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Concerning the Motion of a Solid Circular Cylinder in a Fluid Contained in a Concentric Cylindrical Boundary in Response to a Blunt Impact

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An analysis of the motion of a solid circular cylinder in a fluid contained in a concentric cylindrical boundary for the case of a blunt impact is presented. Special attention is paid to the influence of time parameter and amplitude of circulation of fluid on the pressure distribution of the circumferential coordinate, especially when the gap is quite small. It is shown that there is a „critical" time at which the pressure in the opposite direction of motion can change its sign. Maximum value of pressure gradient with respect to the angular coordinate is discussed. Results of the present analysis can be applied to actual biomechanical and biomedical problems.

1 Introduction

Hydrodynamic processes of a more or less regular kind are seen in very diverse forms in all parts of the biomedical world. Some of these are closely associated with impact problems. The latter case may arise, for instance, when a solid body immersed in the fluid is suddenly set in motion, or if the boundary conditions suddenly change. We shall not attempt a review of the pertinent literature, but we refer to fundamental results by Lamb (1945), Birkhoff (1955), Prandtl and Tietjens (1957), Van Dyke (1964), and Batchelor (1974). Whereas in Lamb and in Batchelor, for example, the force impulse was determined which must be applied to the rigid body in order to generate the given unbounded fluid motion from rest as was the resultant of the distributed force impulse that must be applied to a limited portion of the fluid in order to generate the whole of the given motion from rest. The resultant force impulse was called the „fluid impulse" of the flow field.
In the following an analysis of the mathematical model for the impulsive motion of a solid circular cylinder in a fluid contained in a concentric fixed cylindrical boundary is discussed here. The instanteneous motion of a cylinder is specified fully by the velocity of its center and the angular circulation function of a fluid. The fluid viscosity effect in this analysis will be ignored. Thus the velocity potential and stream function required are those describing flow due to a circular cylinder.

The main features of the present discussion include the verification that there exists a critical time for the cylinder motion which corresponds to the effect of changing the sign of the pressure with respect to the motion and pressure gradient at that time.

2 Formulation of Problem

Consider the motion produced in a liquid contained between a solid circular cylinder of radius a and a fixed concentric cylindrical boundary of radius b, when the cylinder is moving under impulsive loading with given velocity U perpendicular to its length.

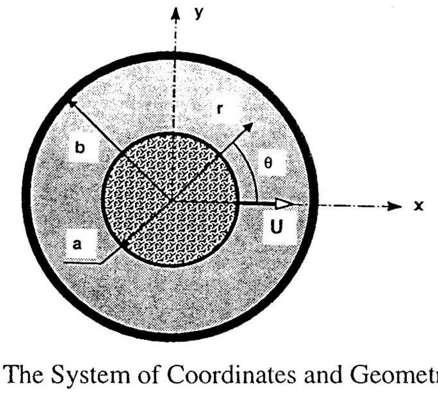

Figure 1. The System of Coordinates and Geometry

If the cylinder is moving with a given velocity

$$U = U_0(1-t^*) \qquad t^* = \frac{t}{t_1} = \tau \tag{1}$$

the velocity potential can be written as (Lamb, 1945)

$$\phi = U_0(1-\tau)\frac{1}{1-\alpha^2}\left[r(1+\alpha^2)+\frac{a^2}{r}\right]\cos\theta + k(\theta) \tag{2}$$

where the circulation function $k(\theta)$ of the fluid is assumed as follows:

$$k(\theta) = k_0 \tanh(\theta) \tag{3}$$

Here k_0 is a constant.

The velocity q of the fluid relative to the axes of the moving cylinder can be evaluated as

$$q = \frac{1}{\mathrm{r}}\frac{\partial\phi}{\partial\theta} = -U_0(1-\tau)\frac{1}{1-\alpha^2}\left[\left(1+\alpha^2\right)+\frac{a^2}{r^2}\right]\sin\theta + \frac{k_0}{r}\operatorname{sech}^2\theta \tag{4}$$

The derivative of the velocity function (2) with respect to time is

$$\frac{\partial\phi}{\partial\tau} = -U_0\frac{1}{1-\alpha^2}\left[r\left(1+\alpha^2\right) + \frac{\alpha^2}{r}\right]\cos\theta \tag{5}$$

At $r = a$ and $r = b$ we obtain

$$\begin{aligned} q\Big|_{r=a} &= \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r=a} = -U(1-\tau)\frac{2+\alpha^2}{1-\alpha^2}\sin\theta+\frac{k_0}{a}\operatorname{sech}^2\theta \\ q\Big|_{r=b} &= \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r=b} = -U(1-\tau)\frac{1+2\alpha^2}{1-\alpha^2}\sin\theta+\frac{k_0}{b}\operatorname{sech}^2\theta \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\partial\phi}{\partial\tau}\Big|_{r=a} &= -U_0 a\frac{2+\alpha^2}{1-\alpha^2}\cos\theta \\ \frac{\partial\phi}{\partial\tau}\Big|_{r=b} &= -U_0 b\frac{1+2\alpha^2}{1-\alpha^2}\cos\theta \end{aligned} \tag{7}$$

For impulsive motion of a solid cylinder, the second term in the formula for the pressure in the fluid (Batchelor, 1974) is

$$\begin{aligned} \frac{p}{\rho} &= \frac{\partial\phi}{\partial\tau} - \frac{1}{2}I + E \\ p^* &= 1 - \frac{1}{2E}I + \frac{1}{E}\frac{\partial\phi}{\partial\tau} \end{aligned} \tag{8}$$

where $p^* = \dfrac{p}{\rho E}$ has to be evaluated from

$$
\begin{aligned}
I\big|_{r=a} &= \int^{\tau} q^2 d\tau = U_0^2\left(\tau-\tau^2+\frac{\tau^3}{3}\right)\left(\frac{2+\alpha^2}{1-\alpha^2}\right)^2 \sin^2\theta \\
&\quad -U_0\left(\tau-\frac{\tau^2}{2}\right)\left(\frac{2+\alpha^2}{1-\alpha^2}\right)\sin\theta\frac{k_0}{a}\operatorname{sech}^2\theta+\frac{k_0{}^2}{a^2}\tau\operatorname{sech}^4\theta \\
I\big|_{r=b} &= \int^{\tau} q^2 d\tau = U_0^2\left(\tau-\tau^2+\frac{\tau^3}{3}\right)\left(\frac{1+2\alpha^2}{1-\alpha^2}\right)^2 \sin^2\theta \\
&\quad -U_0\left(\tau-\frac{\tau^2}{2}\right)\left(\frac{1+2\alpha^2}{1-\alpha^2}\right)\sin\theta\frac{k_0}{b}\operatorname{sech}^2\theta+\frac{k_0{}^2}{b^2}\tau\operatorname{sech}^4\theta
\end{aligned}
\tag{9}
$$

At $\theta = 0$ and $\theta = \pi$ we will have

$$
\begin{aligned}
I\big|_{r=a,\,\theta=0} &= \frac{k_0{}^2}{a^2}\tau & I\big|_{r=a,\,\theta=\pi} &= 5.5\cdot 10^{-5}\frac{k_0{}^2}{a^2}\tau \\
I\big|_{r=b,\,\theta=0} &= \frac{k_0{}^2}{a^2}\alpha^2\tau & I\big|_{r=b,\,\theta=\pi} &= 5.5\cdot 10^{-5}\frac{k_0{}^2}{a^2}\alpha^2\tau
\end{aligned}
\tag{10}
$$

Final formulas for the pressure parameter in the fluid are

$$
p^*\big|_{r=a,\theta=0} = 1-\tilde{U}_0-\tilde{k}\tau \qquad p^*\big|_{r=a,\theta=\pi} = 1+\tilde{U}_0-5.5\cdot 10^{-5}\,\tilde{k}\,\tau \tag{11}
$$

$$
\begin{aligned}
p^*\big|_{r=b,\,\theta=0} &= 1-\tilde{U}_0\frac{1}{\alpha}\frac{1+2\alpha^2}{2+\alpha^2}-\tilde{k}\,\alpha^2\,\tau \\
p^*\big|_{r=b,\,\theta=\pi} &= 1+\tilde{U}_0\frac{1}{\alpha}\frac{1+2\alpha^2}{2+\alpha^2}-5.5\cdot 10^{-5}\tilde{k}\,\alpha^2\,\tau
\end{aligned}
\tag{12}
$$

$$
\tilde{U}_0 = U_0 a\frac{2+\alpha^2}{1-\alpha^2} \qquad \tilde{k} = \frac{k_0{}^2}{2Ea^2}
$$

where

$$
\frac{1}{\alpha}\frac{1+2\alpha^2}{2+\alpha^2}\Big|_{\alpha=0.95} = 0.69 \tag{13}
$$

$$
5.5\cdot\alpha^2 = 4.9638
$$

3 Numerical Results for the Pressure Distribution

The results of calculations for the pressure distribution vs. time and angular coordinate θ are presented in Figures 2 to 10 (here denoted $p^* = p,\ \tilde{U}_0 = U,\ \tilde{k} = k,\ \tau = t,\ \theta = T$).

$r = a, U = 1.5, \alpha = 0.95, \tilde{k} = 10^5, \theta = \pi$

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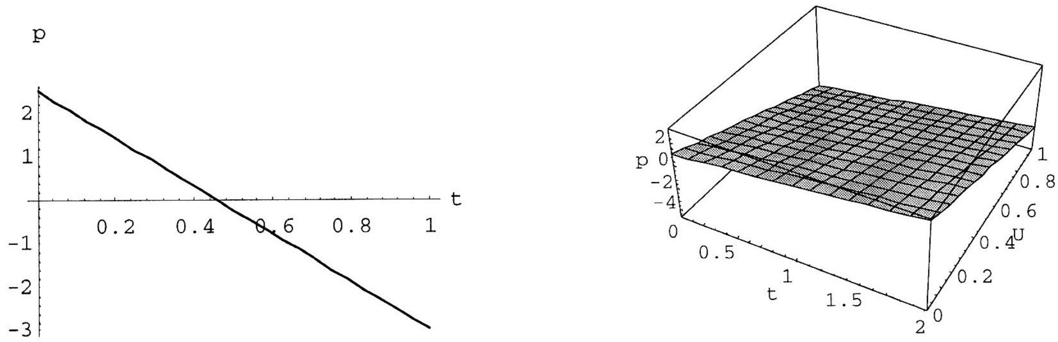

Figure 2. Pressure as Function of Time and Velocity

$r = a, r = b, \theta = \pi, U = 1.5,$
$\alpha = 0.95, \tilde{k} = 10^5$

$r = a, r = b, \theta = 0, U = 1.5,$
$\alpha = 0.95, \tilde{k} = 10^5$

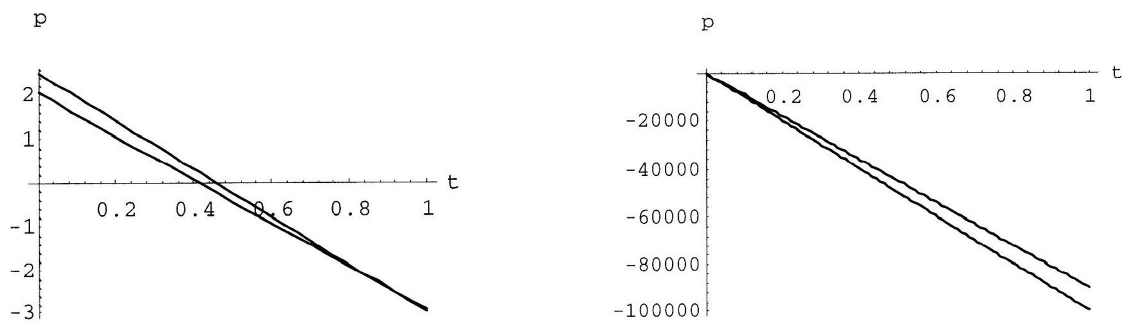

Figure 3. Pressure as Function of Time

$r = b, \theta = 0, U = 1.5,$
$\alpha = 0.1, \tilde{k} = 10^5$

$r = a, r = b, \theta = 0, U = 1.5,$
$\alpha = 0.1, \tilde{k} = 10^5$

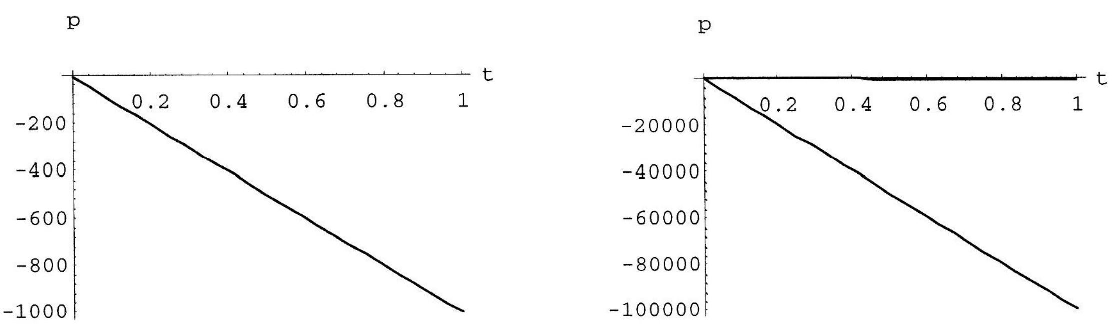

Figure 4. Pressure as Function of Time

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = 1,\ \tau = 1,\ \alpha = 0.95$

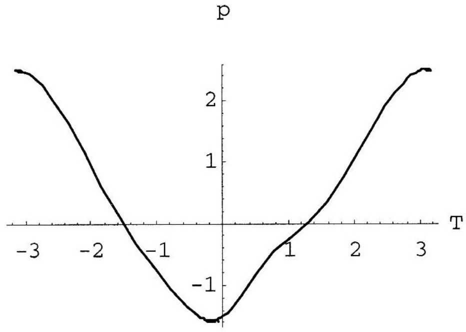

Figure 5. Pressure as Function of Coordinate θ

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = 10^5,\ \theta = 0$

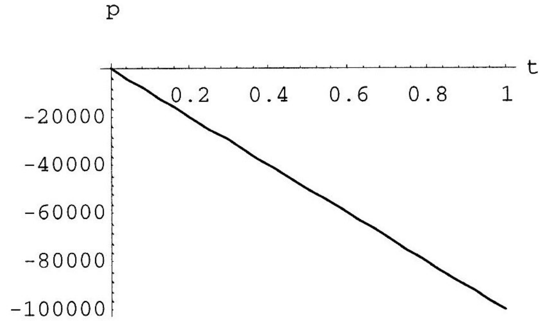

Figure 6. Pressure as Function of Time at $\theta = 0$

$r = a,\ U = 1.5,\ A = \left[0,\ 10^5\right],\ \tilde{k} = 10^5,\ \theta = \pi$

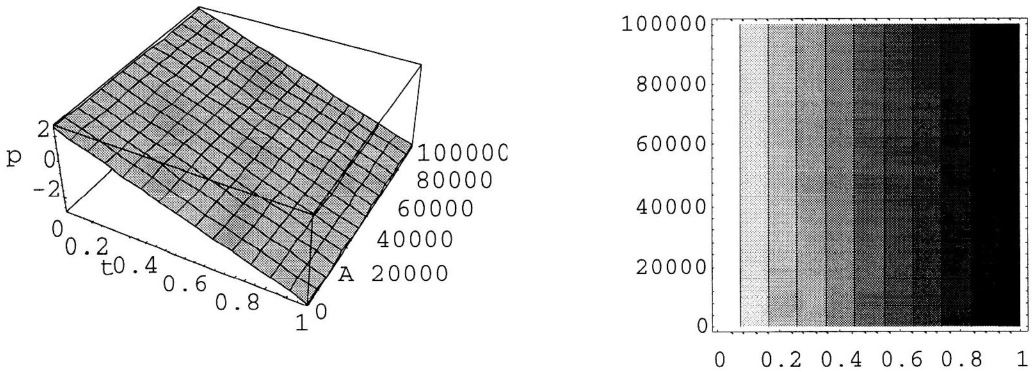

Figure 7. Pressure as Function of τ and A

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = \left[0,\ 10^5\right],\ \theta = \pi$

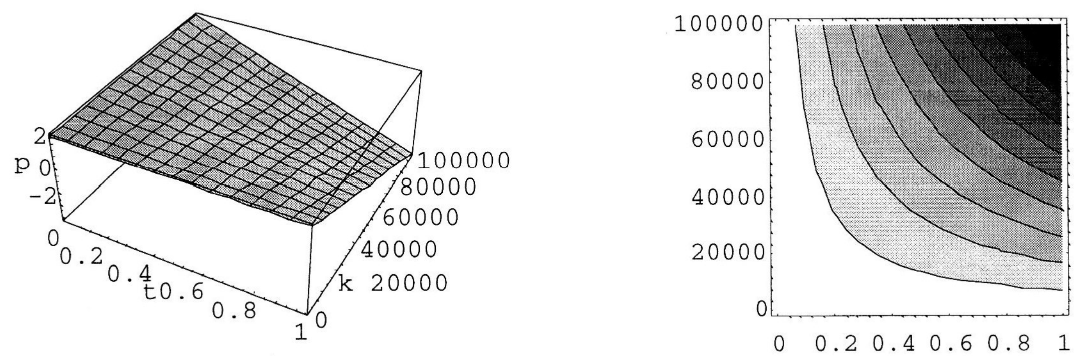

Figure 8. Pressure as Function of τ and $\tilde{k}$

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = [0,\ 1],\ [0,\ 10],\ \tau = 1$

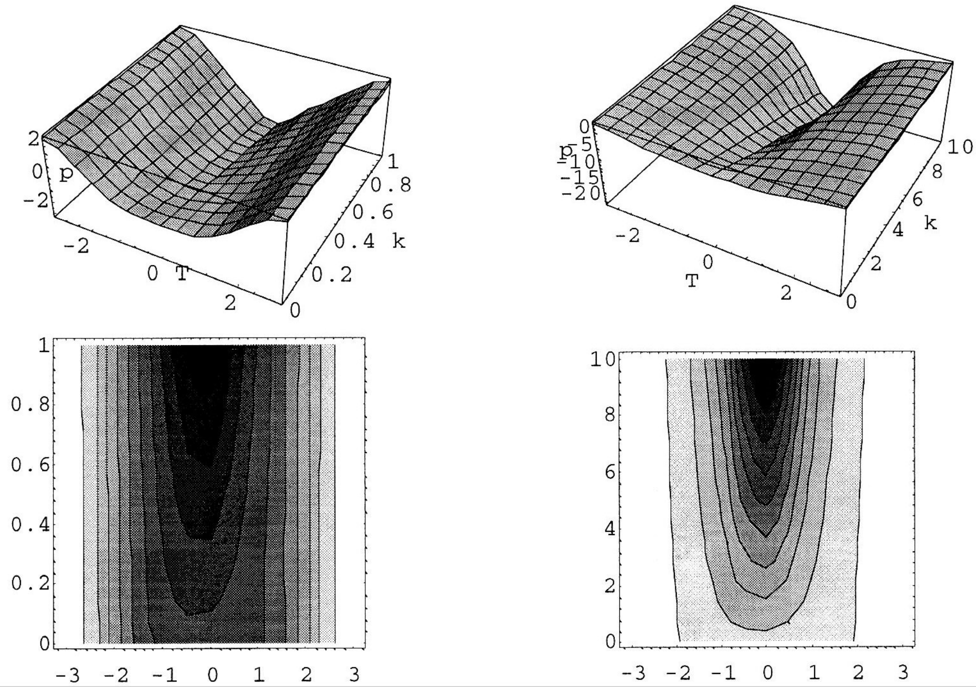

Figure 9. Pressure as Function of θ and $\tilde{k}$ at $\tau = 1$

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = [0,\ 100],\ \left[0,\ 10^5\right],\ \tau = 1$

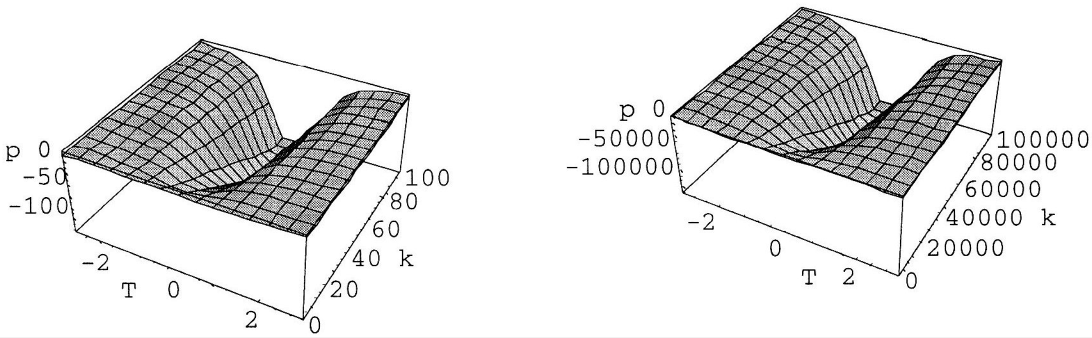

Figure 10. Pressure Parameter as Function of θ and $\tilde{k}$ at $\tau = 1$

4 Numerical Analysis for the Pressure Gradient

On the basis of equations (5) and (7) the final formula for the pressure as function of the coordinate θ at $r = a$ can be written as

$$p^*\Big|_{r=a} = 1 - \tilde{U}_0 \cos\theta - A\,\tilde{U}_0^2\left(\tau - \tau^2 - \frac{\tau^3}{3}\right)\sin^2\theta + \sqrt{A\,\tilde{k}}\,\tilde{U}_0\left(\tau - \frac{\tau^2}{2}\right)\sin\theta\,\mathrm{sech}^2\theta - \tilde{k}\,\tau\,\mathrm{sech}^4\theta \tag{14}$$

where $$\tilde{U}_0 = \frac{U_0}{E}\,a\,\frac{2+\alpha^2}{1-\alpha^2} \qquad \tilde{k} = \frac{k_0^{\,2}}{2E\,a^2} \qquad A = \frac{E}{2a^2} \tag{15}$$

The derivative of the function (14) with respect to θ is

$$\begin{aligned}\left.\frac{\partial p^*}{\partial\theta}\right|_{r=a} &= \tilde{U}_0 \sin\theta - A\tilde{U}_0^2\left(\tau-\tau^2+\frac{\tau^3}{3}\right)\sin(2\theta)\\ &+\sqrt{A\,\tilde{k}\,\tilde{U}_0}\left(\tau-\frac{\tau^2}{2}\right)\cos\theta\ \mathrm{sech}^2\ \theta\\ &-\sqrt{A\,\tilde{k}\,\tilde{U}_0}\left(\tau-\frac{\tau^2}{2}\right)\sin\theta\ \mathrm{sech}^2\ \theta\ \tanh\ \theta\\ &+4\,\tilde{k}\ \tau\,\mathrm{sech}^4\,\theta\ \tanh\theta\end{aligned} \tag{16}$$

The pressure parameter at $r = a, b;\ \theta = 0, \pi$ can be written as follows

$$\left.p^*\right|_{r=a,\theta=0} = 1 - \tilde{U}_0 - \tilde{k}\,\tau \qquad \left.p^*\right|_{r=a,\theta=\pi} = 1 + \tilde{U}_0 - 5.5\cdot 10^{-5}\tilde{k}\,\tau \tag{17}$$

$$\begin{aligned}\left.p^*\right|_{r=b,\theta=0} &= 1 - \tilde{U}_0\,\frac{1}{\alpha}\,\frac{1+2\alpha^2}{2+\alpha^2}-\tilde{k}\,\alpha^2\,\tau\\ \left.p^*\right|_{r=b,\theta=\pi} &= 1 + \tilde{U}_0\frac{1}{\alpha}\,\frac{1+2\alpha^2}{2+\alpha^2}-5.5\cdot 10^{-5}\tilde{k}\,\alpha^2\,\tau\end{aligned} \tag{18}$$

The results of calculations are given in Figures 11 to 15.

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = 1,\ \tau = 1$

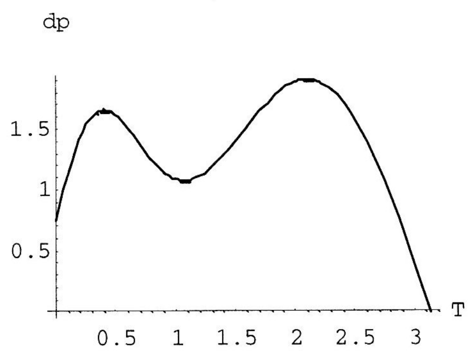

Figure 11. $dp = \frac{\partial p^*}{\partial\theta}$ as Function of θ

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = 1,\ \tau^{cr} = 0.44$

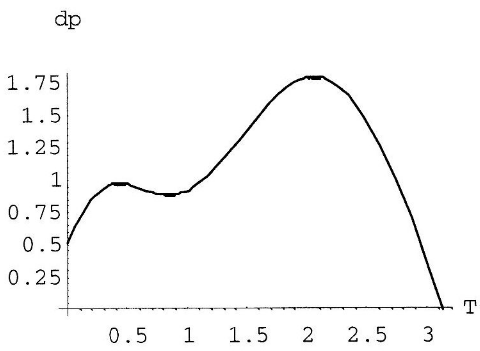

Figure 12 $dp = \frac{\partial p^*}{\partial\theta}$ as Function of θ at τ^{cr}

The values of the coordinate θ that corresponds to $\left.\frac{\partial p}{\partial\theta}\right|^{\max}$ at $\tau = 1$ are

$$\theta_1^{cr} = 0.556 = 31.8565^0 \qquad \theta_2^{cr} = 2.08 = 119.175^0$$

The values of the coordinates θ that corresponds to $\left.\frac{\partial p^*}{\partial\theta}\right|^{\max}$ at $t^{cr} = 0.44$ are

$$\theta_1^{cr} = 0.449 = 25.7258^0 \qquad \theta_2^{cr} = 2.076 = 118.946^0$$

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = [0, 1],\ \tau^{cr} = 0.44$

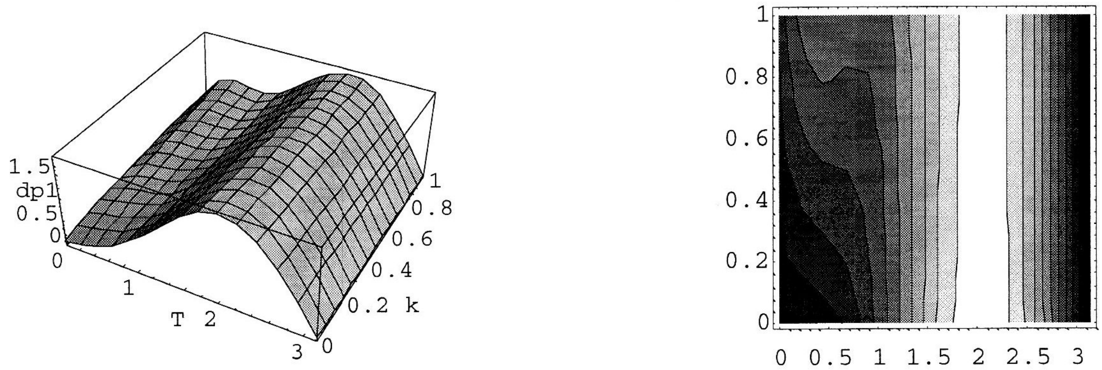

Figure 13a) $dp = \dfrac{\partial p^*}{\partial \theta}$ as Function of θ and $\tilde{k}$ at τ^{cr}

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = [0, 1],\ \tau = 1$

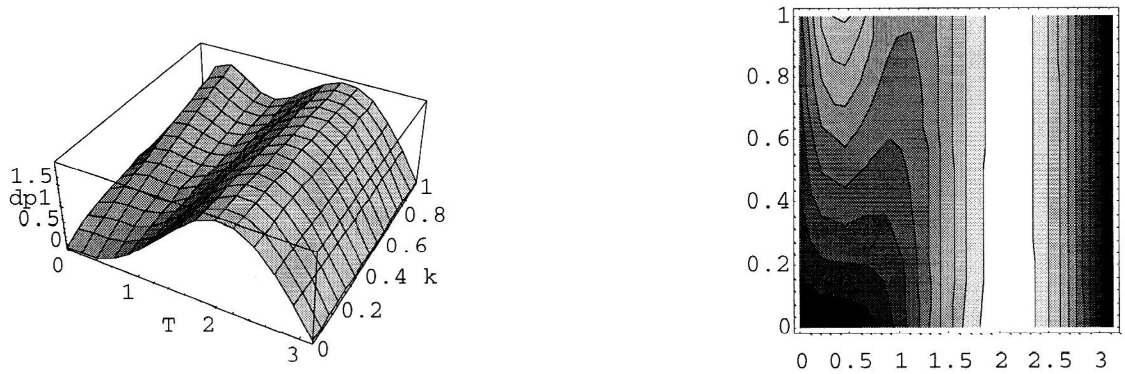

Figure 13b) $dp = \dfrac{\partial p^*}{\partial \theta}$ as Function of θ and $\tilde{k}$ at τ^{cr}

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = [0, 1],\ \tau = 0$

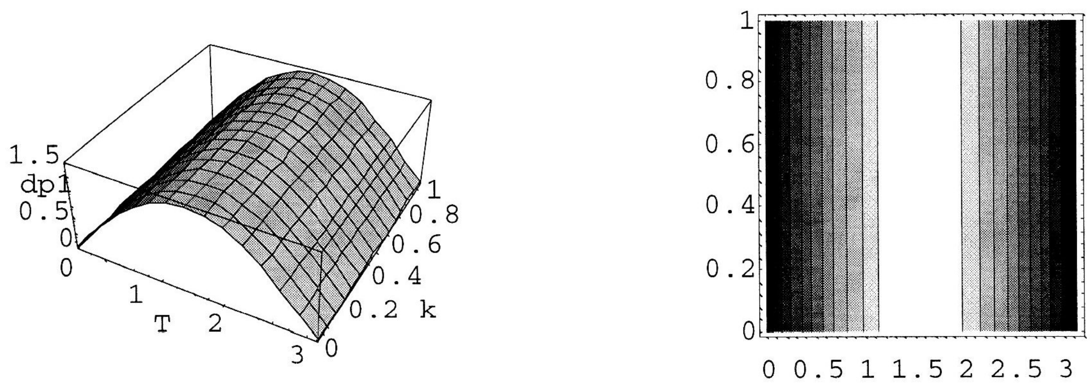

Figure 13c) Pressure Derivative $dp = \dfrac{\partial p^*}{\partial \theta}$ as Function θ and $\tilde{k}$ at τ^{cr}

$r = a,\ U = 1.5,\ A = 1,\ \tilde{k} = [0, 1], [0, 5],\ \tau^{cr} = 0.44$

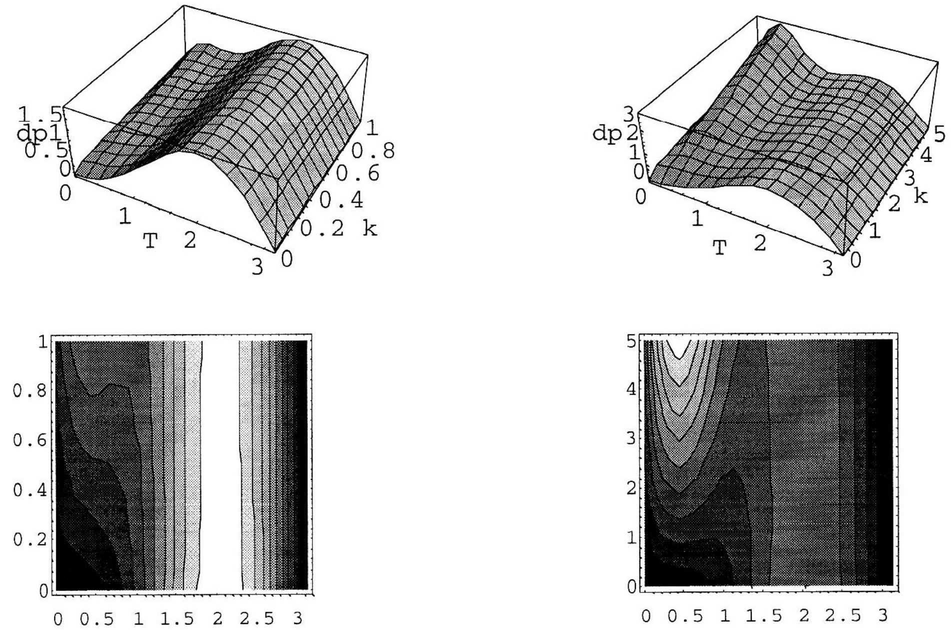

Figure 14. $dp = \dfrac{\partial p^*}{\partial \theta}$ as Function of θ and $\tilde{k}$ at τ^{cr}

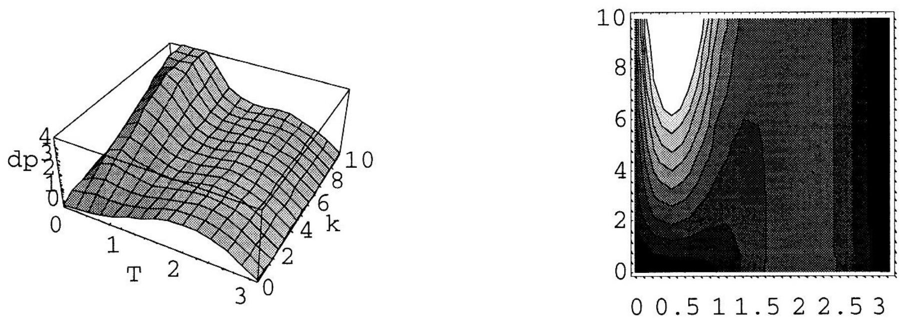

Figure 15a) $dp = \dfrac{\partial p^*}{\partial \theta}$ as Function of θ and $\tilde{k}$ at τ^{cr}

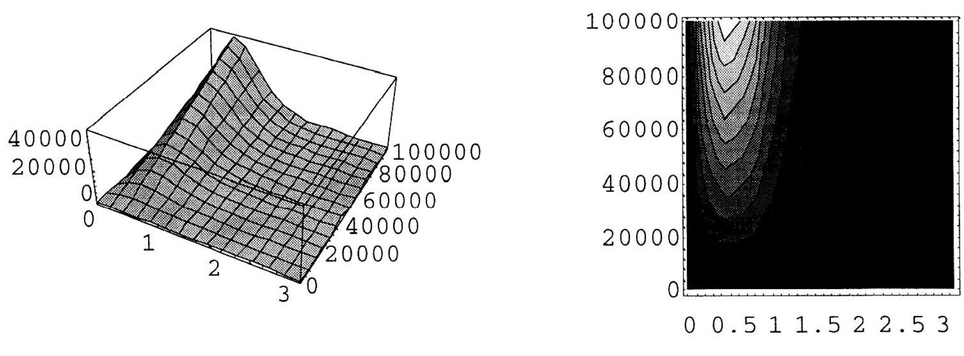

Figure 15b) $dp = \dfrac{\partial p^*}{\partial \theta}$ as Function of θ and $\tilde{k}$ at τ^{cr}

The results of calculations for the pressure gradient at various values of the time parameter τ and angular coordinates θ in the some limiting cases are

$$\tau = 0$$

$$\frac{\partial p^*}{\partial \theta}(0) = 0 \qquad \frac{\partial p^*}{\partial \theta}(0.96\pi) = 0{,}188 \approx 0.19 \qquad \frac{\partial p^*}{\partial \theta}(\pi) = 0$$

$$\tau = 1$$

$$\theta_1^{cr} = 0.556 = 31.86^0 \approx 32^0 \qquad \theta_2^{cr} = 2.08 = 119.175^0 \approx 119^0$$

where θ_i^{cr} corresponds to $\left.\frac{\partial p^*}{\partial \theta}\right|^{\max}$

$$\frac{\partial p^*}{\partial \theta}(0) = 0 \qquad \frac{\partial p^*}{\partial \theta}(0.96\pi) = 0{,}3749 \qquad \frac{\partial p^*}{\partial \theta}(\pi) = 0{,}00022$$

$$\tau^{cr} = 0.44$$

$$\theta_1^{cr} = 0.44865 = 25.72^0 \approx 26^0 \qquad \theta_2^{cr} = 2.0764 = 118.946^0 \approx 119^0$$

$$\frac{\partial p^*}{\partial \theta}(0) = 0.3696 \qquad \frac{\partial p^*}{\partial \theta}(0.96\pi) = 0{,}3375 \qquad \frac{\partial p^*}{\partial \theta}(\pi) = -0{,}0026$$

$$\frac{\partial p^*}{\partial \theta}\left(\theta_1^{cr}\right) = \frac{\partial p^*}{\partial \theta}(0.449) = 0.8319 \qquad \frac{\partial p^*}{\partial \theta}\left(\theta_2^{cr}\right) = \frac{\partial p^*}{\partial \theta}(2.076) = 1.7935$$

5 Concluding Remarks

Impulsive motion of a solid circular cylinder moving in a fluid contained in a concentric cylindrical boundary is discussed. Special attention is paid to the influence of a geometrical parameter α on pressure distribution in the fluid at various values of velocity U^* and amplitude k^* of liquid circulation. It is shown that there is a critical parameter τ^{cr} of time at which the sign of the pressure is changed in the area $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Two „critical" coordinates (for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$) where the partial derivative $\frac{\partial p^*}{\partial \theta}$ has an extreme value are noted. Further efforts might be directed to the examination of elastic deformation of circular cylindrical and more complex boundary shapes with the purpose of improving the agreement between theoretical and experimental data.

Acknowledgment

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