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# A 2-D Analysis of the Stability of Boundary Layer Flow on Single and Multilayer Compliant Coatings

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*The paper deals with a theoretical study of the stability of a laminar boundary layer flow on single or multilayer compliant coatings, where the main flow is the classical Blasius boundary layer flow. Using linear stability analysis, the disturbance modes in the boundary layer flow are governed by the Orr-Sommerfeld equation which is integrated numerically employing a method based on a Riccati matrix. The dynamics of the compliant coating, composed of one or more viscoelastic sublayers, is analyzed using an original formalism, the central result being a recurrent relation for the displacements of successive sublayers. After the implementation of the matching conditions between the perturbations in the fluid flow and in the compliant coating, the model is tested successfully with known results from the open literature.*

## 1 Introduction

Studies concerning the interaction between boundary layer flows and compliant coatings have their origin in the experimental work of Kramer (1957, 1959) and are based on his observation of a dolphin swimming in water. In these papers, Kramer obtained substantial drag reduction for towed underwater bodies covered with compliant coatings, which simulated the dolphin skin. The early papers by Benjamin (1959, 1963), Landahl (1962) and Kaplan (1964) tried to elucidate the physical mechanisms of the favourable effects of the compliance on the transition delay. Then a number of theoretical and experimental studies extended the knowledge of this phenomenon. However, subsequent work showed that none of the experimental tests considered to simulate the coatings proposed by Kramer (1957, 1959) were satisfactory (see Riley et al., 1988). The research is rapidly continuing in the direction of more sophisticated numerical formulation and of more careful and competent experimental work. Several papers showing the state-of-the-art on this subject have been published in the last years by Gad-el-Hak (1986), Riley et al. (1988) and Carpenter (1990, 1994).

In the present context, we are interested here only in the 2-D analysis of the stability of a laminar, viscous and incompressible boundary layer flow on single and multilayer compliant coatings. The first paper attempting to treat this topic is due to Duncan et al. (1985), where the compliant coating was modelled as a single isotropic layer of finite thickness bounded by a rigid half-space. For the flow region these authors have considered a model based on the potential flow theory modified to account for the characteristics found in the previous studies of boundary layer flow over wavy walls. This kind of analysis was refined by Yeo (1988), Carpenter et al. (1990) and Dixon et al. (1994). Using a normal mode analysis, under the assumption of local parallelism of the flow in the boundary layer and a dimension of the coupled solid-flow system much greater than the length of the perturbations developed in the boundary layer, Carpenter et al. (1990) and Dixon et al. (1994) were able to analyze and optimize such a configuration, in the sense of delaying the transition from laminar to turbulent flow. These studies have used linear theory of hydrodynamic stability flow. The Orr-Sommerfeld equation is integrated numerically following a compound-matrices technique as was used by Yeo (1988), or the Chebyshev-Tau method (based on Chebyshev polynomials) presented in Dixon et al. (1994). Thus, the Tollmien-Schlichting instabilities (TSI) have been determined. The dynamic instability in the compliant coating of travelling wave flutter (TWF) and divergence type instabilities were also predicted by different methods.

In this paper, we present some new results concerning the problem of linear stability of a 2-D boundary layer flow on single or multiple compliant coatings. Like in other studies mentioned above, the basic flow is the Blasius boundary layer on a semi-infinite flat plate. In section 2, the basic equations are given and an analysis for a single compliant wall model is presented. The method used is a displacement-stress one, being similar to other techniques found in the literature (see Duncan, 1985; Yeo, 1988). Then, a rigidity matrix is introduced for a viscoelastic solid (or liquid viscous) sublayer and the matching conditions between the sublayers are established. In this way, the analysis is directed to a recurrent relation between the displacements of successive sublayers. The flow analysis is presented in section 3, while the interface conditions between the coating wall and the boundary layer flow are established in section 4. In section 5 a technique based on a Riccati matrix is proposed for the numerical integration of the Orr-Sommerfeld equation, with the appropriate boundary conditions. The present model is tested in sections 6 and 7 using a single compliant coating wall as in Yeo (1988). Concluding remarks follow in section 8. Because one of the major aims of the present paper is the use of an alternative scheme for the integration of the Orr-Sommerfeld equation, the attention is focussed only on instabilities of Tollmien-Schlichting type. The actual model has been adapted to incorporate the asymptotic method, introduced by Carpenter and Gajjar (1990), for the prediction of travelling wave flutter and divergence instabilities; more details on this subject are given by Postelnicu (1996).

## 2 The Compliant Wall Analysis

The geometry of the problem is illustrated in Figure 1, where $\bar{x}$ and $\bar{z}$ are Cartesian coordinates, with the $\bar{x}$-axis measured along the rigid base and the $\bar{z}$-axis normal to it, $h$ is the height of the solid material and $U_\infty$ is the free stream velocity. The compliant walls are composed of $m$ isotropic viscoelastic layers, some of which may be viscous liquid sublayers. The solid and the fluid regions are considered to be theoretically unlimited in the $\bar{x}$-direction and practically they correspond to a length along the $\bar{x}$-axis much greater than the wave length of perturbation $2\pi/\alpha$, where $\alpha$ is the wave number of the perturbation developed in the boundary layer flow.

### 2.1 Governing Equation for the Solid Region

For a material, which obeys the model of an ideal viscoelastic solid, the equation which governs the displacement $\eta$ can be written in non-dimensional form as

$$\frac{\partial^2 \eta}{\partial t^2} = c_L^2 \Delta \eta + \left(c_L^2 - c_T^2\right) grad(div\,\eta) \tag{1}$$

where the non-dimensional variables are defined as $t = U_\infty \bar{t}/L$, $(x,z) = (\bar{x},\bar{z})/L$, $\eta = \bar{\eta}/\delta^*$ and $(c_L, c_T) = (\bar{c}_L, \bar{c}_L)/U_\infty$. Here $\bar{t}$ is the time, $\Delta$ is the Laplacian operator, $\bar{c}_L$ and $\bar{c}_T$ are the longitudinal (bulk) and transversal (shear) wave speeds respectively, $L$ is a characteristic length of the rigid base, $U_\infty$ is the velocity of the free stream flow and $\delta^*$ is the displacement thickness of the boundary layer flow. In this paper, we shall use the following expressions for the wave speeds:

$$c_T = \sqrt{\frac{G}{\rho_s}} = \sqrt{\frac{E}{2(1+\mu)\rho_s}} \qquad c_L = \sqrt{(\lambda+G)/\rho_s} = \sqrt{2}c_T\sqrt{\frac{1-\mu}{1-2\mu}} \tag{2}$$

where $\lambda$, $G$ and $\rho_\sigma$ are the Lamé constant, shear modulus and density of the solid material respectively, $E$ is the elastic modulus and $\mu$ is the Poisson ratio of the material; these expressions can be found in Dixon et al. (1994). The quantities $E$, $G$, $\lambda$ and $\rho_s$ are nondimensionalized as $(E,G,\lambda) = (\bar{E},\bar{G},\bar{\lambda})/(\bar{\rho}_f U_\infty^2)$, and $\rho_s = \bar{\rho}_s/\bar{\rho}_f$ where $\bar{\rho}_f$ is the fluid density.

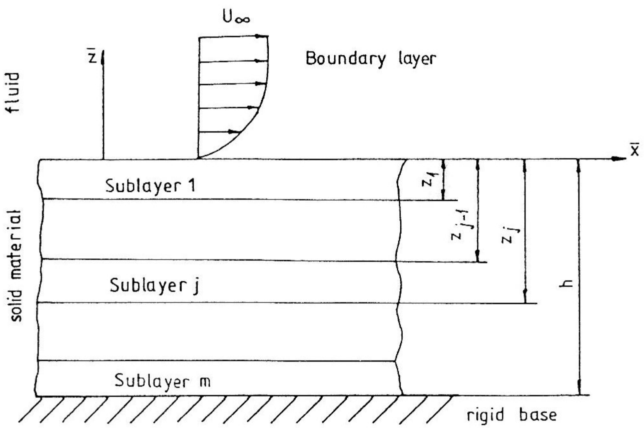


Figure 1 Coordinate System and Physical Model

The effect of the wall elasticity is introduced in a manner similar to that of Carpenter et al. (1990) and Dixon et al. (1994), namely $E = E' + iE'' = E'(1 - i\gamma_s)$, where $E'$ is the conventional elasticity modulus, $\gamma_s$ is a non-dimensional damping coefficient and $i = \sqrt{-1}$. The expression for $c_T$ used by Yeo (1988) is slightly different than that of equation (2). He used for $G$ the relation $G = \rho_s C_t^2 - i\omega d$, where $C_t$ is the wave speed for the viscoelastic material, $d$ is a damping coefficient, and $\omega = c/\alpha$ is the circular frequency with $c$ being the wave speed; $C_d$, $D$, $\gamma_s$ and $\omega$ are nondimensional quantities. However, there is an equivalence between his formulation and the present one. Thus if we express $G$ as $G = \rho_s C_t^2 - i\omega d = G'(1 - i\gamma_s)$, with $G' = E'/[2(1+\mu)]$ *and* $\gamma_s = \omega d/(\rho_s C_t^2)$, then there results the expression (10) given in Dixon et al. (1994). A similar case can be made for $c_L$, as well.

### 2.2 Displacements and Shear Stresses in the Solid Material

We notice that for the case in which we are interested here, there exists a plane state of displacement of the form $\eta = \{\eta_1, 0, \eta_3\}^T$. Thus, the Stokes-Helmholtz decomposition $\eta = \nabla\phi + \nabla \times \psi$, with $\psi = \{0, \psi, 0\}^T$ gives

$$\eta_1 = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} \qquad \eta_3 = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{3}$$

where $\phi$ and $\psi$ satisfy the wave equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c_L^2}\frac{\partial^2\phi}{\partial t^2} \qquad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c_L^2}\frac{\partial^2\psi}{\partial t^2} \tag{4}$$

Considering the waves which propagate in the $x$-direction, we look for a solution of equations (4) of the form

$$\phi = \hat{\phi}(z)e^{i(\alpha x - \omega t)} \qquad \psi = \hat{\psi}(z)e^{i(\alpha x - \omega t)} \tag{5}$$

Then, equations (4) become

$$\frac{d^2\hat{\phi}}{dz^2} - b_L^2\hat{\phi} = 0 \qquad \frac{d^2\hat{\psi}}{dz^2} - b_T^2\hat{\psi} = 0 \tag{6}$$

where $b_L^2 = \alpha^2 - \omega^2/c_L^2$ and $b_T^2 = \alpha^2 - \omega^2/c_T^2$ .

The solutions of equations (6) are

$$\hat{\phi} = A_1 e^{b_L z} + A_2 e^{-b_L z} \qquad \hat{\psi} = A_3 e^{b_T z} + A_4 e^{-b_T z} \tag{7}$$

where $A_1$, $A_2$, $A_3$, $A_4$ are complex constants. If we now substitute equations (5) and (7) into equations (3), the displacements $(\eta_1, \eta_3)$ can be expressed as

$$\eta_1 = \hat{\eta}_1 e^{i(\alpha x - \omega t)} \qquad \eta_3 = \hat{\eta}_3 e^{i(\alpha x - \omega t)} \tag{8}$$

where

$$\begin{aligned} \hat{\eta}_1 &= i\alpha e^{b_L z}A_1 + i\alpha e^{-b_L z}A_2 - b_T e^{b_T z}A_3 + b_T e^{-b_T z}A_4 \\ \hat{\eta}_3 &= b_L e^{b_L z}A_1 - b_L e^{-b_L z}A_2 + i\alpha e^{b_T z}A_3 + i\alpha e^{-b_T z}A_4 \end{aligned} \tag{9}$$

For the problem of a 2-D configuration, the stresses can be written in non-dimensional form as

$$\tau_{xz}^{(s)} = r\rho c_T^2\left(\frac{\partial\eta_1}{\partial z} + \frac{\partial\eta_3}{\partial x}\right) \qquad \sigma_z^{(s)} = r\rho\left[c_L^2\frac{\partial\eta_3}{\partial z} + \left(c_L^2 - 2c_T^2\right)\frac{\partial\eta_1}{\partial x}\right] \tag{10}$$

where $r = \delta^* / L = \mathrm{Re}_{\delta^*} / \mathrm{Re}_L$ and $\rho = \rho_s / \rho_f$. Here $\mathrm{Re}_{\delta^*}$ and $\mathrm{Re}_L$ are Reynolds numbers defined as $\mathrm{Re}_{\delta^*} = U_\infty \delta^* / \nu_f$ and $\mathrm{Re}_L = U_\infty L / \nu_f$ with $\nu_f$ being the kinematic viscosity of the fluid. Further, introducing equations (8) and (9) into equations (10), we obtain

$$\hat{\tau}_{xz}^{(s)} = r\rho c_T^2\left(\hat{\eta}_1' + i\alpha\hat{\eta}_3\right) \qquad \hat{\sigma}_z^{(s)} = r\rho c_T^2\left[c_R^2\,\hat{\eta}_3' + \left(c_R^2 - 2\right)i\alpha\hat{\eta}_1\right] \tag{11}$$

where $c_R = c_L / c_T$. We now substitute equations (9) into equations (11) so that, after some algebra, one gets

$$\frac{\hat{\tau}_{xz}^{(s)}}{r\rho c_T^2} = 2i\alpha b_L e^{b_L z} A_1 - 2i\alpha b_L e^{-b_L z} A_2 - \left(b_T^2 + \alpha^2\right) e^{b_T z} A_3 - \left(b_T^2 + \alpha^2\right) e^{-b_T z} A_4$$
$$\frac{\hat{\sigma}_z^{(s)}}{r\rho c_T^2} = \left(b_T^2 + \alpha^2\right) e^{b_L z} A_1 \left(b_T^2 + \alpha^2\right) e^{-b_L z} A_2 + 2i\alpha b_T e^{b_T z} A_3 - 2i\alpha b_T e^{-b_T z} A_4 \tag{12}$$

### 2.3 The Introduction of a Rigidity Matrix for a Solid Layer

At a given station $z$, we have

$$\{\hat{\eta}_1, \hat{\eta}_3\}^T = \mathbf{D\,A} \qquad \left\{\hat{\tau}_{xz}^{(s)},\ \hat{\sigma}_z^{(s)}\right\}^T = \mathbf{E\,A} \tag{13}$$

where the matrices $\mathbf{D}$ and $\mathbf{E}$ will be specified later and $\mathbf{A} = \{A_1, A_2, A_3, A_4\}^T$. For a $j$th sublayer, $z_{j-1} \le z \le z_j$, we introduce

$$\hat{\eta}_1^{(j)} = \{\hat{\eta}_{1+}, \hat{\eta}_{3+}, \hat{\eta}_{1-}, \hat{\eta}_{3-}\}^{(j)T} \qquad \hat{\sigma}^{(s)(j)} = \left\{\hat{\tau}_{xz^+}^{(s)},\ \hat{\sigma}_{z^+}^{(s)},\ \hat{\tau}_{xz^-}^{(s)},\ \hat{\sigma}_{z^-}^{(s)}\right\}^{(j)T} \tag{14}$$

where plus and minus signs refer to the evaluation at the upperside and at the lower-side of the $j$th sublayer. Using equations (13), we can write

$$\hat{\eta}_+^{(j)} = \{\hat{\eta}_1, \hat{\eta}_3\}_+^{(j)T} = \mathbf{D}_+^{(j)}\mathbf{A}^{(j)} \qquad \hat{\eta}_-^{(j)} = \{\hat{\eta}_1, \hat{\eta}_3\}_-^{(j)T} = \mathbf{D}_-^{(j)}\,\mathbf{A}^{(j)}$$
$$\hat{\sigma}_+^{(s)(j)} = \left\{\hat{\tau}_{xz}^{(s)},\ \hat{\sigma}_z^{(s)}\right\}_+^{(j)T} = \mathbf{E}_+^{(j)}\,\mathbf{A}^{(j)} \qquad \hat{\sigma}_-^{(s)(j)} = \left\{\hat{\tau}_{xz}^{(s)},\ \hat{\sigma}_z^{(s)}\right\}_-^{(j)T} = \mathbf{E}_-^{(j)}\,\mathbf{A}^{(j)} \tag{15}$$

or

$$\hat{\eta}^{(j)} = \mathbf{D}^{(j)}\,\mathbf{A}^{(j)} \qquad \hat{\sigma}_-^{(s)(j)} = \mathbf{E}_-^{(j)}\,\mathbf{A}^{(j)} \tag{16}$$

where $\mathbf{D}^{(j)} = \begin{bmatrix}\mathbf{D}_+^{(j)}\\ \mathbf{D}_-^{(j)}\end{bmatrix}$ and $\mathbf{E}^{(j)} = \begin{bmatrix}\mathbf{E}_+^{(j)}\\ \mathbf{E}_-^{(j)}\end{bmatrix}$. Now, eliminating $\mathbf{A}^{(j)}$ between the relations (16), we finally get

$$\hat{\sigma}^{(s)(j)} = \mathbf{K}^{(j)}\,\hat{\eta} \tag{17}$$

where $\mathbf{K}^{(j)} = \mathbf{E}^{(j)}\,\mathbf{D}^{(j)^{-1}}$ can be called the matrix of rigidity for the *j*th layer. A useful form of equations (17) is also

$$\hat{\sigma}^{(s)(j)}_{+} = \mathbf{K}^{(j)}_{\mathrm{I}}\,\hat{\eta}^{(j)}_{+} + \mathbf{K}^{(j)}_{\mathrm{II}}\,\hat{\eta}^{(j)}_{-} \qquad \hat{\sigma}^{(s)(j)}_{-} = \mathbf{K}^{(j)}_{\mathrm{III}}\,\hat{\eta}^{(j)}_{+} + \mathbf{K}^{(j)}_{\mathrm{IV}}\,\hat{\eta}^{(j)}_{-} \tag{18}$$

where the rigidity matrix **K**, which is 4x4, has been compartmentalized into two 2x2 matrices

$$\mathbf{K}^{(j)} = \begin{bmatrix} \mathbf{K}^{(j)}_{\mathrm{I}} & \mathbf{K}^{(j)}_{\mathrm{III}} \\ \mathbf{K}^{(j)}_{\mathrm{II}} & \mathbf{K}^{(j)}_{\mathrm{IV}} \end{bmatrix} \tag{19}$$

Also, the structure of the matrices **D** and **E** for the *j*th layer are

$$\mathbf{D}^{(j)} = \begin{bmatrix} i\alpha e^{b_L z_{j-1}} & i\alpha e^{-b_L z_{j-1}} & -b_T e^{b_T z_{j-1}} & b_T e^{-b_T z_{j-1}} \\ b_L e^{b_L z_{j-1}} & -b_L e^{-b_L z_{j-1}} & i\alpha e^{b_T z_{j-1}} & i\alpha e^{-b_T z_{j-1}} \\ i\alpha e^{b_L z_j} & i\alpha e^{-b_L z_j} & -b_T e^{b_T z_j} & b_T e^{-b_T z_j} \\ b_L e^{b_L z_j} & -b_L e^{-b_L z_j} & i\alpha e^{b_T z_j} & i\alpha e^{-b_T z_j} \end{bmatrix}$$

$$\mathbf{E}^{(j)} = r\rho c_T^2 \begin{bmatrix} 2i\alpha b_L e^{b_L z_{j-1}} & -2i\alpha b_L e^{-b_L z_{j-1}} & -\left(b_T^2+\alpha^2\right)e^{b_T z_{j-1}} & -\left(b_T^2+\alpha^2\right)e^{-b_T z_{j-1}} \\ \left(b_T^2+\alpha^2\right)e^{b_L z_{j-1}} & \left(b_T^2+\alpha^2\right)e^{-b_L z_{j-1}} & 2i\alpha b_T e^{b_T z_{j-1}} & -2i\alpha b_L e^{-b_T z_{j-1}} \\ 2i\alpha b_L e^{b_L z_j} & -2i\alpha b_L e^{-b_L z_j} & -\left(b_T^2+\alpha^2\right)e^{b_T z_j} & -\left(b_T^2+\alpha^2\right)e^{-b_T z_j} \\ \left(b_T^2+\alpha^2\right)e^{b_L z_j} & \left(b_T^2+\alpha^2\right)e^{-b_L z_j} & 2i\alpha b_T e^{b_T z_j} & -2i\alpha b_T e^{-b_T z_j} \end{bmatrix} \tag{20}$$

It is worth mentioning that the notion of a matrix of rigidity for a solid layer was introduced previously by Evrensel and Kalnins (1985, 1988), but with a different meaning and only for the case of a single layer. On the other hand, one can easily prove that the relations (8) and (12) can be reduced to relations (2.12a) and (2.12b) from the paper by Yeo (1988).

### 2.4 The Matching Conditions

The matching conditions to be imposed adjacent to the two viscoelastic sublayers are the kinematic conditions

$$\hat{\eta}^{(j)}_{-} = \hat{\eta}^{(j+1)}_{+} \tag{21}$$

and the dynamic conditions

$$\hat{\sigma}^{(s)(j)}_{-} = \hat{\sigma}^{(s)(j+1)}_{+} \tag{22}$$

At the interface between the solid layer and the rigid base, we have $\eta_{-3}{}^{(m)} = 0$ and $\tau_{xz-}{}^{(m)} = 0$ (see Kaplan, 1964) if displacement at the rigid base is permitted and

$$\eta^{(m)}_{-} = 0 \qquad \text{or} \qquad \hat{\eta}^{(m)}_{-} = 0 \tag{23}$$

if no displacement at the rigid base is permitted. This last condition is encountered in all the modern studies (Duncan et al., 1985; Yeo, 1988; Dixon et al., 1994) and therefore, we shall use it here. Relation (22) can also be written as

$$\mathbf{K}_{\text{III}}^{(j)}\,\hat{\eta}_{+}^{(j)} + \mathbf{K}_{\text{IV}}^{(j)}\,\hat{\eta}_{-}^{(j)} = \mathbf{K}_{\text{I}}^{(j+1)}\,\hat{\eta}_{+}^{(j+1)} + \mathbf{K}_{\text{II}}^{(j+1)}\,\hat{\eta}_{-}^{(j+1)} \tag{24}$$

if equations (17) and (22) are used. We also get, after some manipulation, the final relation

$$\hat{\eta}_{-}^{(j+1)} = \mathbf{A}^{(j)}\,\hat{\eta}_{-}^{(j)} + \mathbf{B}^{(j)}\hat{\eta}_{-}^{(j-1)} \tag{25}$$

where

$$\mathbf{A}^{(j)} = \mathbf{K}_{\text{II}}^{(j+1)^{-1}}\left(\mathbf{K}_{\text{IV}}^{(j)} - \mathbf{K}_{\text{I}}^{(j+1)}\right) \quad \text{and} \quad \mathbf{B}^{(j)} = \mathbf{K}_{\text{II}}^{(j+1)^{-1}}\,\mathbf{K}_{\text{III}}^{(j)} \tag{26}$$

We notice that equation (25) is a recurrent relation of 3 levels, which requires the knowledge of two starting terms, $\hat{\eta}_{-}^{(1)}$ and $\hat{\eta}_{-}^{(2)}$ , where the first term can be related through equation (24) to the displacement of the coating at the interface with the boundary layer flow and the second term can be easily calculated, writing equations (17) and (19) for the first and the second sublayers and imposing the kinematic and dynamic matching conditions (21) and (22). Thus, we get, after some short algebra,

$$\hat{\eta}_{-}^{(1)} = \mathbf{L}^{(1)}\,\hat{\sigma}^{(s)}{}_0 + \mathbf{M}^{(1)}\,\hat{\eta}_0 \tag{27}$$

where $\mathbf{L}^{(1)} = \mathbf{K}_{\text{II}}^{(1)^{-1}}$ and $\mathbf{M}^{(1)} = -\mathbf{K}_{\text{II}}^{(1)^{-1}}\,\mathbf{K}_{\text{I}}^{(1)}$

## 3 The Analysis of the Fluid Flow

### 3.1 The Boundary Layer Flow

We will consider for the basic fluid flow, the classical Blasius boundary layer flow past a semi-infinite flat plate. As it is known, the governing equation of this flow is

$$2f''' + ff'' = 0 \tag{28}$$

where $f'(z) = U(z)$ is the reduced velocity in $x$-direction and primes denote differentiation with respect to $z$. However, equation (28) can be also written as (see Jaffe et al., 1970; Drazin and Reid, 1982)

$$2f''' + a_B{}^2 ff'' = 0 \tag{29}$$

where $a_B = 1.72078$ is the Blasius constant. The boundary conditions of equation (29) are

$$f(0) = 0 \qquad f'(0) = 0 \qquad f'(Z_2) = 1 \tag{30}$$

where $Z_2 = 6$ is an optimum value.

### 3.2 The Stability Analysis of the Hydrodynamic Problem

We know that a study of the flow stability can be made using the Orr-Sommerfeld equation (see Drazin and Reid, 1982)

$$(U - c)\left(\hat{\psi}'' - \alpha^2\hat{\psi}\right) - U''\hat{\psi} = \frac{1}{i\alpha \mathrm{Re}_{\delta^*}}\left(\hat{\psi}^{\text{IV}} - 2\alpha^2\hat{\psi}'' + \hat{\psi}\right) \tag{31}$$

the pressure $\hat{p}$ being given by

$$\hat{p} = -(U - c)\,\hat{\psi}' + U'\hat{\psi} - \frac{i}{\alpha \mathrm{Re}_{\delta^*}}\left(\hat{\psi}''' - \alpha^2\hat{\psi}'\right) \tag{32}$$

The non-dimensional fluid shear stresses for the present problem (2-D case) can be written as

$$\tau_{xz}^{(f)} = \frac{1}{\mathrm{Re}_{\delta^*}}\left(\frac{dU}{dz} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \sigma_z^{(f)} = -p + \frac{2}{\mathrm{Re}_{\delta^*}}\frac{\partial w}{\partial z} \tag{33}$$

or, in terms of the non-dimensional stream function,

$$\tau_{xz}^{(f)} = \frac{1}{\mathrm{Re}_{\delta^*}}\left(U' + \psi'' + \alpha^2\psi\right) \qquad \sigma_z^{(f)} = \frac{i}{\alpha\,\mathrm{Re}_{\delta^*}}\psi''' + \left(U - c - \frac{3i\alpha}{\mathrm{Re}_{\delta^*}}\right)\psi' - U'\psi \tag{34}$$

where $w$ is the non-dimensional fluid velocity component in $z$-direction. We shall proceed here to a spatial stability analysis, so that $\alpha$ is real and $\omega$ is complex (see Drazin and Reid, 1982).

## 4 The Interface Conditions

These conditions are to be imposed at the interface between the sublayer 1 of the compliant wall and the Blasius boundary layer flow, and they are analysed in detail in the papers by Yeo (1988) and Postelnicu (1996). The second author has analysed these conditions on the basis of the form given by Dowell and Ilgamov (1988). Thus, we consider here the interface conditions of the following form: the kinematic conditions

$$\dot{\eta}_1 = u + \eta_3 U_0' \qquad \dot{\eta}_3 = w \tag{35}$$

and the dynamic conditions

$$\left[\delta_z^{(s)}\right]_0 = \eta_3\left[\frac{\partial\sigma_z^{(f)m}}{\partial z}\right]_0 + \left[\sigma_z^{(f)}\right]_0 \qquad \left[\tau_{xz}^{(s)}\right]_0 = \eta_3\left[\frac{\partial\tau_{xz}^{(f)m}}{\partial z}\right]_0 + \left[\tau_{xz}^{(f)}\right]_0 \tag{36}$$

where the subscript 0 denotes the value of the quantities at the interface between the sublayer 1 and the fluid flow. Using now equations (34), (35) and (36), we obtain

$$\begin{aligned}
&\hat{\psi}_0' + \left[\hat{\eta}_3\right]_0 U_0' = -i\omega\left[\hat{\eta}_1\right]_0 \\
&\frac{1}{c}\hat{\psi}_0 = \left[\hat{\eta}_3\right]_0 \\
&\frac{1}{\mathrm{Re}_{\delta^*}}\left(\hat{\psi}_0'' + \alpha^2\hat{\psi}_0\right) = \left[\hat{\tau}_{xz}^{(s)}\right]_0 \\
&\frac{i}{\alpha\,\mathrm{Re}_{\delta^*}}\hat{\psi}_0''' - \left(c + \frac{3i\alpha}{\mathrm{Re}_{\delta^*}}\right)\hat{\psi}_0' - U_0'\hat{\psi}_0 = \left[\hat{\sigma}_z^{(s)}\right]_0
\end{aligned} \tag{37}$$

which can be also written as

$$\mathbf{Q}_D\,\hat{\boldsymbol{\psi}}_0 = \hat{\boldsymbol{\eta}}_0 \qquad \mathbf{Q}_E\,\hat{\boldsymbol{\psi}}_0 = \hat{\boldsymbol{\sigma}}^{(s)}{}_0 \tag{38}$$

where we have used the notations $\hat{\boldsymbol{\psi}}_0 = \left\{\hat{\psi}_0, \hat{\psi}_0', \hat{\psi}_0'', \hat{\psi}_0'''\right\}^T$ and

$$\mathbf{Q}_D = \begin{bmatrix} \dfrac{iU_0'}{\omega c} & \dfrac{i}{\omega} & 0 & 0 \\ \dfrac{1}{c} & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q}_E = \begin{bmatrix} \dfrac{\alpha^2}{\mathrm{Re}_{\delta^*}} & 0 & \dfrac{1}{\mathrm{Re}_{\delta^*}} & 0 \\ -U_0' & -\left(c + \dfrac{3i\alpha}{\mathrm{Re}_{\delta^*}}\right) & 0 & \dfrac{1}{\alpha\,\mathrm{Re}_{\delta^*}} \end{bmatrix}$$

The matrix $\mathbf{Q}_C = \begin{bmatrix} \mathbf{Q}_D \\ \mathbf{Q}_E \end{bmatrix}$ can be found in Yeo (1988) where it is called the coupling matrix. We notice that equation (25) is a recurrent relation which needs, for a wall with $m$ sublayers, 2 starting conditions, namely, $\eta_0$ and $\eta_1$, which can be obtained from equations (23) and (27). Thus, using these relations, we get

$$\mathbf{L}\,\hat{\sigma}^{(s)}{}_0 + \mathbf{M}\,\hat{\eta}_0 = 0 \tag{39}$$

or, with equation (38) in mind,

$$\mathbf{Q}_0\,\hat{\psi}_0 = 0 \tag{40}$$

where $\mathbf{Q}_0 = \mathbf{L}\,\mathbf{Q}_D + \mathbf{M}\,\mathbf{Q}_E$.

## 5 The Solution of the Stability Problem Using the Technique of a Riccati Matrix

### 5.1 General Considerations

This technique is, in principle, described by Drazin and Reid (1982) for the case of rigid walls with application to the Poiseuille flow. The main idea of this method is to transform the linear eigenvalue problem into a nonlinear one. We will follow this technique here by adopting it for the case of compliant coatings. Thus, the Orr-Sommerfeld equation is written in matrix form as

$$\mathbf{U}' = \mathbf{A}\,\mathbf{V} \qquad\qquad \mathbf{V}' = \mathbf{U} \tag{41}$$

with

$$\mathbf{U} = \left\{\hat{\psi}', \hat{\psi}''' - \alpha^2\hat{\psi}'\right\}^T \qquad\qquad \mathbf{V} = \left\{\hat{\psi}, \hat{\psi}'' - \alpha^2\hat{\psi}\right\}^T$$

$$\mathbf{A} = \begin{bmatrix} \alpha^2 & 1 \\ -i\alpha \mathrm{Re}_{\delta^*} U'' & \alpha^2 + i\alpha \mathrm{Re}_{\delta^*}(U-c) \end{bmatrix}$$

Further, on using the transformation

$$\mathbf{U} = \mathbf{R}\;\mathbf{V} \tag{42}$$

one introduces Riccati´s matrix, **R**, which satisfies the differential equation

$$\mathbf{R}' + \mathbf{R}^2 = \mathbf{A} \tag{43}$$

where

$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \tag{44}$$

and $r_1, r_2, r_3, r_4$ are unknown variables. The matrix equation (43) is equivalent to the following 4 scalar equations:

$$r_1' + r_1^2 + r_2 r_3 = \alpha^2 \qquad r_2' + r_1 r_2 + r_2 r_4 = 1 \qquad r_3' + r_1 r_3 + r_3 r_4 = -i\alpha\mathrm{Re}_{\delta_*} U''$$
$$r_4' + r_2 r_3 + r_4^2 = \alpha^2 + i\alpha\mathrm{Re}_{\delta_*}(U - c) \tag{45}$$

with the boundary conditions that we shall specify below. We mention to this end that from equation (42), we have

$$\hat{\psi}' = r_1\hat{\psi} + r_2\left(\hat{\psi}'' - \alpha^2\hat{\psi}\right) \qquad \hat{\psi}''' - \alpha^2\hat{\psi}' = r_3\hat{\psi} + r_4\left(\hat{\psi}'' - \alpha^2\hat{\psi}\right) \tag{46}$$

### 5.2 The Boundary Conditions

To determine these conditions, we follow the procedure where the integration starts from the wall (there exists also the inverse way, from the outer edge of the boundary layer). With the notation $|Q_{ij}|$ (which is the determinant of the submatrix formed with the *i*th row and the *j*th column of the matrix $\mathbf{Q}_0$), from equation (40) we get

$$\hat{\psi}_0'' = F_1\hat{\psi}_0 + F_2\hat{\psi}_0' \qquad \hat{\psi}_0''' = F_3\hat{\psi}_0 + F_4\hat{\psi}_0' \tag{47}$$

where $F_1 = -|Q_{14}|/|Q_{34}|$, $F_2 = -|Q_{24}|/|Q_{34}|$, $F_3 = -|Q_{13}|/|Q_{34}|$ *and* $F_4 = -|Q_{23}|/|Q_{34}|$. We assume here that $|Q_{34}| > \varepsilon$, where $\varepsilon$ has an appropriate value for computation convenience. This is, of course, not a unique situation. Thus Postelnicu (1996) has discussed other cases too, such as, for example, when $|Q_{34}| < \varepsilon$ and $|Q_{12}| < \varepsilon$ with appropriate values for $\varepsilon$. Using equations (47), equations (46) become at the wall

$$\begin{aligned} &\left[r_1 + r_2\left(F_1 - \alpha^2\right)\right]\hat{\psi}_0 + (r_2F_2 - 1)\hat{\psi}_0' = 0 \\ &\left[r_3 - F_3 + r_4\left(F_1 - \alpha^2\right)\right]\hat{\psi}_0 + \left(r_4F_2 - F_4 + \alpha^2\right)\hat{\psi}_0' = 0 \end{aligned} \tag{48}$$

which lead to

$$\begin{aligned} &r_1 + r_2\left(F_1 - \alpha^2\right) = 0 \\ &r_2F_2 - 1 = 0 \\ &r_3 - F_3 + r_4\left(F_1 - \alpha^2\right) = 0 \\ &r_4F_2 - F_4 + \alpha^2 = 0 \end{aligned} \tag{49}$$

This is an algebraic system of equations in the variables $r_1, r_2, r_3$ and $r_4$. The solution of equation (49) is

$$r_1 = \left(\alpha^2 - F_1\right)/F_2 \qquad r_2 = 1/F_2 \qquad r_3 = F_3 + \left(\alpha^2 - F_1\right)\left(\alpha^2 - F_4\right)/F_2 \qquad r_4 = \left(\alpha^2 - F_4\right)/F_2 \tag{50}$$

But, it is known from the paper by Drazin and Reid (1982) that

$$\hat{\psi} = C_1e^{-\alpha z} + C_2e^{-\chi z} \tag{51}$$

where $C_1$ and $C_2$ are constants yet unknown. Here $\chi^2 = \alpha^2 + i\alpha(1-c)\mathrm{Re}_{\delta^*} = \alpha^2 + i(\alpha - \omega)\mathrm{Re}_{\delta^*}$ and the real parts of $\chi$ and $\alpha$ must be positive. Substituting equation (51) into equation (46), the following homogeneous system of algebraic equations is obtained for $C_1$ and $C_2$

$$(\alpha + r_1)e^{-\alpha z}C_1 + \left[\chi + r_1 + (\chi^2 - \alpha^2)r_2\right]e^{-\chi z}C_2 = 0$$
$$r_3 e^{-\alpha z}C_1 + \left[\chi^3 - \alpha^2\chi + r_3 + (\chi^2 - \alpha^2)r_4\right]e^{-\chi z}C_2 = 0 \tag{52}$$

Imposing the condition that this system must have a nontrivial solution, we obtain the following eigenvalue problem:

$$i\alpha(1-c)\mathrm{Re}_{\delta^*}\left[(\chi + r_4)(\alpha + r_1) - r_2 r_3\right] + r_3(\alpha - \chi) = 0 \tag{53}$$

It is worth mentioning that Postelnicu (1996) has also proposed a procedure for solving the Orr-Sommerfeld equation by using the matrix $\mathbf{S} = \mathbf{R}^{-1}$, where $\mathbf{R}^{-1}$ is the inverse of Riccati´s matrix.

## 6 Numerical Details

The Blasius and the Orr-Sommerfeld equations are integrated simultaneously. For this purpose, a column vector $\mathbf{Y} = \{f, f', f'', r_1, r_2, r_3, r_4\}^T$ is formed. Also, a system of first order differential equations for the variable $Y$ is formed from equations (29) and (45). The integration of this system was performed using a subroutine in Matlab, based on the Runge-Kutta-Fehlberg method. A solver, based on the secant method along with the „regula falsi" procedure, was also worked out in Matlab for equation (53). The starting scheme was performed by a special implemented procedure. The fixed level of convergence for the present work was $10^{-8}$ for the system of ordinary differential equations (29) and (45), and also $10^{-8}$ for the nonlinear algebraic equation (53).

## 7 Results for the Tollmien-Schlichting Instabilities

The results obtained in this paper are for a single compliant viscoelastic sublayer, which is characterized by $h = 1$, $C_t = 0.7$, $\rho_s = 1$, $d = 0.0049$, $\mathrm{Re}_L = 20000$ and $K = 500$, where $K$ is the bulk modulus, which is given by $K = E'/3(1-2\mu)$. These results refer first to the curve of neutral stability which is shown in Figure 2.

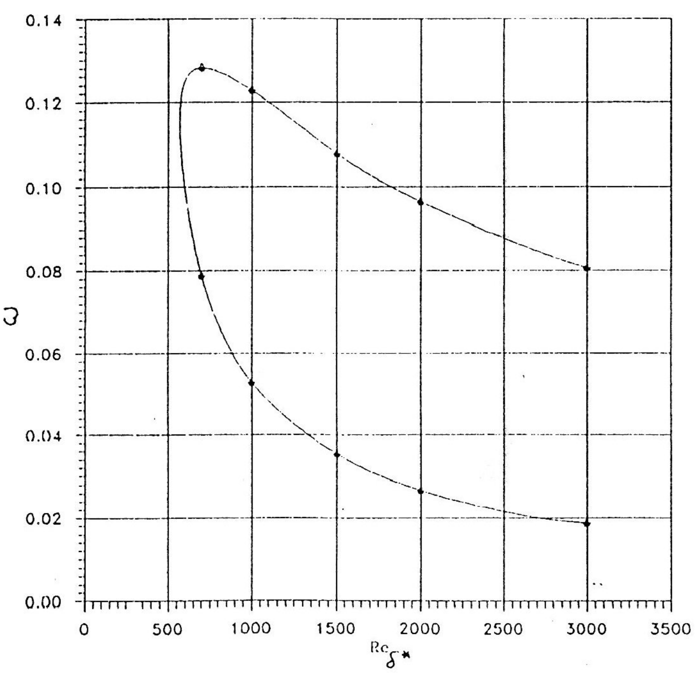


Figure 2. The Curve of Neutral Stability

The next step is to determine the maximum constant amplification envelope. To do this, we use the $e^n$ criterion expressed as

$$n = \ln\left[A\left(Re_{\delta^*}\right) / A\left(\left(Re_{\delta^*}\right)_0\right)\right] = -\frac{2}{a_B^2} \int_{\left(Re_{\delta^*}\right)_0}^{Re_{\delta^*}} \alpha_i\left(Re_{\delta^*}\right) d\, Re_{\delta^*} \tag{54}$$

where $n$ is the exponential growth factor, $\alpha_i$ is the imaginary part of $\alpha$, $\left(Re_{\delta^*}\right)_0$ is the Reynolds number corresponding to the lower branch of the marginal stability curve at a fixed value of the frequency parameter $F = \omega \times 10^6 / Re_{\delta^*}$, and $A$ signifies the amplitude of the Tollmien-Schlichting instabilities corresponding to $Re_{\delta^*}$ and $\left(Re_{\delta^*}\right)_0$, respectively.

We should notice that it is generally accepted that a value of $n$ in the range (8,..,10) characterizes correctly the transition stage, which is studied within the linear hydrodynamic stability analysis. However, Carpenter et al. (1990) have used a value of $n = 7$ and we have also used this value in the present paper. The total amplification curves are plotted in Figure 3 for some values of $F$.

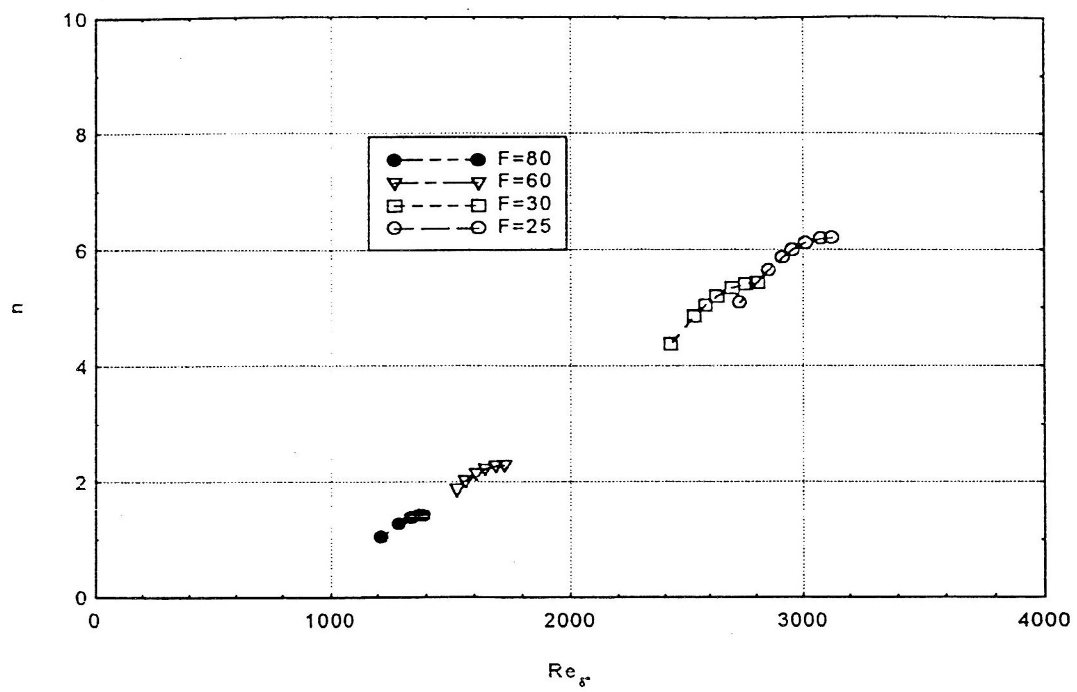


Figure 3. Total Amplification Curves

Finally, using the results shown in Figure 3, we have presented in Figure 4 the variation of the maximum constant amplification envelope for the compliant wall and for the compliant coating cases. Thus, for $n = 7$, we found $Re_{\delta^*} = 3376$. On the other hand, it can be read from Figure 6 of the paper by Yeo (1988) that $Re_{\delta^*} = 3300$. Therefore, the value of $Re_{\delta^*}$ determined employing the present method compares favourably with that from Yeo (1988).

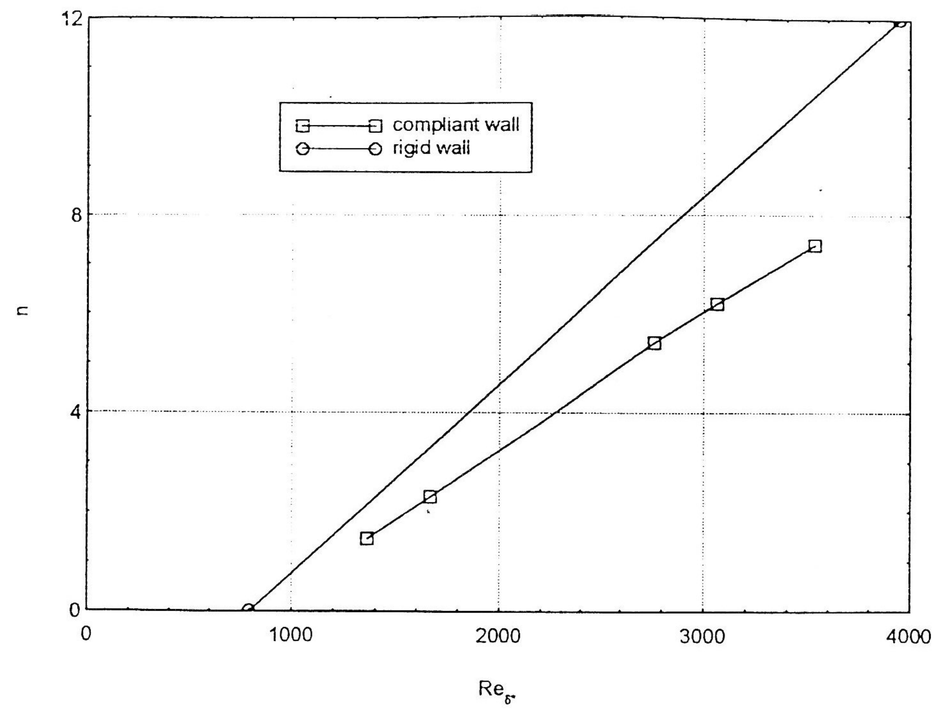


Figure 4. The Maximum Constant Amplification Envelope

## 8 Conclusions

A new method has been developed in this paper for the analysis of the stability of a 2-D boundary layer flow on single and multilayer compliant coatings. The originality of the dynamical behaviour analysis of the compliant walls lies in the introduction of a rigidity matrix for the solid viscoelastic sublayers. After partitioning of this matrix, a suitable recurrent relation between the displacements of successive sublayers is obtained. The two conditions necessary to close this recurrent relation are the zero displacements of sublayers at the interface with the rigid base and also the interface conditions with the boundary layer flow. This method possesses the advantage of accelerating the numerical computation and, on the other hand, the possibility to properly treat discontinuities between adjacent sublayers. Such situations are left for future work and we mention that they are, to this date, not found in the literature devoted to this topic.

For the fluid region, the present method offers a successful alternative to numerical integration of the Orr-Sommerfeld equation; the method of Riccati's matrix works more rapidly in comparison with other methods; its rapidity is comparable only with that of the spectral method (see Postelnicu, 1996). These are important arguments for such problems of interaction which require quite a long computation time. Other test cases concerning the TSI and TWF/divergence predictions can be found in Postelnicu (1996). Our model has incorporated the asymptotic technique for the prediction of the TWF characteristics, originally presented by Carpenter and Gajjar (1990). Thus, a new way to analyse the linear stability of the boundary layer over single and multilayer coatings has been established in this paper.

## Literature


1. Benjamin, B.: Effects of a Fexible Boundary on Hydrodynamic Stability, J. Fluid Mech., 9, (1959), 514

2. Benjamin, B.: The Threefold Classification of Unstable Disturbances in Flexible Surfaces Bounding Inviscid Flow, J. Fluid Mech., 16, (1963), 463

3. Carpenter, P.W.: Status of Transition Delay Using Compliant Walls, Viscous Drag Reduction in Boundary Layers, Bushnell, D.M.; Hefner, J.N. (eds.), AIAA, Washington, D.C., (1990), 79

4. Carpenter, P.W.; Gajjar, J.: A General Theory for 2- and 3-D Wall-Mode Instabilities in Boundary Layers over Isotropic and Anisotropic Compliant Walls. In: Theoret. Comput. Fluid Dynamics, Springer-Verlag, (1990), 349

5. Carpenter, P.W.; Lucey, A.D.; Dixon, A.E.: The Optimization of Compliant Walls for Drag Reduction, In: Recent Developments in Turbulence Management, Choi K.S. (ed.), (1990), 195

6. Carpenter, P.W.: Recent Developments in Laminar-Flow Control Using Compliant Walls, In: Developments in Fluid Dynamics and Aerospace Engineering, Indian Institute of Science, Bangalore, India, (1994)

7. Dixon, A.D.; Lucey, A.D.; Carpenter, P.W.: Optimization of Viscoelastic Compliant Walls for Transition Delay, AIAA J., 32, (1994), 256

8. Dowell, E.H.; Ilgamov, M.: Studies in Nonlinear Aeroelasticity, Springer, (1988)

9. Drazin, P.G.; Reid, W.H.: Hydrodynamic Stability, Cambridge University Press, (1982)

10. Duncan, J.H.; Waxman, A.M.; Tulin, M.P.: The Dynamics of Waves at the Interface Between a Viscoelastic Coating and a Fluid Flow, J. Fluid Mech., 158, (1985), 177

11. Evrensel, C.; Kalnins, A.: Response of a Compliant Slab to Inviscid Incompressible Fluid Flow, J. Acoust. Society of America, 78, (1985), 2034

12. Evrensel, C.; Kalnins, A.: Response of a Compliant Slab to Inviscid Incompressible Fluid Flow, Trans. ASME, J. Appl. Mech., 55, (1988), 660

13. Gad-el-Hak, M.: Boundary Layer Interactions with Compliant Coatings: An Overview, Appl. Mech. Reviews, 39, (1986), 511

14. Jaffe, N.A.; Okamura, T.T.; Smith, A.M.O.: Determination of Spatial Amplification Factors and their Application to Predicting Transition, AIAA J., 8, (1979), 301

15. Kaplan, R.E.: The Stability of Laminar Boundary Layers in the Presence of Compliant Boundaries, Ph. D. Thesis, MIT, (1964)

16. Kramer, M.: Boundary Layer Stabilization by Distributed Damping, J. Aeronaut. Sci., 24, (1957), 459

17. Kramer, M.: Boundary Layer Stabilization by Distributed Damping, J. Aeronaut. Sci., 27, (1960), 69

18. Landahl, M.: On the Stability of a Laminar Incompressible Boundary Layer Over a Flexible Surface, J. Fluid Mech., 13, (1962), 609

19. Postelnicu, A.: Contribution to the Study of Interaction Between the Fluid Flows and the Compliant Structures, Ph. D. Thesis, „Transilvania“ University, Brasov (Romania), (1996)

20. Riley, J.J.; Gad-el-Hak, M.; Metcalfe, R.W.: Compliant Coatings, Ann. Review Fluid Mech., 20, (1988), 393

21. Yeo, K.S.: The Stability of Boundary Layers Flow over Single- and Multilayer Viscoelastic Walls, J. Fluid Mech., 196, (1988), 359

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