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# Asymptotic and Padé Approximants Methods in the Theory of Reinforced Plates and Shells

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*The dynamics of ribbed plates and shells is described by a system of partial differential equations with discontinuous coefficients, when discreteness of ribs is taken into account. Numerical methods are not economical and very often not acceptable for such equations. It is possible to use the homogenization method for the low frequency case. During the first stage, rigidities and densities of lateral ribs are spread along the plate (shell) and the plate (shell) itself is replaced by a smooth orthotropical one with reduced rigidities and densities. Plate without ribbs is used for higher part of spectrum. Homogenization and perturbation procedures give possibility to solve the oscillation problem for a reinforced plate. The lower part of the spectrum is obtained thanks to a homogenization approach and the higher one thanks to a perturbation method. Two-point Padé approximants given provide a possibility to match these solutions. The approach used makes it possible to determine an expansion of the frequencies and oscillation forms to be found with any desirable accuracy and to obtain a closed-form analytical formula for the total spectrum of the base plate's or shell's natural frequencies.*

## 1 Introduction

The dynamics of ribbed plates and shells is described by a system of partial differential equations with discontinuous coefficients. Numerical methods are not economical and very often not acceptable for such problems. It is possible to use the homogenization method (Bensoussan et al., 1978; Andrianov et al., 1985) for the low frequency case. During the first stage, rigidities and densities of lateral ribs are spread along the plate and the plate itself is replaced by a smooth orthotropic one with reduced rigidities. Further on, using the Goldenveizer first approximation approach (Goldenveizer, 1961), corrections to the frequencies and displacements, caused by discreteness, are obtained. In the high frequency case a perturbation method (Nayfeh, 1973) is used. The theory of smooth plates plays the role of the first approximation, then a general perturbation technique is used.

Then the homogenization and perturbation solutions are to be matched by two-point Padé approximants (Baker and Graves-Morris, 1995). As a result of the application of the above-mentioned method an analytic expression has been deduced which describes the dynamics of a ribbed plate on an elastic substratum for a wide range of perturbations. A comparison with the numerical solution was carried out and the accuracy of the method was found to be satisfactory.

## 2 Homogenization Procedure

Let us consider a ribbed plate $(0 \le x \le L_1, \ -L_2 \le x \le L_2)$ (Figure 1) supported by a Winkler elastic substratum with the stiffness characteristic $C_1$ as the model.

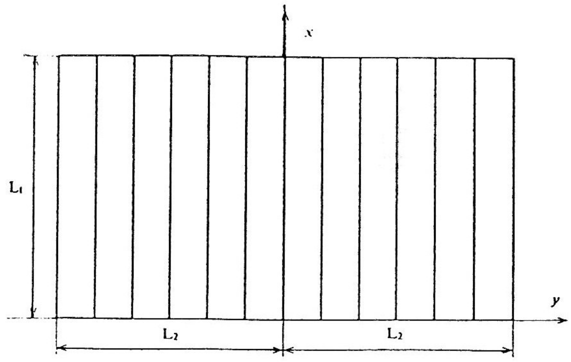


Figure 1. Plate Reinforced in One Direction by Ribs

To a considerable extent this model represents the basic features of the real system's oscillations and at the same time makes it possible to use a rather effective body of mathematics. Here we represent each rib as a one dimension element (this approach may be justified for thin ribs) (Andrianov et al., 1985). The initial equation may then be written as

$$D\nabla^4 W + E_1 IF\phi(y)W_{xxxx} + C_1 W + \left[\rho h + \rho_1 F\phi(y)\right]W_{tt} = 0 \tag{1}$$

where $D = \dfrac{Eh^3}{12\left(l-\nu^2\right)}$; $\phi(y) = \sum_i \delta(y-ib)$; $n = 0.5(N-1)$; $N$ - number of ribs ($N = 2k - 1$); $E$, $\nu$, $\rho$ - modulus of elasticity, Poisson's ratio and mass density of the plate material; $h$ - thickness of the plate; $E_1$, $\rho_1$ - modulus of elasticity and mass density of the rib material; $\delta(\ldots)$ - Dirac function; $t$ - time; $\nabla^4 = \nabla^2\nabla^2$; $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$; $F$ - square rib cross section; $I$ - moment inertia of rib cross section; $w$ - normal displacement. The boundary conditions on the edges of the base plate may be formulated as

$$\begin{aligned} w &= w_{xx} = 0 \quad \text{when} \quad y = \pm L_2 \\ w &= w_{yy} = 0 \quad \text{when} \quad x = 0,\ L_1 \end{aligned} \tag{2}$$

The study of such problems is important from a theoretical as well as a numerical point of view. Because of the complicated structure of the plate, any kind of calculation is difficult to perform. If we treat the boundary value problem we have to impose the boundary condition on the boundary of inclusions which are many in number. So, we would like to approximate the given problem by a homogenized problem on the domain without inclusions. By the method of asymptotic development, a problem on a periodically ribbed domain is reduced to solving problems in the "basic cell" and in a domain without ribs.

Then we use the theory of homogenization, which has been developed by many authors (Bernsoussan et at., 1978; Andrianov et al., 1985).

The homogenization procedure (Bernsoussan et at., 1978; Andrianov et al.,1985) leads to the following boundary value problem:

$$D\nabla^4 W_0 + E_1 I b^{-1} W_{0xxxx} + C_1 W_0 - \left(\rho h + \rho_1 F b^{-1}\right)\lambda_0 W_0 = 0 \tag{3}$$

with boundary conditions

$$\begin{aligned} w_0 &= w_{0xx} = 0 \quad \text{when} \quad y = \pm L_2 \\ w_0 &= w_{0yy} = 0 \quad \text{when} \quad x = 0\ L_1 \end{aligned}$$

Here $$b = \frac{2L_2}{N}$$

### 3 Perturbation Procedure

Now we will investigate the high-frequency oscillations. Let us introduce new "fast" $\xi$, $\eta$, $\tau$ variables as follows

$$\{\xi;\eta\} = \varepsilon^g\{x;y\} \qquad \tau = \varepsilon^{2g} t \qquad g > 0$$

Then the derivatives may be rewritten as follows:

$$\frac{\partial}{\partial x} = \varepsilon^g \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial y} = \varepsilon^g \frac{\partial}{\partial \eta} \qquad \frac{\partial}{\partial t} = \varepsilon^{-2g} \frac{\partial}{\partial \tau} \tag{4}$$

The plate normal displacement asymptotic expansion may be found as

$$W = W_1(\xi, \eta, \tau) + \varepsilon^g W_2(\xi, \eta, \tau) + \ldots \qquad (5)$$

Substituting expressions (4) and (5) into equation (1) and performing $\varepsilon$-splitting, one obtains a parameter $g$ from the conditions of correctness for the limiting systems and a system of equations that determines the unknown expansion coefficients.

$$D \nabla^4 W_1 - \lambda_1 \rho h W_1 = 0 \qquad (6)$$

$$D \nabla^4 W_1 - \lambda_1 \rho h W_2 = -C_1 W_2 - E_1 I\phi W_{1\xi\xi\xi\xi} + \lambda_1 \rho h W_1 + \lambda_1 \rho_1 F\phi W_2 \qquad (7)$$

Here $\nabla^4 = \nabla^2 \nabla^2 \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Equation (6) describes the smooth plate oscillation. The recurrent system of equations (7) allows one to obtain the frequency and mode expansions for any order of $\varepsilon$.

## 4 Two-Point Padé Approximants (TPPA)

First of all we must give some definitions. The notion of TPPA is defined by Baker and Graves-Morris (1995). Let

$$F(\varepsilon) = \sum_{i=0}^{\infty} a_i \varepsilon^i \qquad \text{when} \quad \varepsilon \to 0 \qquad (8)$$

$$F(\varepsilon) = \sum_{i=0}^{\infty} b_i \varepsilon^{-i} \qquad \text{when} \quad \varepsilon \to \infty \qquad (9)$$

The TPPA is represented by the function

$$F(\varepsilon) = \left(\sum_{i=0}^{l} a_k \varepsilon^k\right) / \left(\sum_{i=0}^{j} \beta_k \varepsilon^k\right)$$

in which $l$ coefficients of expansion in the Taylor series when $\varepsilon \to 0$, and $i + j + 2 - l$ coefficients of expansion in the Laurent series when $\varepsilon \to \infty$ coincides with the corresponding coefficients of the series (8) and (9).

Using the TPPA procedure one obtains on the basis of a low and a high frequency approach an analytical expression for the whole frequency spectrum.

## 5 Numerical Results

The values of the parameters used in the numerical analysis are

$$N = 11 \qquad \frac{C_1}{DL_1^4} = 0.1 \qquad \frac{E_1 F}{Db} = 200 \qquad \frac{p_1 F}{\rho b h} = 0.5 \qquad m = 1 \qquad 0 < k < 80$$

Here $m$ and $k$ are the wave numbers in the direction of the $x$ and $y$ axes.

The results are plotted in Figure 2 where the curves correspond to: 1 - the orthotropic construction oscillation frequency; 2 - the case of the smooth plates oscillation frequency; 3 and 4 - the truncated series for the low - and high - frequency asymptotics (only the first two terms of expansion are taken into account). The dotted line represents the exact values of the frequency, which are determined by numerical methods. Curve 5 corresponds to the matched spectrum expression.

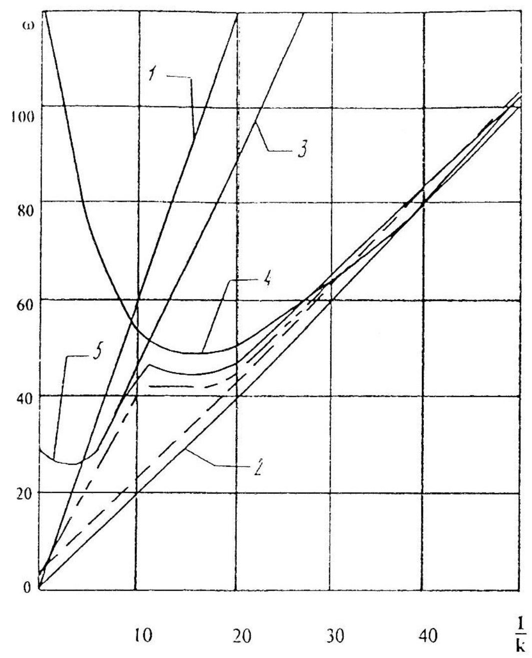


Figure 2. Comparison of Analytical and Numerical Results

First and foremost the plots in Figure 2 show that the values of the frequency obtained with approximate analytical and numerical methods are inside the region limited by the rays 1 and 2. This result is consistent with the physics of the problem and confirms the reliability of the solution. Furthermore, the comparison with the numerical analysis data shows that it is applicable in the interval $0 < k < 80$, the low frequency asymptotics and the expansions. Curve 5 coincides satisfactorily with the numerical solution everywhere in the interval discussed.

## 6 Concluding Remarks

Thus the method proposed makes it possible to obtain the closed analytical formula for the total spectrum of the base plate's natural frequencies.


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