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Stress and Strain Fields in Cracked Damaged Solids

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The crack tip characteristics are analysed for an damaged elastic solid exhibiting softening. A closed-form solution is constructed for an antiplane crack when the residual stress carried by the material is non-zero and the crack tip behavior is well understood. However, when the residual shear stress is zero, difficulties that arise in the construction of the solution are pointed out.

1 Introduction

An adequate knowledge of the crack tip field is necessary for any fracture analysis. For a power-law hardening plastic material, the crack-tip field possesses the so-called HRR singularity. The crack behavior for softening materials is considered in the present paper.

The problem of an antiplane crack in an elastic-damageable material is investigated in detail. The corresponding nonlinear boundary value problem is solved in closed form within the assumption of small scale damage. This solution is obtained via the hodograph transformation. Materials considered here behave linearly up to a given shear strain, harden up to a peak, soften up to a strain beyond which they support a constant residual shear stress.

Two features of the above behavior are important in the construction of the solution. The first one is the presence of the softening which leads to solutions with discontinuous displacement gradients (Knowles and Sternberg, 1981), the other is the value of the residual shear stress carried by the material. When this residual shear stress is different from zero, a full solution is constructed by matching two solutions corresponding to elliptic and hyperbolic regimes of the field equations respectively. The stress, strain and damage states are explicitly obtained and allow to understand the crack behavior for this type of material and in particular the singularity of the strain field. Also the shape and size of the damaged zones ahead of the crack are related to the loading parameter.

When the residual shear stress is zero, however, the situation is much more complex. Fundamental difficulties arise in the construction of the solution. Indeed, the solution constructed (when the residual shear stress is non zero) blows up in the limit when the residual shear stress tends to zero.

2 Formulation of the Basic Equations

2.1 Statement of the Problem

For convenience, we treat the antiplane deformation problem here. We consider a mode III crack in a homogeneous isotropic elastic-damaged material. The semi-infinite crack occupies the negative part of the x-axis. Referring to the rectangular coordinate system (x, y) shown in Figure 1, all stress and strain components are zero except $\bar{\tau}_{xz} \equiv \bar{\tau}_x$, $\bar{\tau}_{yz} \equiv \bar{\tau}_y$, $\bar{\gamma}_{xz} \equiv \bar{\gamma}_x$ and $\bar{\gamma}_{yz} \equiv \bar{\gamma}_y$, where z is the coordinate axis perpendicular to the plane of Figure 1. The equilibrium equation reduces to (no body forces)

$$\frac{\partial \bar{\tau}_x}{\partial x} + \frac{\partial \bar{\tau}_y}{\partial y} = 0 \tag{1}$$

and the strain-displacement equations are written as

$$\bar{\gamma}_x = \frac{\partial w}{\partial x} \qquad \bar{\gamma}_y = \frac{\partial w}{\partial y} \tag{2}$$

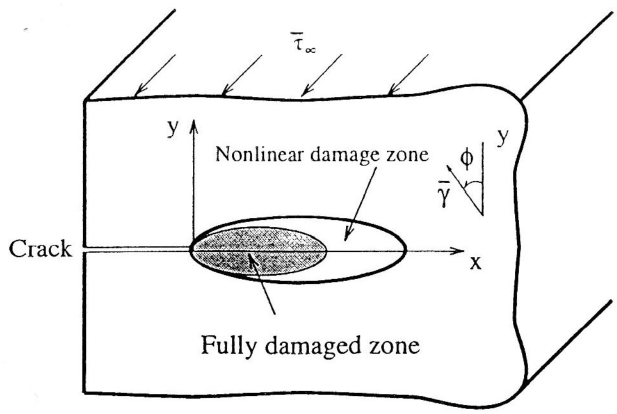


Figure 1. Problem under Consideration

Then we have a single compatibility equation

$$\frac{\partial \bar{\gamma}_x}{\partial_y} = \frac{\partial \bar{\gamma}_y}{\partial_x} \tag{3}$$

The effective stress $\bar{\tau}$ and strain $\bar{\gamma}$ are related to the stress and strain components by

$$\bar{\tau}^2 = \bar{\tau}_x^2 + \bar{\tau}_y^2 \qquad \bar{\gamma}^2 = \bar{\gamma}_x^2 + \bar{\gamma}_y^2$$

The components $\bar{\gamma}_x$ and $\bar{\gamma}_y$ are related to the effective strain $\bar{\gamma}$ by (see inset of Figure 1)

$$\bar{\gamma}_x = -\bar{\gamma}\sin\phi \qquad \bar{\gamma}_y = -\bar{\gamma}\cos\phi \tag{4}$$

We adopt the assumption of proportional loading, so that the components of stress and strain are related by

$$\bar{\tau}_x = \frac{\bar{\tau}(\bar{\gamma})}{\bar{\gamma}}\bar{\gamma}_x \qquad \bar{\tau}_y = \frac{\bar{\tau}(\bar{\gamma})}{\bar{\gamma}}\bar{\gamma}_y \tag{5}$$

3 Elasticity-based Damaged Models

Continuum damage mechanics deals with the load carrying capacity of solids without major cracks but where the material itself is damaged due to the presence of microscopic defects such as microcracks or voids. Here we restrict ourselves to an isotropic damage formulation. The material under consideration is an elastic-damaged softening one. From the hypothesis of strain equivalence the stress tensor σ is given in terms of the strain tensor ε and the damage parameter D by

$$\sigma_{ij} = (1 - D)\mathbf{E}_{ijkl}\,\varepsilon_{kl} \tag{6}$$

where $\mathbf{E}_{ijkl}$ is the fourth-order tensor of the elastic undamaged material. The parameter D reflects the amount of damage which the material has experienced. It starts at zero (sound material) and grows to one (fully damaged material corresponding to complete loss of coherence).

The damage strain energy density is given by

$$Y = \frac{1}{2}\mathbf{E}_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} \tag{7}$$

The damage criterion (or the damage loading function) f depends on Y and the damage variable D

$$f \;=\; f(Y;D)$$

f is such that during progressive damage evolution, the identity $f = 0$ holds, otherwise $f < 0$. The kinetic law of damage evolution is derived from the damage criterion f

$$\dot{D} \;=\; \dot{\lambda}\frac{\partial f}{\partial Y}(Y;D) \tag{8}$$

where $\dot{\lambda}$ is the damage multiplier. In the sequel we firstly limit ourselves to local theory of damage, and secondly we choose for the function f the simple expression

$$f(Y;D) \;\equiv\; \sqrt{Y} \;-\; k(D) \;=\; \sqrt{Y_0} \;+\; M\,D$$

where M is a material constant. The history term $k(D)$ measures the largest value that has been attained by the damage strain energy density release rate Y. Accordingly, it is a non-decreasing function and grows only when $f(Y, D) = 0$. Provided the damage criterion is satisfied, D takes the simple form

$$D \;=\; \frac{\bar{\gamma} \;-\; \bar{\gamma}_o}{\bar{\gamma}_{cr} \;-\; \bar{\gamma}_o} \tag{9}$$

where $\bar{\gamma}_0$ and $\bar{\gamma}_{cr}$ are the yield and critical shear strain, respectively. In the sequel, the material on hand is treated as fully damaged when

$$\bar{\tau} \;=\; \bar{\tau}_u \qquad\qquad D \;=\; \frac{\bar{\gamma}_u \;-\; \bar{\gamma}_o}{\bar{\gamma}_{cr} \;-\; \bar{\gamma}_o} \tag{10}$$

where the material constant $\bar{\gamma}_u \;=\; \frac{1}{2}\left(1 \;+\; \sqrt{1 \;-\; \bar{\tau}_u}\right)$ is the ultimate shear strain beyond which the material is considered fully damaged. At this point we have all ingredients we need to write down the constitutive equations. To this end, it is more convenient, as will become clear later, to introduce the non-dimensional quantities

$$\tau \;=\; 4\gamma_o(1 \;-\; \gamma_o)\frac{\bar{\tau}}{\bar{\tau}_o} \qquad \gamma \;=\; \frac{\bar{\gamma}}{\bar{\gamma}_{cr}} \qquad \gamma_o \;=\; \frac{\bar{\gamma}_o}{\bar{\gamma}_{cr}} \tag{11}$$

The shear stress τ is then given in terms of the shear strain γ by

$$\tau \;=\; \begin{cases} 4(1 \;-\; \gamma_o)\gamma & \text{if } \gamma \;<\; \gamma_o \\ 4(1 \;-\; \gamma)\gamma & \text{if } \gamma_o \;\leq\; \gamma \;<\; \gamma_u \\ \tau_u & \text{if } \gamma_u \;\leq\; \gamma \end{cases} \tag{12}$$

Figure 2 shows the curve (τ, γ) where the peak corresponds to $\gamma = \frac{1}{2}$ and $\tau = 1$. We mention in passing two features of the material considered. First we note that after the peak the stress decreases as the strain increases (softening) and second, when the material is fully damaged the residual shear stress τ_u may go to zero. Indeed, the analysis to follow will be carried out for $\tau_u \neq 0$ and then the limit when $\tau_u \to 0$ will be considered as well. Besides, the previous constitutive relations are only valid for loading cases. We assume in this paper that unloading does not take place.

4 Elliptic and Non-elliptic Solutions

4.1 Hodograph Method

In order to obtain the whole solution to the crack tip fields we use the hodograph transformation first used by Rice (1967) for the problem of a sharp notch in a semi-infinite plane made on with a power-law material and loaded by longitudinal shear. This transformation treats the physical coordinates (x, y) in the physical plane as functions of strain γ_x, γ_y in the hodograph plane. By doing so, the nonlinear problem is reduced to a linear boundary value problem in the hodograph plane. For the following, it is more advantageous to use the polar coordinates system (γ, ϕ) in the hodograph plane shown in Figure 3, where ϕ is the angle, measured positive counterclockwise, between the principal shear strain and the y-axis.

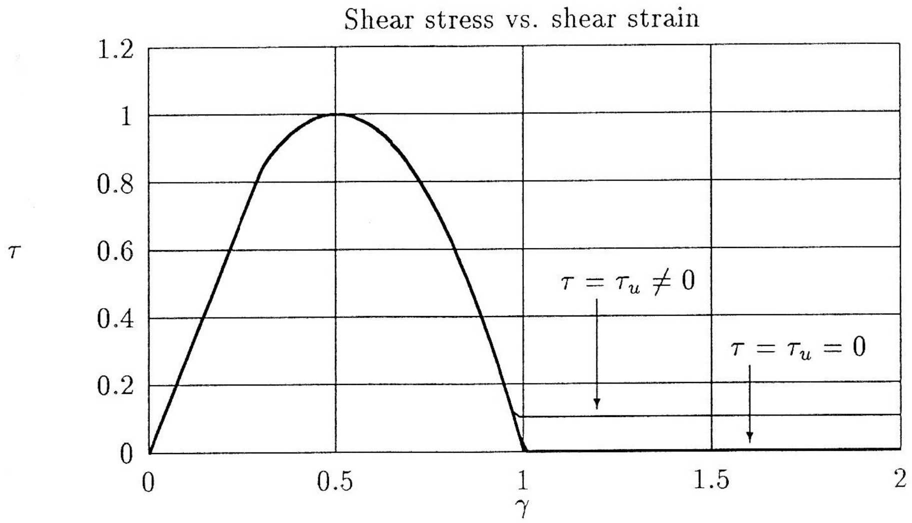


Figure 2. Idealisation of Elastic Damaged Behavior

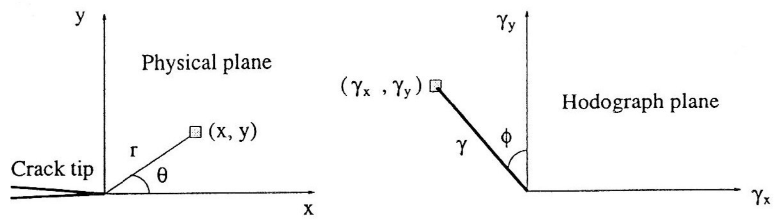


Figure 3. Physical and Hodograph Planes

From the compatibility equation (3), a scalar function $\psi = \psi(\gamma, \phi)$ is found to exist such that the physical coordinates x and y are related to its derivatives by

$$
\begin{aligned}
x &= -\sin\phi\frac{\partial\psi}{\partial\gamma}(\gamma,\phi) - \frac{\cos\phi}{\gamma}\frac{\partial\psi}{\partial\phi}(\gamma,\phi) \\
y &= \cos\phi\frac{\partial\psi}{\partial\gamma}(\gamma,\phi) - \frac{\sin\phi}{\gamma}\frac{\partial\psi}{\partial\phi}(\gamma,\phi)
\end{aligned}
\tag{13}
$$

It can be readily shown that the antiplane displacement w in terms of the potential function $\psi(\gamma, \phi)$ is given by

$$w(\gamma,\phi) = \gamma\frac{\partial\psi}{\partial\gamma}(\gamma,\phi) - \psi(\gamma,\phi) \tag{14}$$

The potential function ψ satisfies (Rice, 1967)

$$\frac{\tau(\gamma)}{\tau'(\gamma)}\frac{\partial^2\psi}{\partial\gamma^2}(\gamma,\phi) + \frac{\partial\psi}{\partial\gamma}(\gamma,\phi) + \frac{1}{\gamma}\frac{\partial^2\psi}{\partial\phi^2}(\gamma,\phi) = 0 \tag{15}$$

and the associate boundary conditions

$$\frac{\partial\psi}{\partial\phi}\left(\gamma,\pm\frac{\pi}{2}\right) = 0 \tag{16}$$

In equation (15) $\tau'(\gamma)$ stands for $\frac{d\tau}{d\gamma}(\gamma)$. Physically, the fully damaged zone is expected to be located in the immediate vicinity of the crack tip, or what amounts to the same thing, the crack tip $x = y = 0$ is the point at infinity in the strain plane. Thus $r \to 0$ as $\gamma \to \infty$.

It must be remembered that the governing differential equation (15) is of mixed elliptic-hyperbolic type and is well known in the theory of transonic flow. Indeed, the type of equation (15) depends on the location of the material point at hand. As outlined for instance by Zhang et al. (1994), equation (15) is elliptic for $\gamma < \frac{1}{2}$ which corresponds to $\tau'(\gamma) > 0$, that is for strain hardening materials, and hyperbolic for $\frac{1}{2} < \gamma$ which corresponds to $\tau'(\gamma) < 0$, that is for strain softening materials.

4.2 Construction of Solutions

A particular solution ψ of the governing equation (15) which meets the boundary conditions $\frac{\partial\psi}{\partial\phi}\left(\gamma,\pm\frac{\pi}{2}\right) = 0$ can be constructed through a separation of variables of the form

$$\psi(\gamma,\phi) = f(\gamma)\sin\phi \tag{17}$$

By means of equation (15) the unknown function f satisfies the ordinary differential equation

$$\gamma\frac{\tau(\gamma)}{\tau'(\gamma)}f''(\gamma) + \gamma f'(\gamma) - f(\gamma) = 0 \tag{18}$$

where (′) and (″) stand for the first and second derivative of f with respect to γ. The fully damaged zone is expected to be located in the immediate vicinity of the crack tip. Thus $r \to 0$ as $\gamma \to \infty$. It ensues from

$$r = \left[f'(\gamma)\sin^2\phi + \frac{1}{\gamma^2}f^2(\gamma)\cos^2\phi\right]^{\frac{1}{2}}$$

that $f(\gamma)$ is such that

$$f'(\gamma) \to 0 \qquad \text{as} \qquad \gamma \to \infty \tag{19}$$

to which we may add another condition to make f definite. It is straightforward to establish by substitution of equation (17) into equation (13) that the physical coordinates $x(\gamma,\phi)$ and $y(\gamma,\phi)$ are given in terms of γ and ϕ by

$$x = X(\gamma) + R(\gamma)\cos 2\phi \qquad y = R(\gamma)\sin 2\phi \tag{20}$$

with

$$R(\gamma) = \frac{1}{2}\left[f'(\gamma) - \frac{1}{\gamma}f(\gamma)\right] \qquad X(\gamma) = -\frac{1}{2}\left[f'(\gamma) + \frac{1}{\gamma}f(\gamma)\right] \tag{21}$$

By means of equation (14) the antiplane displacement $w(\gamma, \phi)$ may be put in the form

$$w(\gamma, \phi) = 2\gamma R(\gamma)\sin\phi \tag{22}$$

The geometrical interpretation of equations (20) is immediate: the lines along which γ and $\tau(\gamma)$ have constant and positive values are circles with radius $R(\gamma)$ and centered on the x-axis at the abscissa $X(\gamma)$. The solution of equation (18) imposes to treat separately the types of behavior that a material point could meet. Let us indicate some striking features of the iso-strain circles inside the yet undetermined elastic, damaged and fully damaged zones. In the sequel, the subscripts *e, d* and *f* for the function *f* stand for elastic, damaged and fully damaged, respectively.

1. **Solution in the Elastic Zone**

   Far away from the crack tip the material is elastic and therefore the potential ψ is harmonic, that is $\Delta_\gamma \psi(\gamma, \phi) = 0$. The corresponding solution is that given by Rice (1967)

$$f_e(\gamma) = -A\left(\frac{1}{\gamma} + B\gamma\right) \tag{23}$$

   with $A = \dfrac{1}{2\pi}\left(\dfrac{K\gamma_o}{\tau_o}\right)^2$, $K = \tau_\infty\sqrt{\pi a}$ and B will be determined later on. Thus, inside the elastic region characterized by $\gamma < \gamma_o$ the iso-strain circles are such that

$$R_e(\gamma) = \frac{A}{\gamma^2} \qquad X_e(\gamma) = AB \tag{24}$$

   Figure 4(a) represents the iso-strain circles corresponding to the elastic zone in the physical plane normalized by the load parameter A. These circles are concentric since $X_e(\gamma)$ is independent of γ.

2. **Solution in the Damaged Zone**

   In the circumstances where $\gamma \geq \gamma_o$, a solution of equation (18) is

$$f_d(\gamma) = C\gamma\int_{\gamma_o}^{\gamma}\frac{du}{u^2\,\tau(u)} + D\gamma \tag{25}$$

   as can readily checked by substitution. The constants C and D are, as will be seen later, related to the constants A and B associated to the elastic solution through the continuity requirement of the physical coordinates $x(\gamma, \phi)$ and $y(\gamma, \phi)$ with respect to γ. Inside the damaged zone, where $\gamma \geq \gamma_o$ the iso-strain lines are, once again, circles whoose radii and centers are respectively given by

$$R_d(\gamma) = \frac{C}{8}\frac{1}{\gamma^2(1-\gamma)}$$

$$X_d(\gamma) = \frac{C}{8}\left\{\frac{(1 - 2\gamma)}{\gamma(1 - \gamma)} + 2\ln\frac{\gamma_o(1 - \gamma)}{\gamma(1 - \gamma_o)} - \frac{1 + 2\gamma_o}{\gamma_o^2}\right\} - D \tag{26}$$

3. **Solution in the Fully Damaged Zone**

   Inside the yet undetermined fully damaged zone for which $\gamma_u \leq \gamma$ and $\tau = \tau_u$, a solution of equation (18) is readily obtained

$$f_f(\gamma) = E\gamma + F \tag{27}$$

   where E and F are constants to be determined. The requirement $f_f'(\gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$ imposes $E = 0$. The constant F will be determined later on from the continuity conditions. Consequently in the (x, y) plane corresponding to the fully-damaged zone we have

$$R_f(\gamma) = -\frac{F}{2\gamma} \qquad\qquad X_f(\gamma) = R_f(\gamma) \qquad (28)$$

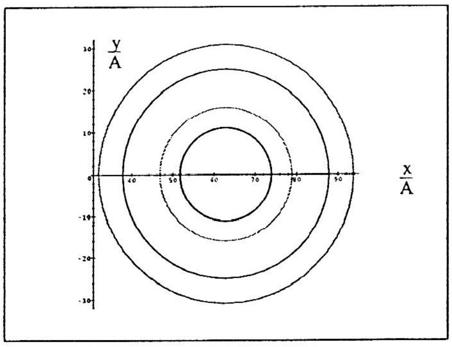
As expected we find that this sub-family of circles are tangent to the y-axis (Figure 4(c)).

We note that for a point located in the neighbourhood of the crack tip, there may exist three iso-strain circles which pass through that point (Figure (4d)). To determine the actual solution, we must account for the matching conditions which in turn impose some constraints. This is the object of the following subsection.

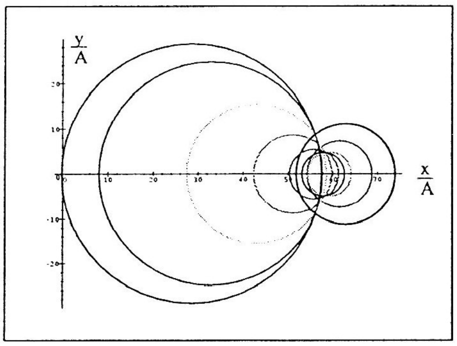
4.3 Matching Conditions

So far we have formally solved the governing equation (15). It remains to specify the constants of integration B, $\frac{C}{A}$, $\frac{D}{C}$ and $\frac{F}{C}$ and the domain of the physical plane where the solutions obtained are valid. First of all, the physical coordinates $x(\gamma,\ \phi)$ and $y(\gamma,\ \phi)$ must be continuous functions with respect to their arguments. Elsewhere, the actual boundaries separating the different zones are characterized by two others constraints

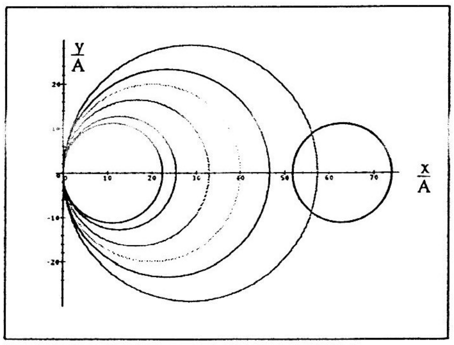
- $w(\gamma,\ \phi)$ as defined by equation (22) is continuous
- the tractions too are continuous accross these boundaries.



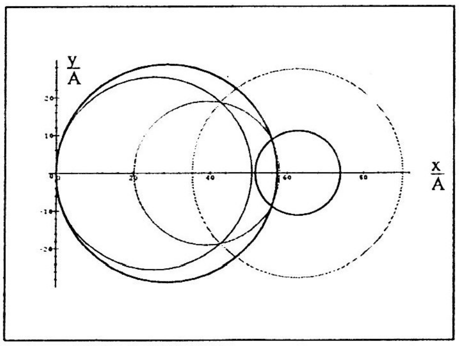
(a) Iso-strain circles in elastic region: $\gamma \leq \gamma_o$



(b) Iso-strain circles in damaged region: $\gamma_o \leq \gamma \leq \gamma_u$



(c) Iso-strain circles in fully damaged region: $\gamma_u \leq \gamma$. Also represented the circle $\gamma = \gamma_o$



(d) Three iso-strain circles pass through some points located in the vicinity of the crack tip

Figure 4. Iso-strain Circles in Different Regions

1. **Continuity of the Physical Coordinates**
The requirement of the continuity of the physical coordinates $x(\gamma, \phi)$ and $y(\gamma, \phi)$ with respect to γ imposes the relations

$$
\begin{aligned}
\frac{C}{A} &= 2\frac{\tau_o}{\gamma_o} = 8(1 - \gamma_o) \\
\frac{D}{C} &= -\frac{1}{\gamma_u \tau_u} - \frac{1}{8}\left[\frac{(1 + 2\gamma_o)}{\gamma_o^2} - \frac{(1 + 2\gamma_u)}{\gamma_u^2} + 2\ln\frac{\gamma_u(1 - \gamma_o)}{\gamma_o(1 - \gamma_u)}\right] \\
B &= -8(1 - \gamma_o)\frac{D}{C} - \frac{1}{\gamma_o^2} \\
\frac{F}{C} &= -\frac{1}{\tau_u}
\end{aligned}
\tag{29}
$$

It is worth noting that

$$
\lim_{\tau_u \to 0} B = +\infty \tag{30}
$$

which means that the centers of the different iso-strain circles corresponding to elastic solutions are extended to infinity. In that case, the small damage assumption is violated. The interface between the damaged zone and the elastic one is the arc of the iso-strain circle $\gamma = \gamma_o$ limited by the intercepts of both circles $\gamma = \gamma_o$ and $\gamma = \gamma_u$.

2. **Continuity of the Displacement and the Tractions**
We turn now to the question of the determination of the interfaces between the different zones around the crack tip where we have at our disposal two or three solutions. These zones which differ one from the other by the level of damage suffered, meet at a common boundaries $\mathcal{S}$ which can be determined from the continuity of the displacement $w(\gamma, \phi)$ and the traction $\tau(\gamma, \phi)$. The shape of $\mathcal{S}$ is not known beforehand except the fact that $\mathcal{S}$ is symmetric with respect to the x-axis. Besides, the curve is expected to be composed with much branches since different material behaviors are met in the neighborhood of the crack tip, and hence the type of solution changes as the branches are crossed.

One of the salient features of the previous geometrical comments is that in the vicinity of the crack tip there exist three or two solutions, depending on the point considered. The number of solutions is exactly the same as the number of iso-strain circles passing through that point. In some regions of the physical plane (x, y) there exists only one iso-strain circle passing through a given point. Hence the solution at that point is completely determined. This is the case, for instance, for remote points for which the material behavior is still elastic. However, as it is the case in the immediate vicinity of the crack tip, there are two or three iso-strain circles which pass through a given point. The actual solution is the one for which the displacement $w(\gamma, \phi)$ and the corresponding traction $\tau(\gamma, \phi)$are continuous across the curve $\mathcal{S}$, yet unknown, which separate two adjacent zones. The expression of the displacement $w(\gamma, \phi)$ is readily obtained from the combination of equations (22) and (20)

$$
w(\gamma, \phi) = \gamma\left[2R(\gamma)\right]^{\frac{1}{2}}\left[X(\gamma) + R(\gamma) - x\right]^{\frac{1}{2}} \tag{31}
$$

A parametric representation of the curve $\mathcal{S}$ may be obtained through the use of the Knowles and Sternberg'procedure outlined in Knowles and Sternberg (1981). For the sake of brevity, this procedure will not be repeated here.

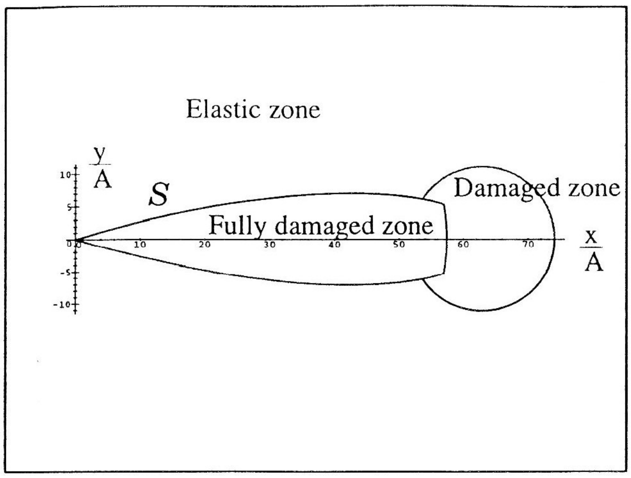


Figure 5. The Shock Curve S together with the Iso-strain Circles $\gamma = \gamma_0$, $\gamma = \frac{1}{2}$, and $\gamma = \gamma_u$

The location of the different zones are displayed graphically in Figure 5. Note in passing that the different branches of the curve S, obtained separately, fit continuously one to each other. Regarding the check of the continuity of the traction accross the curve S, the proof is the same as the one given by Knowles and Sternberg (1981).

The shape for the different zones is shown in Figure 5. The damaged zone is composed with both hardening and softening damaged zones. The foregoing results do not account for unloading cases, although physically plausible. Zhang et al. (1994) have shown that unloading zones prevail with a reduced tangent of the boundary S at the crack tip.

5 The Crack Tip Field Characteristics

One may now determine the nature of the crack tip singularity. Of some interest for fracture mechanics are the fields directly ahead of the crack tip on the x-axis where $\phi = 0$. Inside the fully damaged zone where $\tau = \tau_u$ and which originates at the crack tip, the relations (20) combined with equations (28) simplify to

$$x = X_f(\gamma) + R_f(\gamma) = 2R_f(\gamma) = -\frac{F}{\gamma}$$

which in turn implies that

$$\gamma = \frac{8(1 - \gamma_o)}{\tau_u\left(\frac{x}{A}\right)} \tag{32}$$

To find γ_y $(x, 0)$ and τ_y $(x, 0)$ use will be made of equations (4) and (5) with $\phi = 0$

$$\gamma_y(x,0) = \frac{8(1 - \gamma_o)}{\tau_u\left(\frac{x}{A}\right)} = K^2\left(\frac{\gamma_o}{\tau_o}\right)^2\frac{4(1 - \gamma_o)}{\tau_u}\frac{1}{x}$$

$$\tau_y(x,0) = \tau_u \tag{33}$$

This is an expected result since for $\gamma \geq \gamma_u$ the stresses are constant and equal to τ_u, as for perfect plasticity. Notice in passing from the formula $(33)_1$ that the shear strain γ_y $(x, 0)$ tends to infinity when the residual shear stress τ_u tends to zero. In fact, in this circumstance the solution obtained blows up and is no longer valid since in particular, the small scale damage assumption is violated.

6 Concluding Remarks

While a complete solution has been obtained for the antiplane crack in an elastic damaged solid, difficulties arise when the residual shear stress carried by the material is zero. These difficulties seem to be related to the absence of a singularity (for stress and strain) and to the idealisation of the crack as a line. To overcome these difficulties, two lines are followed: modelling of a crack as a damaged zone with a finite thickness in the spirit of the work by Bui and Ehrlacher (1980) and introduction of nonlocal effects.

Literature

1. Bui, H. D.; Ehrlacher, A.: Propagation dynamique d'une zone endommagée dans un solide élastique en mode III et en régime permanent. Comtes Rendues de l'Académie des Sciences, t. 290:273-276, (1980).

2. Knowles, J. K.; Sternberg, E.: Anti-plane shear fields with discontinuous deformation gradients near the tip of a crack in finite elastostatics. Journal of Elasticity, 11:129-164, (1981).

3. Lemaitre, J. A.: A Course in Damage Mechanics. Springer-Verlag, Berlin, (1992).

4. Rice, J. R.: Stresses due to a sharp notch in a work-hardening elastic-plastic material loaded by longitudinal shear. Journal of Applied Mechanics, 32:287-298, (1967).

5. Zhang, X. T.; Hao, T. H.; Hwang, K. C.: Unloading characteristics of antiplane shear crack in softening material. Theoretical and Applied Fracture Mechanics, 21:233-240, (1994).

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