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Optimal Design of Rotationally-Symmetric Disks in Thermo-Damage Coupling Conditions

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This paper demonstrates a concept of coupling between the rheological and the thermal properties of a material, applied to axisymmetric problems. The damage tensor, which appears in the constitutive equations of creep and damage and, on the other hand, in the combined heat flow-radiation rule, plays a role of the variable introducing coupling. As example, an axisymmetric disk subjected to creep damage under combined mechanical and thermal loadings is considered. The concept of uniform creep strength is extended to optimal design of disks when coupling between the mechanical and the thermal fields is taken into account.

1 Introduction

Creep process and associated material deterioration are temperature sensitive. The classical approach consists in accounting for the effect of temperature on the material constants in constitutive equations of creep and creep damage. When more advanced approach is used, the thermo-damage coupling is required in order to take into account changes of temperature field, caused by deterioration, and vice versa.

1.1 Brittle Rupture Mechanics of Axisymmetric Disks and Plates Subject to Creep under Mechanical and Thermal Loading

Recently, on the basis of continuum damage mechanics, authors have analysed creep rupture mechanisms in axisymmetric disks and plates subject to thermal and mechanical loadings. Ganczarski and Skrzypek (1992) consider clamped annular disk subject to creep under radial tension, body forces due to rotation, and the temperature gradient. The orthotropic damage growth rule is combined with the flow theory and the time hardening hypothesis, in order to analyse failure mechanisms in the disks of a jump-like variable thickness. Influence of the initial prestressing on the localization of first macrocracks in disks of a constant or a variable thickness is examined in Ganczarski and Skrzypek (1992).

The extension to the case of both membrane-bending states, when an axisymmetric sandwich plate of a constant thickness is initially thermally prestressed by the elastic ring or by the cylindrical shell, is done in Ganczarski and Skrzypek (1993). In case of a plate fitted into the cylindrical shell, depending on the prestressing parameters two failure mechanisms are distinguished, when either the circumferential fibres in the central zone of the interior layer, or the radial fibres along the periphery of the exterior layer, suffer first macrocracks. A general formulation in the case of prestressed plates of variable thickness is developed in Ganczarski and Skrzypek (1994). The orthotropic damage growth rule is coupled with the isotropic or the orthotropic creep law (weak or strong creep-damage coupling). The unilaterally coupled Kármán system extended to the case of visco-elastic plates of variable thickness is used to describe membrane-bending states. An additional membrane-bending coupling is due to boundary excitation, when force or displacement type, and membrane or bending type plate prestressing is considered. Further generalization consists in accounting for the effect of transverse shear due to the Reissner's theory of thick plates (cf. Ganczarski at al. (1997)). The shear effect causes, in general, rotation of the principal directions of the damage tensor following the principal directions of tension. Therefore, this formulation requires to associate all constitutive relationships with a corrotational coordinate system and use an objective measure of damage rate tensor.

1.2 Representative Formulations of Optimal Design Problems of Structures under Creep Conditions

Optimization problems under creep conditions have been formulated and classified by Życzkowski (1988). Generally, constraints may be imposed on brittle, ductile or mixed rupture, strength, lifetime, creep stiffness or compliance, residual displacement, stress relaxation, creep buckling, dynamic response, to mention the most important cases. On the basis of continuum damage mechanics the structures of uniform creep strength may be defined (local optimality criterion). When brittle rupture is considered the first macrocracks appear simultaneously either everywhere or, at least, along appropriate lines or surfaces to produce a rupture mechanism (cf. Skrzypek and Egner, 1993; Ganczarski and Skrzypek, 1994). When optimal design with respect to ductile rupture is sought the structures of uniform initial strength or uniform deformability may be defined, where initial principal stress components are equal throughout the structure or principal strain components are equalized at each time step (cf. Szuwalski, 1989).

In general, with geometry changes or non-stationary loadings taken into account, neither structures of uniform creep strength nor structures of uniform deformability are optimal with respect to lifetime. This problem is discussed by Skrzypek and Egner (1993), and by Ganczarski and Skrzypek (1994). In such cases the optimization should be formulated on a basis of a global criterion (maximum lifetime, minimum residual displacement, etc), rather than a local one. On the other hand, under thermal loading, when a brittle rupture mechanism affects the stress relaxation phenomenon, two opposite behaviours result: the decrease of nominal stresses in time due to stress relaxation, and the increase of net stresses with the accumulation of creep damage. Depending on the test temperature the question arises whether the brittle failure occurs at a finite stress level at limited lifetime, or a complete stress relaxation takes place when the lifetime tends to infinity (cf. Bråthe, 1976). Hence, it is necessary to include into an analysis a proper thermo-damage coupling, when a temperature field affects the creep-rupture data and an accumulation of damage influences the temperature gradient by the modified heat transfer law.

2 Basic Equations

2.1 Equation of Heat Transfer through Partially Damaged Solid

Consider an uniaxial representative volume element $dV = \bar{h} r d\theta dr$ (where $\bar{h} = h + \gamma dh$, $\gamma \in (0;1]$ is the mean thickness) as an infinitesimal element which affects brittle rupture at elevated temperature (Figure 1).

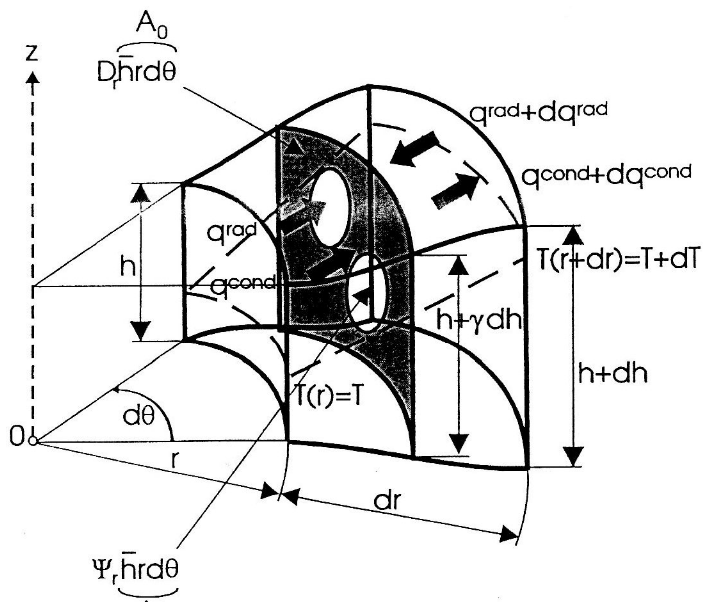

Figure 1. Schematics of Heat Transfer through Partly Damaged Cylindrical RVE

An actual state of damage of the element is determined by the damage parameter D_r (Murakami, 1987). Hence, we can easily interpret the products $D_r\overline{h}rd\theta$ and $\Psi_r\overline{h}rd\theta$ as the damaged and the undamaged portions of the elementary cross section area $\overline{h}rd\theta$ respectively. Due to the dual nature of a partially damaged cross section the total heat flow rate needs to be decomposed into two parts: the classical Fourier conductivity through the undamaged portion of the cross section, and the Stefan-Boltzmann radiation through the damaged portion. The concept of an equivalent coefficient of thermal conductivity λ^{eq} (cf. Ganczarski and Skrzypek, 1997; Skrzypek and Ganczarski, 1998) requires, first of all, to compare the heat flux of radiation through a partially damaged cross section and the heat flux of conductivity through a fictitious undamaged cross section:

$$\sigma\epsilon_o[D_r A_0 T^4(r) \quad - \quad D_r A_0 T^4(r+\Delta r)] \quad = \quad -\Delta\widetilde{\lambda}^{rad}\mathrm{d}A_0\frac{\partial T}{\partial r} \tag{1}$$

Next, expanding the temperature $T(r+\Delta r)$ into a Taylor series around r,

$$T(r) \quad = \quad T \qquad\qquad T(r+\Delta r) \quad = \quad T + \frac{\partial T}{\partial r}\Delta r + \ldots \tag{2}$$

and substituting into equation (1) we have

$$\sigma\epsilon_o\left[D_r T^4\overline{h}r\mathrm{d}\theta \quad - \quad D_r\left(T^4 + 4T^3\frac{\partial T}{\partial r}\Delta r + \ldots\right)\overline{h}r\mathrm{d}\theta\right] \quad = \quad -\Delta\widetilde{\lambda}^{rad}\overline{h}r\mathrm{d}\theta\frac{\partial T}{\partial r} \tag{3}$$

Neglecting higher order terms, the substitutive coefficient of thermal conductivity in pseudoundamaged material $\Delta\widetilde{\lambda}^{rad}$, responsible for the radiation in the damaged material, is expressed by the formula

$$\Delta\widetilde{\lambda}^{rad} \quad = \quad \sigma\epsilon_o 4 D_r T^3\Delta r \tag{4}$$

Therefore, the corresponding equation of uniaxial heat transfer takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda^{eq}\frac{\partial T}{\partial r}\right) \quad + \quad \frac{1}{h}\frac{\partial h}{\partial r}\lambda^{eq}\frac{\partial T}{\partial r} \quad + \quad \dot{q}_v \quad = \quad c_v\varrho\dot{T} \tag{5}$$

where

$$\lambda^{eq} \quad = \quad \widetilde{\lambda} \quad + \quad \Delta\widetilde{\lambda}^{rad} \qquad\qquad \widetilde{\lambda} \quad = \quad \lambda_o(1-D_r) \tag{6}$$

In what follows, the process of growth of microcracks in a real material is assumed to be quasi-static, therefore, the simplified homogeneous form of equation(5) may be considered.

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\lambda^{eq}\frac{\mathrm{d}T}{\mathrm{d}r}\right) \quad + \quad \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\lambda^{eq}\frac{\mathrm{d}T}{\mathrm{d}r} \quad + \quad \dot{q}_v \quad = \quad 0 \tag{7}$$

where

$$\lambda^{eq} \quad = \quad \lambda_o(1-D_r) \quad + \quad \sigma\epsilon_o 4 D_r T^3\Delta r \tag{8}$$

2.2 General Concept of Thermo-Damage Coupling

The essence of thermo-damage coupling is to develop the reciprocal relationship between processes of creep, microcrack growth and evolution of thermal field.

DAMAGE MEASURES	
Scalar	Orthotropic
CREEP - DAMAGE COUPLING	
$\dot{\varepsilon}^c_{kl} = \frac{3}{2}\frac{\dot{\varepsilon}^c_i}{\sigma_i} s_{kl}$	$\dot{\varepsilon}^c_{kl} = \frac{3}{2}\frac{\dot{\varepsilon}^c_i}{\sigma_i^{net}} s^{net}_{kl}$
$\dot{\varepsilon}^c_i = \left(\frac{\sigma_i}{1-D}\right)^m \dot{f}(\tau)$	$\dot{\varepsilon}^c_i = \left(\sigma_i^{net}\right)^m \dot{f}(\tau)$
$\dot{D} = C\left\langle \frac{\sigma_1}{1-D}\right\rangle^n$	$\dot{D}_k = C_k\left\langle \frac{\sigma_k}{1-D_k}\right\rangle^{n_k}$
THERMO - DAMAGE COUPLING	
$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda^{eq}(r,\theta,z;\tau)\frac{\partial T}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\lambda^{eq}(r,\theta,z;\tau)\frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[\lambda^{eq}(r,\theta,z;\tau)\frac{\partial T}{\partial z}\right] + \dot{q}_v = c_v\varrho\dot{T}$	$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda^{eq}_k(r,\theta,z;\tau)\frac{\partial T}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\lambda^{eq}_k(r,\theta,z;\tau)\frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[\lambda^{eq}_k(r,\theta,z;\tau)\frac{\partial T}{\partial z}\right] + \dot{q}_v = c_v\varrho\dot{T}$
$\lambda^{eq}(r,\theta,z;\tau) = \tilde{\lambda}(r,\theta,z;\tau) + \Delta\tilde{\lambda}^{rad}(r,\theta,z;\tau)$	$\lambda^{eq}_k(r,\theta,z;\tau) = \tilde{\lambda}_k(r,\theta,z;\tau) + \Delta\tilde{\lambda}^{rad}_k(r,\theta,z;\tau)$
$\tilde{\lambda}(r,\theta,z;\tau) = \lambda_o(r,\theta,z)\left[1 - D(r,\theta,z;\tau)\right]$	$\tilde{\lambda}_k(r,\theta,z;\tau) = \lambda_o(r,\theta,z)\left[1 - D_k(r,\theta,z;\tau)\right]$
$\Delta\tilde{\lambda}^{rad} = \sigma\epsilon_o 4DT^3(\Delta r + r\Delta\theta + \Delta z)$	$\Delta\tilde{\lambda}^{rad}_k = \sigma\epsilon_o 4D_kT^3(\Delta r + r\Delta\theta + \Delta z)$

Table 1. Concept of Coupling between Creep, Damage and Thermal Properties

Two of the proposals of coupling, based on the flow rule, the time hardening hypothesis and a modified equation of heat transfer are presented in Table 1 (Ganczarski and Skrzypek, 1995, 1997; Kachanov, 1986; Tanigawa, 1995; Skrzypek and Ganczarski, 1998). Two consistent formulations, the scalar as well as the orthotropic, are demonstrated. The orthotropic case results from the general tensorial formulation, when the principal directions of the damage tensor are materially stationary and coincide with the principal directions of stress orthotropy (no shear effects taken into account). On the other hand, in order to take into account the effect of deterioration on the distribution of the thermal field, an extension of the Kassir concept of a thermally nonhomogeneous body to the case of time-dependent thermal non-homogeneity is suggested (Tanigawa, 1995). It is postulated, that the evolution of coefficients of thermal conductivity and the phenomenon of radiation in an extended equation of heat transfer through a partly damaged body are associated with corresponding components of the damage tensor (Ganczarski and Skrzypek, 1995, 1997, 1998).

2.3 General Equations of the Mechanical State

Let us consider the general formulation of an axisymmetric plane stress problem. The geometrically linear theory of small displacements and decomposition of total strains into elastic, creep, and thermal parts: $\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{c}} + \varepsilon^{\mathrm{th}}$ are assumed. The general mixed approach, originally derived for the plate under combined membrane-bending state, where the equation for the membrane state is written by the use of the Airy function, whereas the equation for the bending state is written by the use of the appropriate deflection function (cf. Ganczarski, Skrzypek, 1994), is applied. When membrane-bending equations are reduced to the case of the disk and the following definition of the Airy function $n_r = (F'/r) + U$, $n_\theta = F'' + U$, and potential of body forces $U' = -\varrho\omega^2 rh$ are introduced, where prime symbol stands for the derivative with respect to r, we obtain

$$\begin{aligned} &\mathcal{F}[F] + (1-\nu)\mathcal{B}(r)\nabla^2\left[\frac{U}{\mathcal{B}(r)}\right] + (1-\nu^2)\mathcal{B}(r)\alpha\nabla^2 T = 0 \Big\} && \text{for } \tau = 0 \\ &\mathcal{F}[\dot{F}] + (1-\nu^2)\mathcal{B}(r)\alpha\nabla^2\dot{T} + \mathcal{B}(r)\nabla^2\left[\frac{\dot{n}^c_\theta - \nu\dot{n}^c_r}{\mathcal{B}(r)}\right] + \frac{1+\nu}{r}\mathcal{B}(r)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\dot{n}^c_\theta - \dot{n}^c_r}{\mathcal{B}(r)}\right] = 0 \Big\} && \text{for } \tau > 0 \end{aligned} \quad (9)$$

where the differential operator $\mathcal{F}[...]$ as well as the auxiliary operators ∇^2, ∇^4, which are independent of the circumferential coordinate, take the form

$$\mathcal{F}[...] = \nabla^4 + \mathcal{B}(r)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{\mathcal{B}(r)}\right]\left(2\frac{\mathrm{d}^3...}{\mathrm{d}r^3} + \frac{2-\nu}{r}\frac{\mathrm{d}^2...}{\mathrm{d}r^2} - \frac{1}{r^2}\frac{\mathrm{d}...}{\mathrm{d}r}\right) + \mathcal{B}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[\frac{1}{\mathcal{B}(r)}\right]\left(\frac{\mathrm{d}^2...}{\mathrm{d}r^2} - \frac{\nu}{r}\frac{\mathrm{d}...}{\mathrm{d}r}\right)$$
$$\nabla^2... = \frac{\mathrm{d}^2...}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}...}{\mathrm{d}r} \qquad \nabla^4... = \frac{\mathrm{d}^4...}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3...}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2...}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}...}{\mathrm{d}r} \tag{10}$$

The inelastic membrane forces expressed in terms of inelastic strains and membrane stiffness are defined as follows:

$$n^c_{r/\theta} = \mathcal{B}(r)(\varepsilon^c_{r/\theta} + \nu\varepsilon^c_{\theta/r}) \qquad \mathcal{B}(r) = \frac{E(r)h(r)}{1-\nu^2} \tag{11}$$

2.4 Constitutive Equations for Coupled Creep-damage Problem

In general, as loadings are nonproportional, an orthotropic damage rule causes the creep process to be orthotropic as well (damage included creep orthotropy). Hence, the strong formulation of the creep damage coupling, where net-stress components are used instead of simple stress components, and the time hardening hypothesis governs the creep strain-rate intensity, is required (cf. discussion by Ganczarski and Skrzypek, 1994):

$$\dot{\varepsilon}^c_{kl} = \frac{3}{2}\frac{\dot{\varepsilon}^c_i}{\sigma^{net}_i}s^{net}_{kl} \qquad \dot{\varepsilon}^c_i = \left(\sigma^{net}_i\right)^{m(T)}\dot{f}(\tau) \qquad s^{net}_{r/\theta} = \frac{2}{3}\left(\frac{\sigma_{r/\theta}}{1-D_{r/\theta}} - \frac{\sigma_{\theta/r}}{2(1-D_{\theta/r})}\right) \qquad k,l = r,\theta \tag{12}$$

The orthotropic damage-growth rule is applied to describe damage accumulation (Kachanov, 1986):

$$\dot{D}_k = C_k(T)\left\langle\frac{\sigma_k}{1-D_k}\right\rangle^{n_k(T)} \tag{13}$$

Symbol $f(\tau)$ denotes a given time function, $\langle\rangle$ Macauley bracket, whereas the strain rate intensity, and the net stress intensity, are defined by the following formulae (Ganczarski and Skrzypek, (1991)):

$$\dot{\varepsilon}^c_i = \sqrt{\frac{2}{3}\dot{e}^c_{kl}\dot{e}^c_{kl}} \qquad \sigma^{net}_i = \sqrt{\left(\frac{\sigma_r}{1-D_r}\right)^2 + \left(\frac{\sigma_\theta}{1-D_\theta}\right)^2 - \left(\frac{\sigma_r}{1-D_r}\right)\left(\frac{\sigma_\theta}{1-D_\theta}\right)} \tag{14}$$

The orthotropic damage is described, in general, by different material functions $C_k(T)$, $n_k(T)$ and independently cumulating principal components of the damage tensor D_k. In what follows we consider material isotropy $C_r = C_\theta = C$ and $n_r = n_\theta = n$, but admit for independent evolution of microcracks for both principal directions D_r, D_θ. The other problem arises when the temperature dependence of creep rupture functions $m(T)$, $C(T)$, $n(T)$, which introduces material nonhomogeneity in the inelastic range, is considered (Ganczarski and Skrzypek, (1991)). Following Ganczarski and Skrzypek (1991) it is required in the present paper the quantities $m(T)$, $n(T)$ to be linearly interpolated, whereas $C(T)$, which strongly depends on local temperature, logarithmically interpolated.

2.5 Formulation of Thermo-Mechanical Boundary Problems

Suppose a disk of variable thickness, which is thin enough to assure the plane state of stress, is considered. The disk is subject to constant (in time) temperature at the outer edge, its upper and lower faces are cooled by a stream of fluid, and a centrifugal body force as well as stretching by a constant force at the edge, are applied (Figure 2).

The mechanical state fulfills equations (9) and the following boundary conditions:

$$\begin{array}{lll} n_r(0) = n_\theta(0) & n_r(R) = p_o h_o & \text{for } \tau = 0 \\ \dot{n}_r(0) = \dot{n}_\theta(0) & \dot{n}_r(R) = 0 & \text{for } \tau > 0 \end{array} \tag{15}$$

Radial stress is always predominant and non-negative, hence, the corresponding radial component of the damage tensor D_r plays an essential role in the equation of heat transfer. The equation of heat transfer (10) requires an explicit formula for the inner heat source intensity:

$$\dot{q}_v \stackrel{\text{def}}{=} -\frac{\dot{Q}_v}{\mathrm{d}V} = -\frac{\dot{Q}_v}{r\mathrm{d}\theta\bar{h}\mathrm{d}r} \tag{16}$$

where the surface element and its slope are

$$\mathrm{d}A = \frac{r\mathrm{d}\theta\mathrm{d}r}{\cos\vartheta} \qquad \cos\vartheta = \frac{1}{\sqrt{1+\tan^2\vartheta}} = \frac{1}{\sqrt{1+(\mathrm{d}h/\mathrm{d}r)^2}} \tag{17}$$

To express the overall effect of convection through both disk faces, the classical Newton law of cooling is applied (Holman, 1990):

$$\dot{Q}_v = 2\beta\mathrm{d}A(T-T_\infty) \tag{18}$$

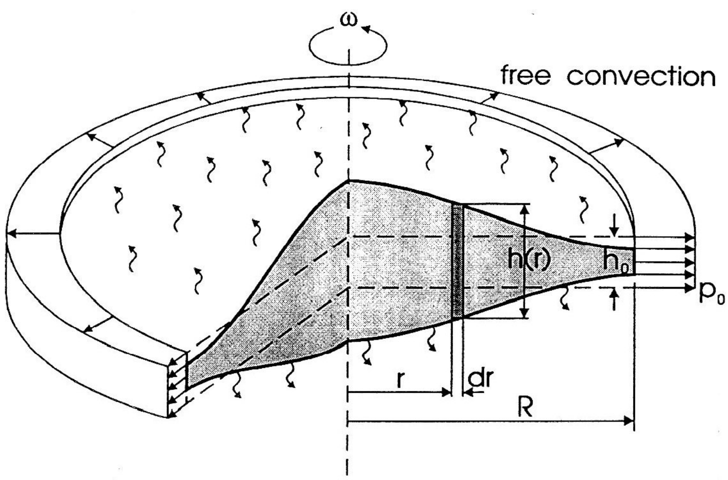

Figure 2. Rotating Disk of Variable Thickness (versus Constant Thickness Disk of the Same Volume) Streched at Periphery and Cooled through Faces

It furnishes the final formula for the inner heat source intensity

$$\dot{q}_v = -2\beta\frac{\sqrt{1+(\mathrm{d}h/\mathrm{d}r)^2}}{\bar{h}}(T-T_\infty) \tag{19}$$

The appropriate boundary conditions of the thermal problem are

$$\begin{array}{lll} \mathrm{d}T/\mathrm{d}r|_{r=0} = 0 & T(R) = T_o & \text{for } \tau = 0 \\ \mathrm{d}\dot{T}/\mathrm{d}r\Big|_{r=0} = 0 & \dot{T}(R) = 0 & \text{for } \tau > 0 \end{array} \tag{20}$$

In case of an initial elastic solution ($\tau = 0$, $D_r \equiv 0$) and a constant disk thickness ($h(r) \equiv h_o$) the boundary problem of heat transfer (7), (20) can easily be recognized as the classical cylindrical problem of the associated Bessel equation of order zero.

3 Problem of Optimization

3.1 Optimality Criterion

Structures of the uniform creep strength must satisfy the following local condition:

$$C(n+1)\int_0^{\tau_I}\left[\sigma_{red}(r_j,\tau)\right]^n d\tau \quad = \quad 1 \tag{21}$$

for all points $r_j \in V$, or at least, along appropriate lines or surfaces to produce a rupture mechanism (cf. Życzkowski, 1988). In case when the condition (21) is simultaneously fulfilled at all points of the structure, first macrocracks appear simultaneously everywhere, which defines structures of uniform creep strength with respect to brittle rupture. On the other hand, assuming the orthotropic damage law, the structures of uniform creep strength fulfil the condition:

$$\sup_{\forall r\in V}\{\Omega_{ij}(r)\} \quad \cong \quad 1 \qquad \text{for} \qquad \tau=\tau_I \tag{22}$$

3.2 Constraints

The above formulated optimality criterion requires appropriate constraints, which may take the following form (cf. Ganczarski and Skrzypek, 1994):
The geometric constraint of the maximum and minimum thicknesses

$$h_{\max} \geq h(r) \geq h_{\min} \tag{23}$$

The constraint of maximum local gradient of temperature assuring the assumption of small displacements

$$\max\{\mathrm{d}T/\mathrm{d}r\} \leq (\mathrm{d}T/\mathrm{d}r)_{\max} \tag{24}$$

The condition of constant volume

$$V \quad = \quad 2\pi\int_0^R h(r)r\mathrm{d}r \quad = \quad \text{const} \quad \text{or} \quad \delta V \quad = \quad 2\pi\int_0^R \delta h(r)r\mathrm{d}r \quad = \quad 0 \tag{25}$$

3.3 Decision Variable

The distribution of disk thickness $h(r)$ is considered as the decision variable.

3.4 Optimization Methods

The procedure of optimization, based on iterative corrections of the decision variable, has been suggested. When the optimization with respect to uniform creep strength under constant volume of a structure is performed, increments of decision variable are proportionally chosen to the level of the dominant component of the damage tensor (cf. Ganczarski and Skrzypek, 1994)

$$\Delta h_j \quad = \quad \mathcal{P}\Delta D_j - \Delta h_m \qquad\qquad \Delta D_j \quad = \quad \sup\left\{D_{r/\theta}\right\}_j \tag{26}$$

where the average correction Δh_m must satisfy the constant volume condition

$$\Delta h_m \quad = \quad \frac{\sum_j \mathcal{P}\Delta D_j r_j}{\sum_j r_j} \tag{27}$$

whereas the step factor $\mathcal{P}$ should be chosen experimentally. The process of damage equalization is continued until the following condition is fulfilled:

$$\sup\left\{D_{r/\theta}\right\}_j \quad \leq \quad \text{EPS} \quad \cong \quad 1 \quad \forall j \tag{28}$$

The suggested procedure is essentially relevant to the concept of the fully damage design method. This method leads to exact solutions (optimal with respect to maximal lifetime) when the structure is statically determinate, single loadings are applied, the geometry changes are neglected. If the above assumptions are exceeded, the uniform creep strength solution may occur to be non-optimal. An exact solution may be obtained when more rigorous optimization approaches are used.

4 Numerical Algorithm for Thermo-Creep-Damage Problem

To solve the complete coupled initial-boundary problem we discretize time by inserting the ordered number of N time-intervals $\Delta\tau_k$, where $\tau_0 = 0$, $\Delta\tau_k = \tau_k - \tau_{k-1}$ and $\tau_N = \tau_R$ (rupture). Hence, the initial-boundary coupled problem is reduced to a sequence of quasistatic boundary-value problems, the solution of which determines unknown functions at a given time τ_k, e.g. $F(r,\tau_k) = F^k(r)$, $T(r,\tau_k) = T^k(r)$, etc. To account for primary and tertiary creep regimes a dynamically controlled time step $\Delta\tau_k$ is required, the length of which is defined by the bounded maximum damage increment:

$$\Delta D^{lower} \le \max_{(i,j;r)} \left\{ \left[\dot{D}^k_{ij}(r) - \dot{D}^{k-1}_{ij}(r) \right] \Delta\tau_k \right\} \le \Delta D^{upper} \tag{29}$$

Discretizing also the radial coordinate r_i, by inserting an equal mesh $\Delta r = r_i - r_{i-1}$, and rewriting equations (7) and (9) for a time step τ_k in terms of finite differences of T_i and F_i with respect to the r_i coordinate, we furnish at each time step τ_k the equation for the thermal state

$$\begin{aligned}
&\left[\frac{1}{(\Delta r)^2} - \left(\frac{-\lambda^{eq}_{i-1} + \lambda^{eq}_{i+1}}{2\lambda^{eq}_i \Delta r} + \frac{-h_{i-1} + h_{i+1}}{2h_i \Delta r} + \frac{1}{r}\right)\frac{1}{2\Delta r}\right] T_{i-1} - \left[\frac{2}{(\Delta r)^2} + \frac{2\beta}{\lambda^{eq}_i}\frac{\sqrt{1 + \left(\frac{-h_{i-1}+h_{i+1}}{2\Delta r}\right)^2}}{h_i}\right] T_i \\
&+ \left[\frac{1}{(\Delta r)^2} + \left(\frac{-\lambda^{eq}_{i-1} + \lambda^{eq}_{i+1}}{2\lambda^{eq}_i \Delta r} + \frac{-h_{i-1} + h_{i+1}}{2h_i \Delta r} + \frac{1}{r}\right)\frac{1}{2\Delta r}\right] T_{i+1} = -\frac{2\beta}{\lambda^{eq}_i}\frac{\sqrt{1 + \left(\frac{-h_{i-1}+h_{i+1}}{2\Delta r}\right)^2}}{h_i} T_\infty
\end{aligned} \tag{30}$$

and for the differential operators of the mechanical state

$$\begin{aligned}
\nabla^4 F \cong &\left[\frac{1}{(\Delta r)^4} - \frac{4r - 3\Delta r}{4r(r-\Delta r)(\Delta r)^3}\right] F_{i-2} + \left[-\frac{4}{(\Delta r)^4} + \frac{2}{r(\Delta r)^3}\right] F_{i-1} + \left[\frac{6}{(\Delta r)^4} + \frac{1}{2(r-\Delta r)} \times \right. \\
&\left. \times \frac{1}{(r+\Delta r)(\Delta r)^2}\right] F_i + \left[-\frac{4}{(\Delta r)^4} - \frac{2}{r(\Delta r)^3}\right] F_{i+1} + \left[\frac{1}{(\Delta r)^4} + \frac{4r + 3\Delta r}{4r(r+\Delta r)(\Delta r)^3}\right] F_{i+2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\mathcal{B}}\right)\left(2\frac{\mathrm{d}^3 F}{\mathrm{d}r^3} + \frac{2-\nu}{r}\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} - \frac{1}{r^2}\frac{\mathrm{d}F}{\mathrm{d}r}\right) \cong -\frac{-h_{i-1} + h_{i+1}}{2h_i \Delta r} \times \\
\times \left[-\frac{F_{i-2}}{(\Delta r)^3} + \left(\frac{2}{(\Delta r)^3} + \frac{2-\nu}{r(\Delta r)^2} + \frac{1}{2r^2\Delta r}\right) F_{i-1} - 2\frac{2-\nu}{r(\Delta r)^2} F_i + \right. \\
\left. + \left(-\frac{2}{(\Delta r)^3} + \frac{2-\nu}{r(\Delta r)^2} - \frac{1}{2r^2\Delta r}\right) F_{i+1} + \frac{F_{i+2}}{(\Delta r)^3}\right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{1}{\mathcal{B}}\right)\left(\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - \frac{\nu}{r}\frac{\mathrm{d}F}{\mathrm{d}r}\right) \cong \left[\frac{(-h_{i-1} + h_{i+1})^2}{2h_i^2(\Delta r)^2} - \frac{h_{i-1} - 2h_i + h_{i+1}}{h_i(\Delta r)^2}\right] \times \\
\times \left[\left(\frac{1}{(\Delta r)^2} + \frac{\nu}{2r\Delta r}\right) F_{i-1} - \frac{2}{(\Delta r)^2} F_i + \left(\frac{1}{(\Delta r)^2} - \frac{\nu}{2r\Delta r}\right) F_{i+1}\right]
\end{aligned}$$

$$(1-\nu)\mathcal{B}\nabla^2\left(\frac{U}{\mathcal{B}}\right) \cong (1-\nu)h_i\left\{\left[\frac{1}{(\Delta r)^2}-\frac{1}{2r\Delta r}\right]\frac{U_{i-1}}{h_{i-1}} - \frac{2}{(\Delta r)^2}\frac{U_i}{h_i} + \left[\frac{1}{(\Delta r)^2}+\frac{1}{2r\Delta r}\right]\frac{U_{i+1}}{h_{i+1}}\right\}$$

$$(1-\nu^2)\mathcal{B}\alpha\nabla^2 T \cong E\alpha h_i\left\{\left[\frac{1}{(\Delta r)^2}-\frac{1}{2r\Delta r}\right]T_{i-1} - \frac{2}{(\Delta r)^2}T_i + \left[\frac{1}{(\Delta r)^2}+\frac{1}{2r\Delta r}\right]T_{i+1}\right\}$$

$$\mathcal{B}\nabla^2\left(\frac{n_\theta^c-\nu n_r^c}{\mathcal{B}}\right) \cong h_i\left\{\left[\frac{1}{(\Delta r)^2}-\frac{1}{2r\Delta r}\right]\frac{n^c_{\theta_{i-1}}-\nu n^c_{r_{i-1}}}{h_{i-1}} - \frac{2}{(\Delta r)^2}\frac{n^c_{\theta_i}-\nu n^c_{r_i}}{h_i} + \right. \qquad (31)$$
$$\left. + \left[\frac{1}{(\Delta r)^2}+\frac{1}{2r\Delta r}\right]\frac{n^c_{\theta_{i+1}}-\nu n^c_{r_{i+1}}}{h_{i+1}}\right\}$$

$$\frac{1+\nu}{r}\mathcal{B}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n_\theta^c-n_r^c}{\mathcal{B}}\right) \cong \frac{1+\nu}{2r\Delta r}h_i\left[-\frac{n^c_{\theta_{i-1}}-\nu n^c_{r_{i-1}}}{h_{i-1}} + \frac{n^c_{\theta_{i+1}}-\nu n^c_{r_{i+1}}}{h_{i+1}}\right]$$

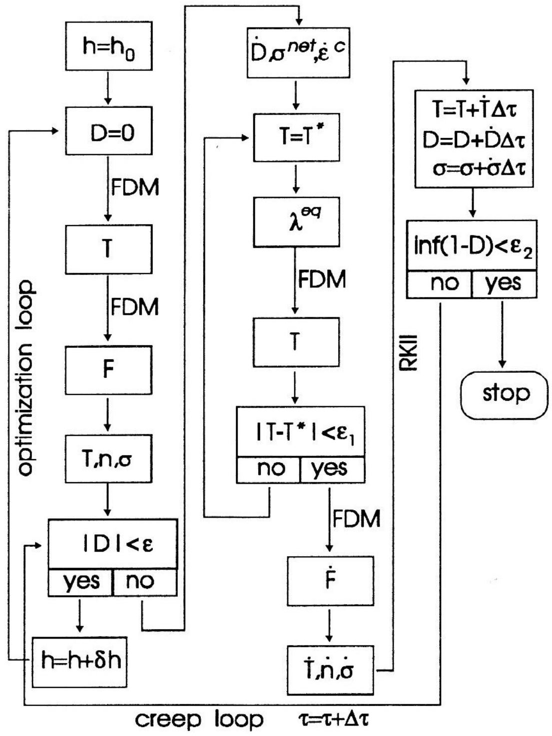

Figure 3. Numerical Algorithm for Coupled Thermo-Mechanical Problem

The numerical procedure begins when the elastic solution of the thermal and mechanical coupled problems is known. Assuming an initially constant thickness of a structure $[h]_j \equiv h_o$, and initial components of the damage tensor $[D_{r/\theta}]_j \equiv 0$, the elastic solution is obtained in the following way. Applying stage algorithm (cf. Figure 3), first, the equation of heat transfer, which is linear for the elastic problem, is solved and the distribution of temperature is found $[T]_j$. Then, equations of the mechanical state are solved, providing the distribution of the Airy function $[F]_j$, and the vector of the elastic state $[T^e, n^e_{r/\theta}, \sigma^e_{r/\theta}]_j$ is determined. Next, the program enters the creep loop, which requires the vector of net stress intensity, and the components of damage tensor and strain rates $[\sigma_i^{net}, \dot{D}_{r/\theta}, \dot{\varepsilon}^c_{r/\theta}]_j$ are computed. The thermal problem is now non-linear, hence, by inserting the previous solution for temperature $[T^*]_j$ to the substitutive coefficient of thermal conductivity λ^{eq} the solution of equation (30) provides the new temperature distribution $[T]_j$, considered next as an aproximate solution for λ^{eq} and subsequent temperature subiteration. The procedure is repeated until the calculated functions $[T]_j$ differs from $[T^*]_j$ by a given amount. In consequence, when rates of change of both the temperature $[\dot{T}]_j$ and the rates of inelastic forces are known $[\dot{n}^c_{r/\theta}]_j$, the rates of the Airy function $[\dot{F}]_j$ are found, and, finally the vector of state can be determined $[\dot{T}, \dot{n}_{r/\theta}, \dot{\sigma}_{r/\theta}]_j$. In the next time step, applying the Runge-Kutta II method for the thermal state and the mechanical state, the 'new' vector of state is computed, and the program jumps to the beginning of the creep loop. This numerical procedure is repeated until the highest value of the damage tensor reaches a certain level, then the program quits the loop, via the conditional statement.

5 Results

All numerical examples presented in this chapter deal with disks made of the ASTM-321 stainless steel (rolled 18 Cr, 8 Ni, 0.45 Si, 0.40 Mn, 0.1 C, Ti/Nb stabilized, austenitic, annealed at 1343K (1070°C), air cooled) of the following mechanical and thermal properties (cf. Holman, 1990): $E = 17.0 \times 10^3 \text{kG/mm}^2$, $\sigma_{0.2} = 12.0$ kG/mm^2, ν=0.3, $\varrho = 7850\text{kG/m}^3$, $\alpha = 1.85 \times 10^{-5}$1/K, λ_o=20W/mK, β=15W/m^{2}K, $R = 1.0$m, $h_o = 0.05$m, $p_o = 0.1 \times \sigma_{0.2}$, $\omega = 100\text{s}^{-1}$, $T_o = 798$K (525°C), $T_\infty = 773$K (500°C), $\sigma = 5.669 \times 10^{-8}\text{W/m}^2\text{K}^4$, ϵ_o=0.5. The temperature dependent material functions for creep rupture are

T (K)	T (°C)	m	n	σ^5_{CB} (kG/mm^2)	C (kG^{-n}s^{-1})
773	500	5.6	3.9	21	4.97×10^{-19}
873	600	4.5	3.1	10	4.27×10^{-16}
923	650	4.0	2.8	6	7.67×10^{-14}

Table 2. Temperature Dependent Material Functions

where σ^5_{CB} denotes the stress necessary to cause creep rupture after 10^5 h. The disk of uniform creep strength, the disk of constant thickness as well as the disk of uniform elastic strength (in sense of the Galileo hypothesis) are shown in Figure 4.

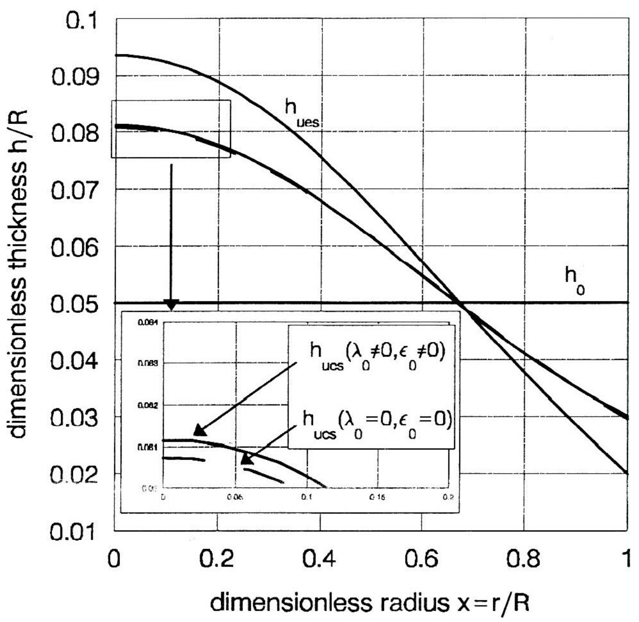

Figure 4. Optimal Profiles of Disks

In case of the disk of uniform creep strength where the thermo-damage coupling is disregarded $h_{ucs}(\lambda_o = 0, \epsilon_o = 0)$, and the disk of uniform creep strength where the thermo-damage coupling is taken into account $h_{ucs}(\lambda_o \neq 0, \epsilon_o \neq 0)$, differences between optimal profiles are barely noticeable (window in Figure 4). However, essential differences of lifetime are observed.

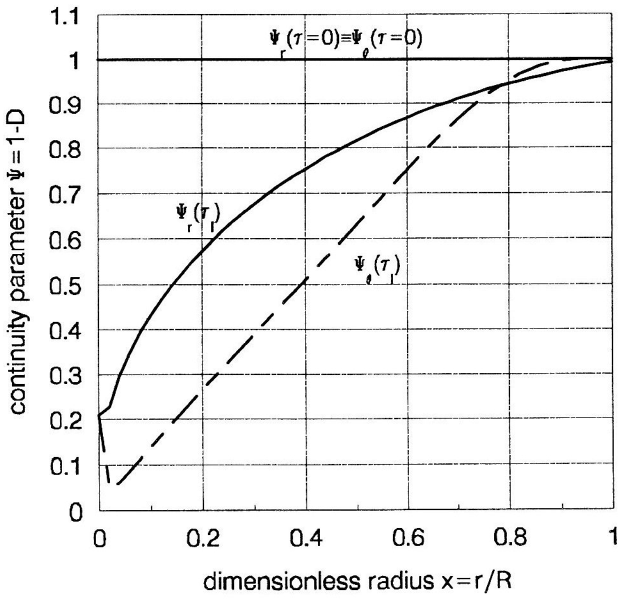

Figure 5. Distribution of Continuity Tensor Components in Disk of Constant Thickness

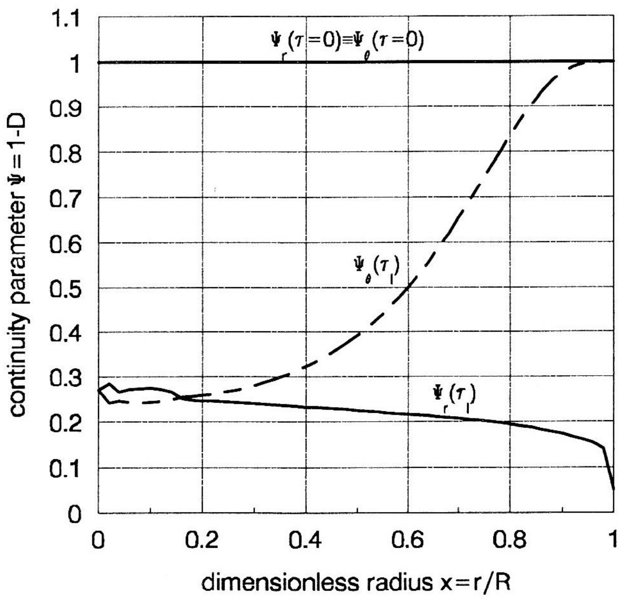

Figure 6. Distribution of Continuity Tensor Components in Disk of Uniform Creep Strength

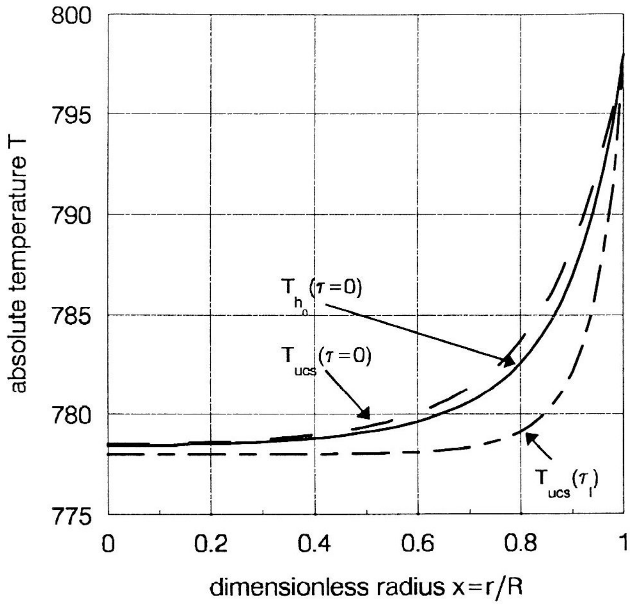

Figure 7. Distribution of Temperature in Disk of Uniform Creep Strngth versus Initial Temperature in Disk of Constant Thickness and Initial Temperature in Disk of Uniform Elastic Strength

Two rupture mechanisms accompany the process of disk design. In case of the disk of constant thickness $h_o(\lambda_o \neq 0, \epsilon_o \neq 0)$, the distribution of the continuity components at the instant of rupture is presented in Figure 5. The damage accumulation with respect to the circumferential component Ψ_θ is concentrated near the centre. Other rupture mechanism, the uniform damage accumulation with respect to the radial component Ψ_r and accompanying the narrow zone of damage with respect to the circumferential component Ψ_θ near the centre, appears in a disk optimally designed $h_{ucs}(\lambda_o \neq 0, \epsilon_o \neq 0)$ (cf. Figure 6). Corresponding distributions of temperature versus initial temperature in a disk of constant thickness are shown in Figure 7. During the creep-damage process the local temperature decreases, lowering damage accumulation, which leads to longer lifetime compared to the case when the thermo-damage coupling is disregarded $h_{ucs}(\lambda_o = 0, \epsilon_o = 0)$. Lifetime of all previously discussed cases are compared in Table 3.

	constant thickness $h(x) = h_o$		uniform elastic strength $h_{ues}(x)$	uniform creep strength $h_{ucs}(x)$	
	no coupling $\lambda_o = 0, \epsilon_o = 0$	coupling $\lambda_o \neq 0, \epsilon_o \neq 0$	coupling $\lambda_o \neq 0, \epsilon_o \neq 0$	no coupling $\lambda_o = 0, \epsilon_o = 0$	coupling $\lambda_o \neq 0, \epsilon_o \neq 0$
life time	τ_I	$1.01\tau_I$	$1.03\tau_I$	$4.43\tau_I$	$4.70\tau_I$

Table 3. Comparison of Lifetime for Optimally Designed Disks

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Symbols

Mechanical Quantities

$\mathcal{B}$	membrane stiffness
$C_k(T), n_k(T)$	constants in damage law
$D, \mathbf{D}$	damage parameter and second-order damage tensor
E, ν	Young's modulus, Poisson's ratio
$\varepsilon, \varepsilon_i$	strain tensor and strain intensity
$f(\tau)$	given time function
F	Airy's function
h, h_o	thickness and reference thickness
$m(T)$	exponent in creep law
n_r, n_θ	components of membrane force
p_o	uniform radial stretching
$\Psi, \mathbf{\Psi}$	continuity parameter and second-order continuity tensor
r, θ	polar coordinates
R	radius of disk
ϱ	mass density
$\mathbf{s}, \sigma$	stress deviator and stress tensor
σ_i	stress intensity
τ	time
U	potential of body forces
V	volume of disk
ω	angular velocity

Thermal Quantities

α	coefficient of thermal expansion
β	coefficient of free convection
c_v	specific heat
ϵ_o	emissivity of a gray body
λ_o	thermal conductivity of virgin solid
λ^{eq}	eqivalent thermal conductivity
$\widetilde{\lambda}$	thermal conductivity of damaged solid
σ	Stefan-Boltzmann's constant
T	absolute temperature
T_o	temperature at outer edge
T_∞	temperature of cooling fluid stream

Additionally, superscript 'c' denotes an inelastic (creep) quantity, superscript '*net*' refers to a quantity with deterioration taken into account, a dot above symbol denotes derivative with respect to time.

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