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Stresses in Twisted Cylindrical Plugs of Non-homogeneous Material Inserted into a Rigid Medium

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The paper deals with an analytical solution for the state of a non-homogeneous short cylinder, having the lateral surface and one of its bases rigidity fixed, the cylinder being acted upon by tangential forced distributed on a circular part of the other end. The approaches illustrated for some functions of elastic modules and results of calculations on an exact and approximate analytical methods are compared.

1 Introduction

The problem of twisted non-homogeneous circular cylindrical plugs embedded into a rigid medium is considered. This is one of the fundamental subjects in the study of applied theory of elasticity. The mathematical tool is an analytical procedure developed using special functions. The last works in this direction (Dutt, 1961; Bhowmick, 1964; Eishinskii, 1976, 1982, 1986) have provided a few attractive results and directed our attention to an analytical solution of the problem. On this basis an approximate analytical WKB-solution for the general case of nonhomogenities of the system is proposed.

2 Governing Equations of Problem

Let r, θ, z be the cylindrical coordinates with the axis of the cylinder along the z-axis, the oringin being at the section twisted by shearing forces. The strain components are given e. g. by Bhowmick (1964) as

$$e_{\theta\theta} = e_{zz} = e_{rr} = 0$$
$$e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} \qquad e_{\theta z} = \frac{\partial v}{\partial z} \tag{1}$$

and the corresponding stresses are

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$$
$$\tau_{r\theta} = G(z)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \qquad \tau_{rz} = G(z)\frac{\partial v}{\partial z} \tag{2}$$

Two equations of equilibrium are identically satisfied and the third equation reduces to the form

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \tag{3}$$

Using relations (2) equation (3) becomes

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} + \frac{G'(z)}{G(z)}\frac{\partial v}{\partial z} = 0 \tag{4}$$

In order to separate the variables we suppose the solution of equation (4) to be of the form

$$v = R(r)\,Z(z) \tag{5}$$

Finally we obtain the following independent equations:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{1}{r^2}\right)R = 0 \tag{6}$$

$$\frac{d^2Z}{dz^2} + \frac{G'(z)}{G(z)}\frac{dZ}{dz} - k^2 Z = 0 \tag{7}$$

where k is the constant of separation.

The solution of the equation (6) has the known form

$$R = Const. \cdot J_1(k\,r) \tag{8}$$

Let the functions $Z_1\,(k,\,z)$ and $Z_2\,(k,\,z)$ be two independent solutions of equation (7), then the solution (5) is given by

$$\mathrm{v} = \left(A_k\, Z_1(k,\,z) + B_k\, Z_2(k,\,z)\right)J_1(k\,r) \tag{9}$$

The boundary conditions of the problem considered are

$$\mathrm{v} = 0 \qquad \text{at} \qquad r = a \qquad z = h \tag{10}$$

$$\left[G(z)\frac{\partial \mathrm{v}}{\partial z}\right]_{z=0} = \begin{cases} F(r) & 0 < r < b \\ 0 & b < r < a \end{cases}$$

From the first boundary condition there follows the equation

$$J_1(k\,a) = 0 \tag{11}$$

with roots $k_1,\, k_2,\, \ldots,\, k_n,\, \ldots$
The second boundary condition for all values of k leads to the correlation

$$B_k = -A_k\frac{Z_1(k,\,h)}{Z_2(k,\,h)} \tag{12}$$

As a general solution of the problem we put

$$\mathrm{v} = \sum_{n=1}^{\infty} A_{k_n}\left(Z_1(k_{n},{}_z) - \frac{Z_1(k_n,h)}{Z_2(k_n,h)}Z_2(k_n,\,z)\right)J_1(k_n z) \tag{13}$$

The corresponding stress function is

$$\left[\tau_{nz}\right]_{z=0} = \left[G(z)\frac{\partial \mathrm{v}}{\partial z}\right]_{z=0} =$$
$$= G(0)\sum_{n=1}^{\infty} A_{k_n}\left[\frac{\partial Z_1(k_n\,0)}{\partial z} - \frac{Z_1(k_n\,h)}{Z_2(k_n\,h)}\frac{\partial Z_2(k_n\,0)}{\partial z}\right]J_1(k_n z) = F(r) \qquad 0 < r < b \tag{14}$$

3 Specific Solutions of the Problem

For practical purposes we will suppose some specific functions of stress distribution.

i) Linear function of stress distribution over a circular area

$$F(r) = Sr \qquad 0 < r < b \qquad F(0) = 0 \qquad b < r < a \tag{15}$$

For this case the coefficients A_{k_n} and B_{k_n} can be given as

$$A_{k_n} = \frac{2\int_0^b r\,F(r)\,J_1(k_n r)dr}{G(0)b^2\left[J_1'(k_n b)\right]^2\left[\frac{\partial Z_1(k_n 0)}{\partial z} - \frac{Z_1(k_n h)}{Z_2(k_n h)}\frac{\partial Z_2(k_n 0)}{\partial z}\right]}$$

$$= B_{k_n}\int_0^b r\,F(r)\,J_1(k_n r)\,dr$$

$$B_{kn} = \frac{2}{G(0)b^2\left[J_1'(k_n b)\right]^2\left[\frac{\partial Z_1(k_n 0)}{\partial z} - \frac{Z_1(k_n h)}{Z_2(k_n h)}\frac{\partial Z_2(k_n 0)}{\partial z}\right]} \tag{16}$$

$$A_{k_n} = B_{k_n} S\int_0^b r^2\,J_1(k_n r)\,dr = B_{kn}\,S\frac{b^2}{k_n}J_2(k_n b)$$

where J_2 is the Bessel function of the first kind and second order.

The stress functions for this specific case can be presented in the form

$$\tau_{\theta z} = \frac{2SG(z)}{G(0)}\sum_{n=1}^{\infty}\frac{J_2(k_n b)\,J_1(k_n r)\left[\frac{\partial Z_1(k_n z)}{\partial z} - \frac{Z_1(k_n h)}{Z_2(k_n h)}\frac{\partial Z_2(k_n z)}{\partial z}\right]}{k_n\left[J_1'(k_n b)\right]^2\left[\frac{\partial Z_1(k_n 0)}{\partial z} - \frac{Z_1(k_n h)}{k_n h}\frac{\partial Z_2(k_n 0)}{\partial z}\right]} \tag{17}$$

$$\tau_{r\theta} = -\frac{2SG(z)}{G(0)}\sum_{n=1}^{\infty}\frac{J_2(k_n b)J_2(k\,r)\left[Z_1(k_n,z) - \frac{Z_1(k_n,h)}{Z_2(k_n,h)}Z_2(k_n,z)\right]}{k_n\left[J_1'(k_n b)\right]^2\left[\frac{\partial Z_1(k_n,0)}{\partial Z} - \frac{Z_1(k_n,h)}{Z_2(k_n,h)}\frac{\partial Z_2(k_n,0)}{\partial z}\right]}$$

Finally at $r = a$ the stresses are

$$\left[\tau_{r\theta}\right]_{r=a} = -\frac{2SG(z)}{G(0)}\sum_{n=1}^{\infty}\frac{J_2(k_n b)J_2(k_n a)\left[Z_1(k_n,z) - \frac{Z_1(k_n,h)}{Z_2(k_n,h)}Z_2(k_n,z)\right]}{k_n\left[J_1'(k_n,b)\right]^2\left[\frac{\partial Z(k_n,0)}{\partial z} - \frac{Z_1(k_n,h)}{Z_2(k_n,h)}\frac{\partial Z_2(k_n,0)}{\partial z}\right]} \tag{18}$$

ii) Constant distribution of stresses over a circular area

$$F(r) = \begin{cases} S, & 0 < r < b \\ 0, & b < r < a \end{cases} \tag{19}$$

For this specific case we obtain the following relations:

$$A_{k_n} = B_{k_n} S \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{k_n}{2} \right)^{2m+1} \frac{b^{2m-3}}{2m+3} \tag{20}$$

$$\tau_{\theta z} = G(z) \sum_{n=1}^{\infty} A_{k_n} \left[\frac{\partial Z_1(k_n, z)}{\partial z} - \frac{Z_1(k_n, h)}{Z_2(k_n, h)} \frac{\partial Z_2(k_n, z)}{\partial z} \right] J_1(k_n r) \tag{21}$$

$$\left[\tau_{\theta z} \right]_{r=a} = G(z) \sum_{n=1}^{\infty} A_{k_n} \left[\frac{\partial Z_1(k_n, z)}{\partial z} - \frac{Z_1(k_n, h)}{Z_2(k_n, h)} \frac{\partial Z_2(k_n, z)}{\partial z} \right] J_1(k_n a) \tag{22}$$

4 The Solution for an Exponential Function of Modulus $G(z)$

Now we assume the function $G(z)$ of the problem in the form

$$G(z) = G_0 \exp\left(Az + \frac{B}{\nu} e^{\nu z} \right) \tag{23}$$

where A, B, $\nu \neq 0$ are arbitrary constants. For this case equation (7) becomes

$$\frac{d^2 Z}{dz^2} + \left(A + B e^{\nu z} \right) \frac{dZ}{dz} - k^2 Z = 0 \tag{24}$$

The solution of equation (24) we represent as

$$Z = Z_1(z) \exp\left(-\frac{Az}{2} - \frac{B}{2\nu} e^{\nu z} \right) \tag{25}$$

From the substitution of equation (25) into equation (24) we obtain

$$\frac{d^2 Z_1}{dz^2} + \left[-\left(k^2 + \frac{A^2}{4} \right) - \frac{B(\nu + A)}{2} e^{\nu z} - \frac{B^2}{4} e^{2\nu z} \right] Z_1 = 0 \tag{26}$$

Taking into account the correlation

$$z_1 = \nu z \tag{27}$$

the governing differential equation (26) becomes

$$\frac{d^2 Z_1}{dz_1^2} + \left[-\frac{k^2 + \frac{A^2}{4}}{\nu^2} - \frac{B(\nu + A)}{2\nu^2} e^{\nu z_1} - \frac{B^2}{4\nu^2} e^{2\nu z_1} \right] Z_1 = 0 \tag{28}$$

and with the relation

$$S = e^{Z_1} \tag{29}$$

equation (28) can be written as

$$S^2 \frac{d^2 Z_1}{dS^2} + S\frac{\partial Z_1}{\partial S} + \left[-\frac{k^2 + \frac{A^2}{4}}{\nu^2} - \frac{B(\nu + A)}{2\nu^2} - \frac{B^2 S^2}{4\nu^2} \right] Z_1 = 0 \tag{30}$$

We introduce a new variable

$$Z_1 = Z_2 \, S^{-1/2} \tag{31}$$

and equation (30) transforms into

$$\frac{d^2 Z_2}{dS^2} + \left[-\frac{B^2}{4\nu^2} - \frac{B(\nu + A)}{2\nu^2 S} + \frac{\frac{1}{4} - \frac{k^2 + \frac{A^2}{4}}{\nu^2}}{S^2} \right] Z_2 = 0 \tag{32}$$

A new variable

$$S_1 = \left| \frac{B}{\nu} \right| S \tag{33}$$

gives the possibility to obtain the differential equation of the problem in the well-known form

$$\frac{d^2 Z_2}{dS_1^2} + \left[-\frac{1}{4} - \frac{|B|(\nu + A)}{2|\nu| B \, S_1} + \frac{\frac{1}{4} - \frac{k^2 + \frac{A^2}{4}}{\nu^2}}{S_1^2} \right] Z_2 = 0 \tag{34}$$

The differential equation (34) is the Whittaker's equation (see Gristchak and Dmitrijeva, 1995)

$$\frac{d^2 Z_2}{dS_1^2} - \left[\frac{1}{4} - \frac{\tilde{k}}{S_1} + \frac{\frac{1}{4} - \mu^2}{S_1^2} \right] Z_2 = 0 \tag{35}$$

where $$\mu^2 = \frac{k^2 + \frac{A^2}{4}}{\nu^2} \qquad \tilde{k} = \frac{|B|(\nu + A)}{2|\nu| B} \tag{36}$$

whose solution is

$$Z = \exp\left[z\left(\nu\sqrt{\frac{k^2+\frac{A^2}{4}}{\nu^2}} - \frac{A}{2}\right) - \frac{1}{2}e^{\nu z}\left(\frac{B}{\nu} + \left|\frac{B}{\nu}\right|\right)\right]$$

$$\times\left[\tilde{C}_1\phi_k\left(\sqrt{\frac{k^2+\frac{A^2}{4}}{\nu^2}} + \frac{(\nu+A)|B|}{2|\nu|B} + \frac{1}{2};\ 2\sqrt{\frac{k^2+\frac{A^2}{4}}{\nu^2}} + 1;\ \frac{1}{2}\left|\frac{B}{\nu}\right|e^{\nu z}\right)\right. \tag{37}$$

$$\left. + \tilde{C}_2\psi_k\left(\sqrt{\frac{k^2+\frac{A^2}{4}}{\nu^2}} + \frac{(\nu+A)|B|}{2|\nu|B} + \frac{1}{2};\ 2\sqrt{\frac{k^2+\frac{A^2}{4}}{\nu^2}} + 1;\ \frac{1}{2}\left|\frac{B}{\nu}\right|e^{\nu z}\right)\right]$$

where $\tilde{C}_1$, $\tilde{C}_2$ are arbitrary constants of integration and the functions ϕ_k, ψ_k are (see Eishinskii, 1986)

$$\begin{aligned}\phi_k(a;\ c;\ z) &= 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\ldots, \\ \psi_k(a;\ c;\ z) &= \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\phi_k(a;\ c;\ z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\phi_k(a - c + 1;\ 2 - c;\ z)\end{aligned} \tag{38}$$

where Γ is Euler's gamma function (Jahnke et al., 1968).

5 Approximate WKB-solution of the Problem

In this section an approximate asymptotic solution of the problem is proposed on the basis of the differential equation (7) with variable coefficient.

$$\frac{d^2Z}{dz^2} + Q(z)\frac{dZ}{dz} - k^2Z = 0 \tag{39}$$

where

$$Q(z) = \frac{G'(z)}{G(z)} \tag{40}$$

In order to exclude the first derivative in equation (39) we introduce a transformation (see Moiseev, 1969) of the type

$$Z = u(z)\exp\int^z\left[-\frac{Q(\zeta)}{2}\right]d\zeta \tag{41}$$

After the substitution of expression (41) into equation (39) we obtain

$$\varepsilon^2\frac{d^2u}{dz^2} - \left(\frac{Q^2}{4k^2} + \frac{Q'}{2k^2} + 1\right)u = 0 \tag{42}$$

where $$\varepsilon^2 = k^{-2} \tag{43}$$

For large values of k $(k > 1)$ the WKB-solution of equation (42) can be presented as a series expansion

$$u(z) = \exp \int^z [\varphi_0 k + \varphi_1 k^0 + \varphi_2 k^{-1} + \ldots] d\zeta \tag{44}$$

Substituting presentation (44) into equation (42) and comparing the coefficients with the same order of k one-term approximation leads to

$$(\varphi_0)_{1,2} = \pm \left(\frac{Q^2}{4k^2} + \frac{Q'}{2k^2} \right)^{1/2} \tag{45}$$

The WKB-solution of equation (42) is given by

$$u(z) = c_1 \sinh[\alpha(z)] + c_2 \cosh[\alpha(z)] \tag{46}$$

where c_1, c_2 are arbitrary constants and

$$\alpha(z) = \int^z \left(\frac{Q^2}{4k^2} + \frac{Q'}{2k^2} + 1 \right)^{1/2} k d\zeta \cong \int^z \left[k + \left(\frac{Q^2}{8k} + \frac{Q'}{4k} \right) \right] d\zeta \left| \frac{1}{k^2} < 1 \right. \tag{47}$$

Finally an approximate analytical solution of the equation (7) can be written as

$$Z = \exp \int^z \left[-\frac{Q(z)}{2} \right] d\zeta \{ c_1 \sinh[\alpha(z)] + c_2 \cosh[\alpha(z)] \} \tag{48}$$

6 Numerical Calculations and Comparison of One-Term WKB-Approximation with Solution

For the specific coefficients $A = -1$, $B = -1$, $\nu = 2$, $G_0 = 1$ in an exponential function (23) we have

$$G(z) = \exp\left[-\left(z + \frac{e^{2z}}{2} \right) \right] \tag{49}$$

In this case the WKB-solution of equation (7) has the form

$$\begin{aligned} Z = \exp\left[\frac{1}{4}\exp(2z) + \frac{z}{2} \right] \Bigg[& c_1 \sinh\left\{ \left(-0.125\exp(2z) + \frac{\exp(4z)}{32} + \frac{(8+k^2)z}{8k^2} \right\} \right. \\ & \left. + c_2 \cosh\left\{ \left(-0.125\exp(2z) + \frac{\exp(4z)}{32} + \frac{(8+k^2)z}{8k^2} \right\} \right] \end{aligned} \tag{50}$$

Using, for example, the boundary conditions

$$Z(0) = 0 \qquad\qquad Z(1) = 1 \tag{51}$$

at $k = 10$, the constants c_1, c_2 are

$$c_1 = 0.0805304358350573876 \qquad c_2 = 0.00752768749889221414 \tag{52}$$

The coefficients in the solution (37) for function (49) and for given boundary conditions (51) are

$$a = 6.25625 \qquad c = 11.0125 \qquad \tilde{c}_1 = 0.356 \cdot 10^{-4} \qquad \tilde{c}_2 = -0.15 \cdot 10^{-14} \tag{53}$$

The results of numerical calculations with using "Mathematica"(see Wolfram, 1988) are given in Figures 1-4.

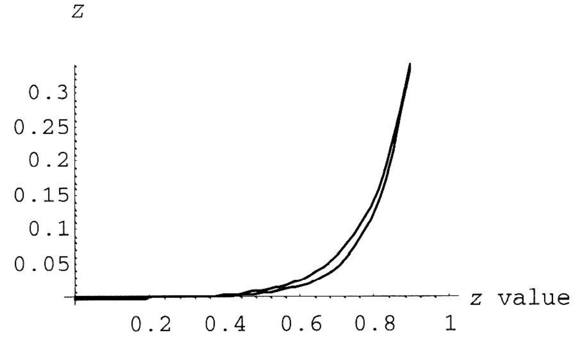

Figure 1. Comparison of One-term WKB-approximation (48) and Solution (37)

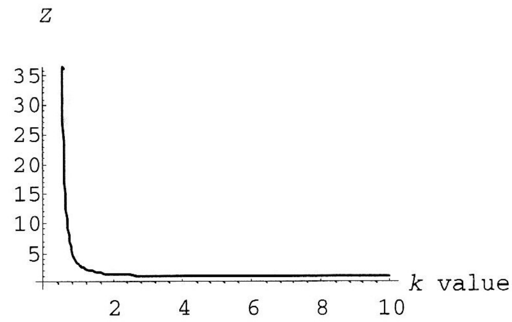

Figure 2. The Dependence of the WKB-solution upon the k value

$0 \le z \le 1 \qquad 1 \le k \le 10$

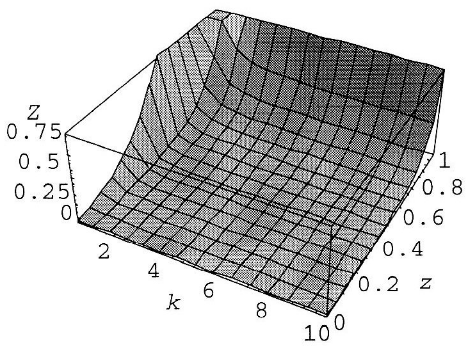

Figure 3. Surface $Z = Z(z, k)$

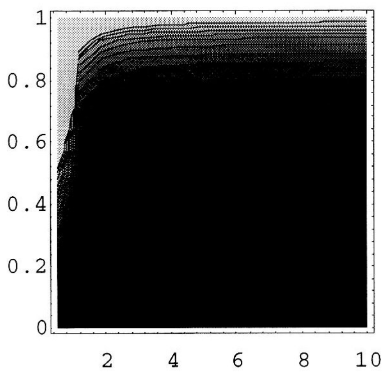

Figure 4. Contour Plot $Z = Z(z, k)$

7 Concluding Remark

A solution for the stress problem in twisted cylindrical plugs of non homogeneous material inserted into rigid medium is proposed. It is shown, for example, that for an exponential function of shear modules of elasticity the solution of problem can be presented through the modfied Whittaker's functions. For the general case of variable parameters of the system, which is discussed in this paper, an approximate asymptotic WKB-solution of problem is given as well. The comparison of exact and approximate solutions for specific function of shear modules showed enough good agreement between numerical results, especially for the parameters $k > 1$. It should be observed here that a hybrid WKB-Galerkin method proposed in Heding (1965) will effectively approximate an analytical approach for the solution of mechanical problems for nonhomogeneous media for "large" as well for "small" parameters of k .

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