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Free and Parametric Vibrations of Cylindrical Shells under Static and Periodic Axial Loads

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This study examines free and parametric vibrations of cylindrical shells subjected to axial loads. In the case of free vibrations, the loading is static. Parametric vibrations are excited by a combined load consisting of one static and two periodic load components. The load is assumed to be nonuniform in the circumferential direction and the shell is noncircular so that vibrations are concentrated near the "weakest" generatrix on the shell surface. By using Tovstik's asymptotic method in combination with the multiple scale method with respect to time, the solutions of the governing equations are found in the form of functions localized near the "weakest" line and growing with time in the case of dynamic instability. The dependence of fundamental frequencies upon the static nonuniform axial force is studied. For the weak parametric excitation, the region of instability of a cylinder is determined directly in terms of its geometry, load intensity, and frequency.

1 Introduction

Vibrations of cylindrical shells under initial static and dynamic axial loads are of great practical interest. The influence of static axial force (as well as of other kinds of loading) on the shell vibration has been examined by Herrmann and Armenakas (1962). Penzes and Kraus (1972). Their studies have shown that accounting of the static axial load, not exceeding the static buckling load, may lead to a significant lowering of fundamental frequencies. Various combinations of the static load and periodic loads have been considered by Yao(1963), Wenzke (1963), Vijayaraghavan and Evan-Iwanowski (1967), and Grundman (1970). In particular, they have investigated the problem of dynamic instability of shells subjected to periodic axial load, and established the regions of instability and stability.

In all the investigations mentioned, shells have been assumed to have constant parameters, and loads have been uniform. The general goal of the present paper is to study free vibrations and the parametric instability of noncircular cylindrical shells which experience static and additional periodic axial loads being nonuniform in the circumferential direction. It is assumed that both the nonhomogeneity of loading and the curvature variability cause the localization of oscillations in a neighborhood of the generatrix which is the "weakest" one. For the first time, similar localization of modes was examined by Tovstik (1983). Afterwards, the approach developed by Tovstik has been used for studying local low-frequency vibrations of elastic and viscoelastic cylindrical shells with slanted edges (Filippov, 1993; Mikhasev, 1992a) and variable thickness (Mikhasev, 1992b), and nonuniformly heated viscoelastic cylindrical shells (Botogova and Mikhasev, 1996) as well. The method of multiple scales has been used by Mikhasev and Kuntsevich (1997) to study local low-frequency thermoparametric vibrations of a noncircular cylinder in a nonstationary temperature field. In the present article, Tovstik's asymptotic method in combination with the multiple scale method over time will be applied for analyzing free and parametric vibrations of a cylinder in the vicinity of the "weakest" generatrix on the shell surface.

2 Basic Equations

The cylindrical shell is assumed to be elastic, isotropic and sufficiently thin for applicability of both the assumptions of the classical shell theory and the asymptotic methods. We introduce an orthogonal coordinate system s, φ, where $s = xR^{-1}$, x is a point coordinate on the generatrix, R is the characteristic size of the middle surface (it will be defined below), φ is a circular coordinate on the shell surface so that the first quadratic form of the middle surface takes the form $R^2\left(ds^2 + d\varphi^2\right)$. In this case the curvature radius is

$R_2 = R\chi^{-1}(\varphi)$. It is assumed that $0 \le s \le l = L/R$, $0 \le \varphi \le \varphi_1$, where L is the shell length. The shell may be not closed in the circumferential direction ($\varphi_1 < 2\pi$); in this case, one has the noncircular cylindrical panel.

Let the shell be under the combined nonuniform axial load

$$F^* = Eh\left\{F_0(\varphi) + \mu\left[F_1(\varphi)\sin\Omega^* t^* + F_2(\varphi)\cos\Omega^* t^*\right]\right\} \tag{1}$$

where E and h are Young's modules and the shell thickness, respectively, $\mu^4 = h^2 / \left[12 R^2 (1-\nu^2)\right]$ is a small parameter, ν is Poisson's ratio, t^* is time, Ω^* is the frequency of the additional periodic axial load. If $F_1^2 + F_2^2 = 0$ for any φ, free vibrations under the initial static load will be considered. In the case $F_1^2 + F_2^2 \ne 0$, where F_1, $F_2 \sim 1$ as $\mu \to 0$, parametric oscillations and instability caused by the small additional periodic forces will be studied below.

We will analyze vibrations which are characterized by large quantity of waves in the both directions on the shell surface. Then the following equations (Bolotin, 1956; Vlasov, 1958; Tovstik, 1995)

$$\mu^4 \Delta^2 W - \mu^2 \chi(\varphi)\frac{\partial^2 \Phi}{\partial s^2} + \mu^2 F(\varphi, t)\frac{\partial^2 W}{\partial s^2} + \frac{\partial^2 W}{\partial t^2} = 0$$

$$\mu^2 \Delta^2 \Phi + \chi(\varphi)\frac{\partial^2 W}{\partial s^2} = 0 \tag{2}$$

written in nondimensional form can be used in our investigation. Here $\Delta = \partial^2/\partial s^2 + \partial^2/\partial\varphi^2$ and the dimensionless magnitudes are introduced as follows:

$$W = CW^*/R \qquad \Phi = C\Phi^* / \left(\mu^2 EhR^2\right) \qquad F = F^* / \left(\mu^{-2} Eh\right)$$

$$t = t^*/t_c \qquad t_c = \sqrt{\rho R^2/E}$$

$$F = F_0(\varphi) + \mu\left[F_1(\varphi)\sin\Omega t + F_2(\varphi)\cos\Omega t\right] \qquad \Omega = \Omega^* t_c \tag{3}$$

where W^* and Φ^* are the normal deflection and the stress function, respectively, ρ is the mass density, t_c is the characteristic time, and C is an arbitrary constant. The functions $\chi(\varphi)$, $F_j(\varphi)$ are supposed to be infinitely differentiable. It is assumed that $F(\varphi, t) < F_b$, where F_b corresponds to the classical value of the static axial buckling load (Timoshenko, 1936).

Let the shell edges be joint supported so that at the edges $s = 0$ and $s = l$ Navier's conditions

$$W = \frac{\partial^2 W}{\partial s^2} = \Phi = \frac{\partial^2 \Phi}{\partial s^2} = 0 \tag{4}$$

are realized. To satisfy boundary conditions (4) the solutions of equations (2) are assumed to be of the form

$$W = w_m(\varphi, t)\sin\left(\mu^{-1} p_m s\right) \qquad \Phi = f_m(\varphi, t)\sin\left(\mu^{-1} p_m s\right) \tag{5}$$

$$p_m = \mu m\pi / l \qquad m = 1, 2, \ldots$$

Substituting (5) into (2) yields the sequence of equations

$$\mu^4 \frac{\partial^4 w_m}{\partial \varphi^4} - 2\mu^2 p_m^2 \frac{\partial^2 w_m}{\partial \varphi^2} + p_m^2 \chi(\varphi) f_m + \left[p_m^4 - p_m^2 F(\varphi, t)\right] w_m + \frac{\partial^2 w_m}{\partial t^2} = 0$$

$$\mu^4 \frac{\partial^4 f_m}{\partial \varphi^4} - 2\mu^2 p_m^2 \frac{\partial^2 f_m}{\partial \varphi^2} + p_m^4 f_m - p_m^2 \chi(\varphi) w_m = 0 \qquad (6)$$

with respect to w_m, f_m for $m = 1, 2, \dots$. The subscript m in p_m, w_m, f_m will be omitted below.

3 The Approach

Taking into account both the inhomogeneity of the axial load and the curvature-variability we suppose that vibrations are concentrated near some generatrix $\varphi = \varphi_0$ which will be defined below. The uniformly valid asymptotic solution of equation (6) in a neighborhood of the line $\varphi = \varphi_0$ may be constructed in the form of WKB-functions (Tovstik, 1983) with amplitudes growing in time (Mikhasev and Kuntsevich, 1997)

$$w(\varphi, t, \mu) \cong \sum_{k=0}^{\infty} \mu^{k/2} w_k(\xi, t_0, t_1) \exp\left\{ i\left[\mu^{-1/2} q \xi + \frac{1}{2} b \xi^2 \right] \right\} \qquad (7)$$

$$f(\varphi, t, \mu) \cong \sum_{k=0}^{\infty} \mu^{k/2} f_k(\xi, t_0, t_1) \exp\left\{ i\left[\mu^{-1/2} q \xi + \frac{1}{2} b \xi^2 \right] \right\} \qquad (8)$$

$$\xi = \mu^{-1/2} (\varphi - \varphi_0) \qquad t_0 = t \qquad t_1 = \mu t$$

where $\mathrm{Im}\, b > 0$, w_k , f_k are polynomials in ξ. Time dilatation (8) is performed here to find the amplitudes of parametric vibrations as the functions of "slow time" t_j (Nayfeh, 1973).

The functions $\chi(\varphi)$ and $F_j(\varphi)$ are expanded into series in the neighborhood of the generatrix $\varphi = \varphi_0$. For example,

$$F_0(\varphi) = F_0(\varphi_0) + \mu^{1/2} F_0'(\varphi_0)\xi + \frac{1}{2}\mu F_0''(\varphi_0)\xi^2 + \dots \qquad (9)$$

Substituting equations (7) and (9) into equation (6), and eliminating the functions f_k , produces the sequence of equations

$$\sum_{j=0}^{k} D_j w_{k-j} = 0 \qquad k = 0, 1, 2, \dots \qquad (10)$$

where

$$D_0 = \frac{\partial^2}{\partial t_0^2} + H(p, q, \varphi_0) \qquad (11)$$

$$H(p,q,\varphi_0)=\left(p^2+q^2\right)^2+\frac{\chi^2(\varphi_0)p^4}{\left(p^2+q^2\right)^2}+F_0(\varphi_0)p^2 \tag{12}$$

$$D_1=\left(b\frac{\partial H}{\partial q}+\frac{\partial H}{\partial \varphi_0}\right)\xi-\mathrm{i}\frac{\partial H}{\partial q}\frac{\partial}{\partial \xi}$$

$$D_2=\frac{1}{2}\left(b^2\frac{\partial^2 H}{\partial q^2}+2b\frac{\partial^2 H}{\partial q\partial\varphi_0}+\frac{\partial^2 H}{\partial\varphi_0^2}\right)\xi^2-\mathrm{i}\left(b\frac{\partial^2 H}{\partial q^2}+\frac{\partial^2 H}{\partial q\partial\varphi_0}\right)\xi\frac{\partial}{\partial\xi}-\frac{1}{2}\frac{\partial^2 H}{\partial q^2}\left(\mathrm{i}b+\frac{\partial^2}{\partial\xi^2}\right) \tag{13}$$

$$-\frac{\mathrm{i}}{2}\frac{\partial^2 H}{\partial q\partial\varphi_0}+2\frac{\partial^2}{\partial t_0\partial t_1}+N,\ \ldots$$

$$N=-p^2\left[F_1(\varphi_0)\sin\Omega t+F_2(\varphi_0)\cos\Omega t\right] \tag{14}$$

3.1 Zeroth and First Order Approximations

In the zeroth order approximation ($k=0$) differential equation (10) is a homogeneous one, which has the solution

$$w_0(\xi,t_0,t_1)=w_{0,c}(\xi,t_1)\cos\omega t_0+w_{0,s}(\xi,t_1)\sin\omega t_0 \tag{15}$$

Here $w_{0,c}$ and $w_{0,s}$ are unknown polynomials in ξ with coefficients being functions of "slow time" t_1, and frequency ω and wave number q satisfy the equation

$$\omega^2=H(p,q,\varphi_0) \tag{16}$$

For $k=1$ in equation (10), one has a nonhomogeneous differential equation. The right part of this equation, with equation (16) in mind, generates secular terms with respect to t_0. The absence conditions of these terms are

$$2\chi'(\varphi_0)-F_0'(\varphi_0)=0 \qquad q^2=\chi^{1/2}(\varphi_0)p-p^2 \tag{17}$$

or

$$2\chi(\varphi_0)\chi'(\varphi_0)-p^2F_0'(\varphi_0)=0 \qquad q=0 \tag{18}$$

Then

$$\omega=p\sqrt{2\chi(\varphi_0)-F_0(\varphi_0)} \tag{19}$$

or

$$\omega=\sqrt{p^4-F_0(\varphi_0)p^2+\chi^2(\varphi_0)} \tag{20}$$

for cases (17) and (18), respectively. Let $\varphi_0 = \varphi_0^\circ$, $q = q^\circ$, $\omega = \omega^\circ$ satisfy equations (17) and (19) or (18) and (20). Then $D_1 w_1 \equiv 0$ and equation (10) for k = 1 admits the solution in the form (15) again, with the subscripts (0, c) and (0, s) being changed to (1, c) and (1, s), respectively.

It may be seen from equations (19) and (20) that $F_0 < F_b$, where $F_b = 2\chi(\varphi_0^\circ)$ in case (17) or $F_b = p^2 + \chi^2(\varphi_0^\circ)p^{-2}$ in case (18). The generatrix $\varphi = \varphi_0^\circ$ is called the "weakest" one (Tovstik, 1995) because under $F_0 = F_b$ the shell buckling takes place in the neighborhood of this line. It is here assumed $R = R_2(\varphi_0^\circ)$ as the characteristic size of the middle surface of a cylinder so that $\chi(\varphi_0^\circ) = 1$.

3.2 Second Order Approximation

Consider herein the case when $\Omega \approx 2\omega^\circ$. It is assumed that

$$\Omega = 2\omega^\circ + \mu\sigma \qquad \sigma \sim 1 \quad \text{as} \quad \mu \to 0 \tag{21}$$

where σ is a detuning parameter (Nayfeh, 1973) for the frequency Ω of the additional periodic axial load.

For k = 2, differential equation (10) is nonhomogeneous again. It is simplified if one notes conditions (17) - (20) giving $D_1 w_1^\circ \equiv 0$ (here and below superscript $()^\circ$ means that all calculations are carried out for $\varphi_0 = \varphi_0^\circ$, $q = q^\circ$, $\omega = \omega^\circ$). The right part in equation (10) for k = 2 generates secular terms. The absence conditions of these terms, taking into account equation (21), gives the system of differential equations with respect to $\mathbf{X} = \left(w_{0,s}, w_{0,c}\right)^{\mathrm{T}}$.

$$-\frac{1}{2}\frac{\partial^2 H^\circ}{\partial q^2}\frac{\partial^2 \mathbf{X}}{\partial \xi^2} + a\xi\frac{\partial \mathbf{X}}{\partial \xi} + \left(c\xi^2 + \frac{1}{2}a\right)\mathbf{X} - \mathbf{G}\mathbf{X} - 2\omega^\circ \mathbf{E}_{-1}\frac{\partial \mathbf{X}}{\partial t_1} = 0 \tag{22}$$

where

$$a = -\mathrm{i}\left(b\frac{\partial^2 H^\circ}{\partial q^2} + \frac{\partial^2 H^\circ}{\partial q \partial \varphi_0}\right) \qquad c = \frac{1}{2}\left(b^2\frac{\partial^2 H^\circ}{\partial q^2} + 2b\frac{\partial^2 H^\circ}{\partial q \partial \varphi} + \frac{\partial^2 H^\circ}{\partial \varphi_0^2}\right) \tag{23}$$

$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \mathbf{E}_{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$g_{21} = g_{12} \qquad g_{22} = -g_{11} \qquad F_j^\circ = F_j(\varphi_0^\circ)$$

$$g_{11} = -\frac{1}{2}p^2\left(F_1^\circ \sin\sigma t_1 + F_2^\circ \cos\sigma t_1\right) \qquad g_{12} = \frac{1}{2}p^2\left(F_1^\circ \cos\sigma t_1 - F_2^\circ \sin\sigma t_1\right)$$

Equation (22) has a solution in polynomial form in ξ if c = 0. Hence

$$b = \frac{1}{8}p\left\{F_0' + \mathrm{i}\left[4\left(\chi'^2 + \chi''\right) - \left(F_0'^2 + 2F_0''\right)\right]^{1/2}\right\}\left(p - p^2\right)^{-1/2}\Bigg|_{\varphi_0 = \varphi_0^\circ} \tag{24}$$

or

$$b = \mathrm{i}\frac{p}{2}\left[2\left(\chi'^2+\chi''\right)-p^2F_0''\right]^{1/2}\left(p^4-1\right)^{-1/2}\Big|_{\varphi_0=\varphi_0^\circ} \tag{25}$$

for φ_0° satisfying equation (17) or (18), respectively. One can see that Im $b > 0$ if

$$p<1 \qquad 4\left(\chi'^2+\chi''\right)-F_0'^2-2F_0''>0 \tag{26}$$

or

$$p>1 \qquad 2\left(\chi'^2+\chi''\right)-p^2F_0''>0 \tag{27}$$

Therefore, in the case $p < 1$, the "weakest" generatrix $\varphi = \varphi_0^\circ$ is found from conditions (17) and (26), and for $p > 1$ it is determined from conditions (18) and (27). It should be noted that the inequality $p < 1$ corresponds to a long cylinder. The case $p = 1$ is not considered here. The following particular cases may be picked out: (A) for $\chi \equiv 1$ and $F_0 = F_0(\varphi)$ (the circular cylinder under the nonuniform axial load) vibrations are localized near the line where $F_0(\varphi)$ is maximum; (B) for $\chi = \chi(\varphi)$ and a constant F_0 (the noncircular cylinder subjected to a uniform force) oscillations are concentrated near the generatrix along which the curvature $\chi(\varphi)$ is minimum.

Taking into account equation (24) or (25), equation (22) has the solution

$$\mathbf{X} = \sum_{k=0}^{n}\mathbf{Y}_k\,\xi^k \qquad \mathbf{Y}_k = (S_k, C_k)^{\mathrm{T}} \tag{28}$$

where $S_k(t_1)$ and $C_k(t_1)$ are functions satisfying the following system of differential equations with periodic coefficients:

$$\dot{\mathbf{Y}}_k - \mathbf{A}_k\mathbf{Y}_k = \mathbf{B}_k\mathbf{Y}_{k+2} \qquad k = n, n-1, \ldots, 0 \tag{29}$$

Here

$$\mathbf{B}_n = \mathbf{B}_{n-1} = 0 \qquad \mathbf{B}_k = \frac{(1+k)(2+k)}{4\omega^\circ}\frac{\partial^2 H^\circ}{\partial q^2}\mathbf{E}_{-1} \qquad \text{for } k = n-2, \ldots, 0$$

$$\mathbf{A}_k(t_1) = \begin{pmatrix} -a_1\sin(\sigma t_1-\theta) & a_1\cos(\sigma t_1-\theta)-a_{2,k} \\ a_1\cos(\sigma t_1-\theta)+a_{2,k} & a_1\sin(\sigma t_1-\theta)\end{pmatrix} \tag{30}$$

$$a_1 = \frac{p^2\sqrt{\left(F_1^\circ\right)^2+\left(F_2^\circ\right)^2}}{4\omega^\circ} \qquad a_{2.k} = -\frac{\mathrm{i}(1+2k)}{4\omega^\circ}\left(b\frac{\partial^2 H^\circ}{\partial q^2}+\frac{\partial^2 H^\circ}{\partial q\,\partial\varphi_0}\right)$$

$$\sin\theta = \frac{F_1^\circ}{\sqrt{\left(F_1^\circ\right)^2+\left(F_2^\circ\right)^2}} \qquad \cos\theta = \frac{F_2^\circ}{\sqrt{\left(F_1^\circ\right)^2+\left(F_2^\circ\right)^2}}$$

The procedure of seeking functions w_k may be continued indefinitely.

4 Free Vibration

If the additional periodic axial forces are absent ($F_1^2+F_2^2=0$ for any φ), then system (22) admits the following solution:

$$w_{0,s}(\xi,t_1) = H_n(\xi)\left(c_1\cos\omega_1^{(n)}t_1+c_2\sin\omega_1^{(n)}t_1\right)$$

$$w_{0,c}(\xi,t_1) = H_n(\xi)\left(c_2\cos\omega_1^{(n)}t_1-c_1\sin\omega_1^{(n)}t_1\right) \tag{31}$$

where $H_n(\xi)$ is Hermite's polynomial of the nth degree, c_j are arbitrary constants, and

$$\omega_1^{(n)} = \frac{1}{2\omega^\circ}\left(\frac{1}{2}+n\right)a \tag{32}$$

Then

$$W^* = CR\sin\frac{\pi m x}{L}\exp\left\{i\mu^{-1}\left[q^\circ\left(\varphi-\varphi_0^\circ\right)+\frac{1}{2}b\left(\varphi-\varphi_0^\circ\right)^2\right]\right\}$$
$$\times\left\{H_n\left[\mu^{-1/2}\left(\varphi-\varphi_0^\circ\right)\right]\left[c_1\cos\left(\omega^* t^*\right)+c_2\sin\left(\omega^* t^*\right)\right]+O\left(\mu^{1/2}\right)\right\} \tag{33}$$

is the mode of free vibrations with the fundamental frequency

$$\omega^* = \sqrt{\frac{E}{\rho R^2}}\left[\omega^\circ+\mu\,\omega_1^{(n)}+O\left(\mu^2\right)\right] \tag{34}$$

where, in particular for a noncircular cylinder under the constant force F_0 (case (B)), parameters ω° and $\omega_1^{(n)}$ are

$$\omega^\circ = p\sqrt{2-F_0} \qquad \omega_1^{(n)} = 4(1+2n)\sqrt{\frac{\left(p-p^2\right)\chi''\left(\varphi_0^\circ\right)}{2-F_0}} \qquad \text{for } p<1 \tag{35}$$

$$\omega^\circ = \sqrt{p^4-F_0p^2+1} \qquad \omega_1^{(n)} = \frac{1+2n}{2p}\sqrt{\frac{2\chi''\left(\varphi_0^\circ\right)\left(p^4-1\right)}{p^4-F_0p^2+1}} \qquad \text{for } p>1 \tag{36}$$

It may be seen that increasing the axial load F_0 leads to decreasing ω° and increasing the correction $\mu\omega_1^{(n)}$, i.e., the influence of curvature-variability on fundamental frequencies grows with the axial force. However, for formula (34) to be asymptotically valid it is necessary to require $\omega_1^{(n)} \sim 1$ as $\mu \to 0$; it is not fulfilled if $F_0 \to F_b$.

5 Parametric Instability

For $F_1^2+F_2^2 \neq 0$, the approximate formula for the normal deflections

$$W^* = CR\sin\frac{\pi m x}{L}\exp\left\{i\mu^{-1}\left[q^\circ\left(\varphi-\varphi_0^\circ\right)+\frac{1}{2}b\left(\varphi-\varphi_0^\circ\right)^2\right]\right\}$$

$$\times\left\{\sum_{k=0}^{n}\mu^{-k/2}\left(\varphi-\varphi_0^\circ\right)^k\left[S_k(\mu t)\sin\omega^\circ t + C_k(\mu t)\cos\omega^\circ t\right]+O\left(\mu^{1/2}\right)\right\} \tag{37}$$

describes parametric vibrations of a shell in a neighborhood of the line $\varphi=\varphi_0^\circ$. Depending on the correlation of parameters a_1, $a_{2,k}$, σ, this vibration will be stable or not. The stability region for system (29) has been established and is plotted in Figure 1 of Mikhasev and Kuntsevich (1997). For points $\left(\sigma/a_1, a_{2,k}/a_1\right)$ lying in the shadowed area, the amplitudes of parametric oscillations are functions growing infinitely with time. Outside of the shadowed area, where $\sigma, a_1, a_{2,k} \sim 1$, amplitudes are bounded.

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