n \qquad \mathbf{x}_{n+1} = \mathbf{K}_E^{-1}\psi_n$$

i.e.

$$\mathbf{x}_{n+1} = \mathbf{K}_E^{-1}\mathbf{K}_P\mathbf{x}_n \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

The vector of the unbalanced nodal forces ψ_n can be defined without forming the global matrix $\mathbf{K}_P$ of the changes in the stiffness by using the changes in the matrices of the separate finite elements affected by the changes in the body. The unbalanced forces decrease and the iterative process is convergent when the body stiffness alteration is not great and the maximum eigenvalue of matrix $\mathbf{K}_E^{-1}\mathbf{K}_P$ is less than 1 (Baltov, 1981).

The efficiency of this method of solution, known as the initial load method (initial stiffness method), can be increased considerably by convergence acceleration. For this purpose, an approximation of the precise increment is constructed (on the basis of simple increments obtained from equation (1.1)), which could be obtained after $\mathbf{K}_{EP}$ is worked out and inverted. In this way, the number of iterations is reduced and the convergence area is considerably extended (Nickolov, 1989).

To simplify the recording of the dependences, we shall write down the successive simple increments without any indices with the succesive letters **a**, **b**, **c**, **d**, ...

$$\mathbf{b} = \mathbf{K}_E^{-1}\mathbf{K}_P\mathbf{a} \qquad \mathbf{c} = \mathbf{K}_E^{-1}\mathbf{K}_P\mathbf{b} \tag{1.2}$$

and their corresponding precise increments with **p**, **q**, **r**, **s**, ...

$$\psi = \mathbf{K}_E\mathbf{a} = \mathbf{K}_{EP}\mathbf{p} \qquad \mathbf{a} = \mathbf{K}_E^{-1}(\mathbf{K}_E - \mathbf{K}_P)\mathbf{p}$$

$$\mathbf{a} = \mathbf{p} - \mathbf{K}_E^{-1}\mathbf{K}_P\,\mathbf{p} \qquad \mathbf{b} = \mathbf{q} - \mathbf{K}_E^{-1}\mathbf{K}_P\,\mathbf{q} \qquad \mathbf{c} = \mathbf{r} - \mathbf{K}_E^{-1}\mathbf{K}_P\,\mathbf{r} \tag{1.3}$$

$$\mathbf{p} = \mathbf{q} + \mathbf{a} \qquad \mathbf{q} = \mathbf{r} + \mathbf{b} \tag{1.4}$$

It is obvious that from equations (1.3) and (1.4) there follows

$$\mathbf{q} = \mathbf{K}_E^{-1}\,\mathbf{K}_P\,\mathbf{p} \qquad \mathbf{r} = \mathbf{K}_E^{-1}\mathbf{K}_P\,\mathbf{q} \tag{1.5}$$

i.e. the ordinary iteration, with a given precise increment, will lead to the next precise increment. If, however, instead of the precise increment $\mathbf{q}$, some approximation $\mathbf{q}_0$ is constructed, then the connection (1.5) between $\mathbf{q}_0$ and the next approximation $\mathbf{r}_0 = \mathbf{q}_0 - \mathbf{b}$ will be satisfied with a certain error ξ:

$$\mathbf{r}_0 + \xi = \mathbf{K}_E^{-1}\mathbf{K}_P\,\mathbf{q}_0 \tag{1.6}$$

2 Acceleration on the Basis of Two Simple Increments

Let us assume that only the first two iterations have been carried out i.e. the increments $\mathbf{a}$ and $\mathbf{b}$ are present. We shall express the increment $\mathbf{b}$ through increment $\mathbf{a}$ with the dependence

$$\mathbf{b} = \alpha\mathbf{a} + \mathbf{y}_1 \tag{2.1}$$

After substituting it in equation (1.2) we can see that it is valid for the following simple increments

$$\mathbf{c} = \alpha\,\mathbf{K}_E^{-1}\,\mathbf{K}_P\,\mathbf{a} + \mathbf{K}_E^{-1}\,\mathbf{K}_P\,\mathbf{y}_1 = \alpha\,\mathbf{b} + \mathbf{z}_1 \tag{2.2}$$

when

$$\mathbf{z}_1 = \mathbf{K}_E^{-1}\,\mathbf{K}_P\,\mathbf{y}_1 \tag{2.3}$$

If we multiply term-by-term all the terms on the left hand side in equation (2.1) of matrix $\mathbf{K}_{EP}^{-1}\mathbf{K}_E$ and keep in mind that

$$\mathbf{K}_{EP}^{-1}\,\mathbf{K}_E\,\mathbf{a} = \mathbf{p} \qquad \mathbf{K}_{EP}^{-1}\,\mathbf{K}_E\,\mathbf{b} = \mathbf{q} \qquad \mathbf{K}_{EP}^{-1}\,\mathbf{K}_E\,\mathbf{y}_1 = \Delta_{y_1} \tag{2.4}$$

we obtain

$$\mathbf{q} = \alpha\,\mathbf{p} + \Delta_{y_1} \tag{2.5}$$

and having in mind that $\mathbf{p} = \mathbf{q} + \mathbf{a}$, we find

$$\mathbf{q} = \frac{1}{\omega_1}(\alpha\,\mathbf{p} + \Delta_{y_1}) \qquad \omega_1 = 1 - \alpha \tag{2.6}$$

If coefficient α is defined so that vector $\mathbf{y}_1$ is the least possible, according to equation (2.4) vector Δ_{y_1} will be the least possible too, and then to find the approximation of the precise increment $\mathbf{q}$, it could be neglected in the expression (2.6).
The norm of vector $\mathbf{y}_1 = \mathbf{b} - \alpha\,\mathbf{a}$ will be minimal if

$$F = \sum_i (b_i - \alpha a_i)^2 \quad \Rightarrow \quad \min \tag{2.7}$$

i.e., if $\frac{dF}{d\alpha} = 0$, from which we find

$$-2\sum_i (b_i - \alpha a_i) a_i = -2\sum_i y_{1i} a_i = 0 \tag{2.8}$$

$$\alpha = \frac{\sum_i a_i b_i}{\sum_i a_i^2} \tag{2.9}$$

If we carry out an ordinary iteration by the method of initial stiffness with approximation $\mathbf{q}_\alpha = \frac{1}{\omega_1}\alpha\,\mathbf{a}$, according to equation (1.6) we shall obtain

$$\begin{aligned} &\mathbf{q}_\alpha - \mathbf{b} + \delta_1 = \mathbf{K}_E^{-1}\mathbf{K}_p\,\mathbf{q}_\alpha = \frac{1}{\omega_1}\alpha\,\mathbf{K}_E^{-1}\mathbf{K}_p\,\mathbf{a} = \frac{1}{\omega_1}\alpha\,\mathbf{b} \\ &\delta_1 = \frac{1}{\omega_1}(\mathbf{b} - \alpha\,\mathbf{a}) = \frac{1}{\omega_1}\mathbf{y}_1 \end{aligned} \tag{2.10}$$

The latter dependence shows that with approximation $\mathbf{q}_\alpha$ it is not necessary to carry out an ordinary iteration as it gives no new information - having found coefficient α, vector $\mathbf{y}_1 = \mathbf{b} - \alpha\,\mathbf{a}$ as well as vector δ_1 are completely defined.

If for approximation we choose

$$\mathbf{q}_0 = \mathbf{q}_\alpha + \delta_1 = \frac{1}{\omega_1}\alpha\,\mathbf{a} + \frac{1}{\omega_1}\mathbf{y}_1 = \frac{1}{\omega_1}\mathbf{b} \tag{2.11}$$

the error

$$\mathbf{q} - \mathbf{q}_0 = \frac{1}{\omega_1}(\alpha\,\mathbf{a} + \Delta_{y_1} - \mathbf{b}) = \frac{1}{\omega_1}(\Delta_{y_1} - \mathbf{y}_1) = \frac{1}{\omega_1}\Delta_{z_1} \tag{2.12}$$

will obviously be smaller, and with ordinary iteration we shall obtain

$$\begin{aligned} &\mathbf{q}_0 - \mathbf{b} + \varepsilon_1 = \mathbf{K}_E^{-1}\,\mathbf{K}_p\,\mathbf{q}_b = \frac{1}{\omega_1}\,\mathbf{K}_E^{-1}\,\mathbf{K}_p\,\mathbf{b} = \frac{1}{\omega_1}\mathbf{c} \\ &\omega_1\,\varepsilon_1 = \mathbf{c} - \alpha\,\mathbf{b} = \mathbf{z}_1 \end{aligned} \tag{2.13}$$

from which we can find the simple increment $\mathbf{c}$ without a new iteration. Obtaining equation (2.12), we have had in mind that according to equations (2.4) and (2.3), Δ_{y_1} and Δ_{z_1} are the precise increments covering the series of simple ones, beginning from $\mathbf{y}_1$, respectively $\mathbf{z}_1$ and therefore $\Delta_{y_1} = \Delta_{z_1} + \mathbf{y}_1$.

So, when two succesive simple increments of the displacements $\mathbf{a}$ and $\mathbf{b}$ are present, with equation (2.9) we find coefficient α, with equation (2.11) we construct the approximation $\mathbf{q}_0$ of the precise increment and with it, instead of with $\mathbf{b}$, we carry out the next ordinary iteration. If vector ε_1, obtained as a result of the iteration, is not sufficiently small, to consider the iteration process complete, we can calculate the simple increment $\mathbf{c}$ with equation (2.13) and repeat the attempt for acceleration - this time with increments $\mathbf{b}$ and $\mathbf{c}$. In this way we shall obtain a generally better approximation, because approximation of Rayleigh's coefficient is actually defined with equation (2.9) (i.e. α is approximation of the greatest eigenvalue λ_1 of matrix $\mathbf{K}_E^{-1}\mathbf{K}_P$), and it is proved that with the increase of the number of iterations it tends to its precise value (Zurmühl, 1950).

However, it turns out that it makes sense to look for another, more efficient way for the approximation of the next precise increment, on the basis of already known three successive simple increments of the displacements.

3 Acceleration on the Basis of Three or More Simple Increments

We shall follow the same approach and the last simple increment of the displacements will be expressed by the previous two

$$\mathbf{c} = \beta_1 \mathbf{a} + \beta_2 \mathbf{b} + \mathbf{y}_2 \qquad \mathbf{d} = \beta_1 \mathbf{b} + \beta_2 \mathbf{c} + \mathbf{z}_2 \quad \ldots \tag{3.1}$$

Analogically with the expressed above, we obtain

$$\mathbf{r} = \beta_1 \mathbf{p} + \beta_2 \mathbf{q} + \Delta_{y2} \tag{3.2}$$

$$\mathbf{r}_2 = \frac{1}{\omega_2}[\beta_1 \mathbf{a} + (\beta_1 + \beta_2)\mathbf{b} + \Delta_{y_2}] \qquad \omega_2 = 1 - \beta_1 - \beta_2 \tag{3.3}$$

From the condition that vector $\mathbf{y}_2$ should be minimal

$$F = \sum_i (c_i - \beta_1 a_i - \beta_2 b_i)^2 \qquad \Rightarrow \qquad \min$$

we obtain a system of linear equations to define coefficients β_1 and β_2

$$\begin{aligned} \frac{dF}{d\beta_1} &= 0 & \sum_i a_i c_i &= \beta_1 \sum_i a_i^2 + \beta_2 \sum_i a_i b_i \\ \frac{dF}{d\beta_2} &= 0 & \sum_i b_i c_i &= \beta_1 \sum_i a_i b_i + \beta_2 \sum_i b_i^2 \end{aligned} \tag{3.4}$$

After we find them, vector $\delta_2 = \frac{1}{\omega_2}\mathbf{y}_2$ is completely defined. We shall accept

$$\mathbf{r}_0 = \frac{1}{\omega_2}[\beta_1 \mathbf{a} + (\beta_1 + \beta_2)\mathbf{b}] + \delta_2 = \frac{1}{\omega_2}(\beta_2 \mathbf{b} + \mathbf{c}) \tag{3.5}$$

as an approximation of the precise increment which substitutes the simple increment $\mathbf{c}$. We carry out the next iteration with it and obtain error ε_2

$$\varepsilon_2 = \frac{1}{\omega_2}(\mathbf{d} - \beta_1 \mathbf{b} - \beta_2 \mathbf{c}) = \frac{1}{\omega_2}\mathbf{z}_2 \tag{3.6}$$

If this error is not sufficiently small to consider the iteration process finished, then with it we define the following, fourth, simple increment, without a new iteration.

Following the same logic, we can use the four successive simple increments $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ and build an analogous approximation $\mathbf{s}_b$, substituting $\mathbf{d}$

$$\mathbf{s}_b = \frac{1}{\omega_3}[\gamma_1 \mathbf{b} + (\gamma_1 + \gamma_2)\mathbf{c} + \mathbf{d}] \qquad \omega_3 = 1 - \gamma_1 - \gamma_2 - \gamma_3 \tag{3.7}$$

in which the coefficients are defined by the system

$$\begin{aligned}
\sum_i a_i d_i &= \gamma_1 \sum_i a_i^2 + \gamma_2 \sum_i a_i b_i + \gamma_3 i \sum a_i c_i \\
\sum_i b_i d_i &= \gamma_1 \sum_i a_i b_i + \gamma_2 \sum_i b_i^2 + \gamma_3 \sum_i b_i c_i \\
\sum_i c_i d_i &= \gamma_1 \sum_i a_i c_i + \gamma_2 \sum_i b_i c_i + \gamma_3 \sum_i c_i^2
\end{aligned} \tag{3.8}$$

In carrying out the following iteration with approximation (3.7) instead of $\mathbf{d}$, we obtain error ε_3 through which the following, fifth, simple increment

$$\mathbf{e} = \gamma_1 \mathbf{b} + \gamma_2 \mathbf{c} + \gamma_3 \mathbf{d} + \omega_3 \varepsilon_3 \tag{3.9}$$

can be defined.

In an analogous way, approximations of the consecutive precise increments can be built on the basis of five, six or more simple increments, obtained through the error after iteration is carried out with them. The coefficients in these approximations can be found by solving systems of linear equations with four, five or more equations which structurally are analogous to equations (3.8). Naturally, a basis approximation can be constructed too, for example for only the last four increments, leaving out the preceding ones. Naturally, these approximations improve with the increase of the number of iterations.

4 The Point of the Approach

The main idea of the approach is based on the circumstance that the connection (1.5) between the precise increments is absolutely the same as the connection (1.2) between the simple ones. The last one of the known simple increments of the displacements is represented as a linear combination of the preceding ones and vector-error $\mathbf{y}_n$. An analogous linear combination is obtained between the precise increments too. It allows an approximation of any of the precise increments to be constructed as an accelerated increment, the error being defined synonymously by the vector $\mathbf{y}_n$. To achieve good quality of the approximation, the coefficients in the linear combination should be defined in such manner that the vector $\mathbf{y}_n$ is the shortest possible one. A circumstance which is very favourable is that if, after an ordinary iteration has been carried out with an accelerated increment and the accuracy is not satisfactory, it is possible to define, without new iteration, a new simple increment which can be used together with those found by that time, in creating a subsequent accelerated increment.

It is obvious that conditions (2.8), (3.4), (3.8) and so on, from which the coefficients of the linear combination are defined, are actually conditions for orthogonalizing the vectorerror $\mathbf{y}_n$ to the simple increments found by this time

$$\begin{array}{lll}
 & (\mathbf{y}_1, \mathbf{a}) = 0 & \Rightarrow \quad \alpha \\
(\mathbf{y}_1, \mathbf{a}) = 0 & (\mathbf{y}_2, \mathbf{b}) = 0 & \Rightarrow \quad \beta_1, \beta_2 \\
(\mathbf{y}_1, \mathbf{a}) = 0 & (\mathbf{y}_2, \mathbf{b}) = 0 & (\mathbf{y}_2, \mathbf{c}) = 0 \quad \Rightarrow \quad \gamma_1, \gamma_2, \gamma_3
\end{array} \tag{4.1}$$

Hence, the vectorerror $\mathbf{y}_n$ will decrease rapidly with the increase of the number of simple increments participating in the construction of the approximation.

The vectorerror $\mathbf{y}_n$ can turn out to be strictly equal to zero if the (n+1) simple increment proves to be linearly dependent on the preceding ones, i. e. if the rank of the simple increments vector system is n. In this case, with the coefficients found in the linear combination we construct not an approximation but the precise increment, and the approach becomes accurate.

The most important merit of the approach is that it functions even when the ordinary iteration process of the initial stiffness method is divergent.

This can be seen at the attempt of constructing an accelerated increment on the basis of two simple increments. As it was already shown, an approximation of the greatest in modulus eigenvalue λ_1 of matrix $\mathbf{K}_E^{-1}\mathbf{K}_P$ is actually found with equation (2.9). It is known (Baltov, 1981) that all eigenvalues λ_k of matrix $\mathbf{K}_E^{-1}\mathbf{K}_P$ are smaller than 1. Their values depend on the changes in the body undertaken with the optimization. When the changes cause decrease of the stiffness in the separate finite elements, all λ_k are positive and the ordinary iteration process is convergent. When, however, the stiffness of separate finite elements increases some of the eigenvalues become negative and it is possible for some of them to become smaller than -1. In such case, the ordinary iteration process is divergent and the successive simple increments increase, alternately changing their algebraic sign. It can be proved that the positively defined symmetric matrix $\mathbf{K}_{EP}^{-1}\mathbf{K}_E$, participating in the expression $\mathbf{q} = \mathbf{K}_{EP}^{-1}\mathbf{K}_E\mathbf{b}$ is diagonalized with the eigenvectors of matrix $\mathbf{K}_E^{-1}\mathbf{K}_P$ into a diagonal matrix containing the numbers $\mu_k = \frac{1}{1-\lambda_k}$. Therefore the values of the eigenvalues λ_k are not an obstacle to find an approximation of the connection between $\mathbf{q}$ and $\mathbf{b}$.

The rank of the system of simple increment m-dimensional vectors cannot be higher than m and consequently, theoretically the approach provides an explicit solution, regardless of the fact whether the successive simple increments decrease or increase, i.e. whether the ordinary iteration process is convergent or divergent.

How many of the simple increment vectors are linearly independent depends on the singularities of the particular problem and is defined by the fact how many global degrees of freedom with the accepted discretization are affected by the changes undertaken in the geometry of the body or in its material. When the changes are locally limited, the stiffness matrices of only a limited number of finite elements will change and the rank of the vector system is comparatively low. Besides, we must have in mind that the change of the coordinates of only one node can cause a change in the stiffness matrices of several finite elements (of all in which the node takes place) and in this way, other global degrees of freedom are indirectly affected by the change.
Naturally, the search for an approximation on the basis of a continously and practically infinitely growing number of simple increments, is not expedient. The reason for this is that on the one hand additional computation can grow a lot and on the other hand, defining the coefficients is connected with inevitable loss of accuracy due to errors of rounding when storing the calculation results with a limited number of significant digits.

5 Efficiency of the Approach

The approach was subjected to a thorough experimentation by solving a set of check examples. Juxtaposing the approximated solution with the accurate one showed its very high efficiency.

For the case when the ordinary iteration process is convergent, usually a very small number of iterations are sufficient (three, four or more rarely five simple displacement increments) to find a sufficiently good approximation of the precise increment, no matter how local the changes in the body are. When the ordinary iteration process is divergent and the rank of the vector system of simple increments is high, then the number of iterations necessary for the solution grows, and the sensitivity towards the loss of accuracy increases. In such cases the coefficients in the systems of the type (3.8) can prove to be of different order, and the accuracy of the roots and the stability of the successively constructed approximations decrease. That is why, when optimizing and dimensioning, as a most appropriate strategy should be accepted the one that is based on an object with increased stiffness and possibilities should be sought successively for making it higher by decreasing its stiffness.

Experimentation confirmed that in local changes in the body, the explicit solution is obtained with the first few iterations, no matter whether the ordinary iteration process is convergent or not. It has been established that building an approximation of the precise increment on the basis of more than six simple increments is not expedient. That is why, if the required accuracy is not achieved after the sixth iteration, then, with the further iterations, it is advisable to construct an approximation only on the basis of the last six increments.

The conception accepted to express the last simple increment as a linear combination of the preceding ones is certainly not the only one possible. It is possible to build other solutions too, where another simple increment is expressed by the ones known at that time. Then all dependences describing the calculation of the coefficients in the linear combination and the construction of the accelerated increment change but, as experimentation has shown, the accuracy of calculations and the approach efficiency practically remain unchanged.

Literature

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